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Perfect 2-Colorings of k-Regular Graphs

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Abstract. We study perfect 2-colorings of regular graphs. In particular, we consider the 4-regular case. We obtain a characterization of perfect 2-colorings of toroidal grids.

1. Introduction

Throughout this article, G is a finite connected simple graph with vertex set V and edge set E. A *perfect m-coloring* of G with matrix $S = [s_{ij}]\,; i, j = 1, 2, \ldots, m$ is a coloring of V with the colors $\{1, 2, \ldots, m\}$ such that every vertex of color i has s_{ij} neighbors of color j. Note that if some entry $s_{ii} > 0$, then these are not proper colorings. The matrix S is called the *m-coloring matrix*. Two m-coloring matrices S_1 and S_2 are called equivalent if there exists a permutation matrix R such that $S_2 = R^t S_1 R$; this corresponds to permuting the colors. We call a matrix $S_{m\times m}$ *admissible* for G if there exists a perfect m-coloring of G with the parameters $s_{ij}; i, j = 1, \ldots, m$.

According to the definition, if G admits a perfect m-coloring, then all vertices of the same color are of the same degree. So a necessary condition for the existence of a perfect m-coloring of G is that the degree sequence of G contains at most m different numbers.

In this article we study perfect 2-colorings. We call the first color white, and the second color black. For a perfect 2-coloring of G, we denote the sets of white and black vertices by W and B, respectively. The coloring matrix of a perfect 2-coloring is of the form

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}.$$

The first row and column belong to the white color, and the second row and column belong to the black color. That means, every white vertex is adjacent to

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s_{11} white vertices and s_{12} black vertices, and every black vertex is adjacent to s_{21} white vertices and s_{22} black vertices.

2. Perfect 2-Colorings of k-Regular Graphs

In this section we assume that k is a positive integer and G is k-regular. Let $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$ be a 2-coloring matrix of G. Naturally $s_{11}+s_{12} = k$ and $s_{21}+s_{22} = k$. Therefore, the number of possible cases for the first row is $k+1$. For each of them, the number of possible cases for the second row is at most $k+1$. So, in general, there are at most $(k+1)^2$ matrices some of which are impossible. If $s_{12} = 0$, then no white vertex has a black neighbor. Then the only connected graphs that admit a perfect 2-coloring with matrix S are those in which all vertices have the same color. Thus, we assume $s_{12} > 0$ and $s_{21} > 0$. On the other hand, by interchanging colors, we have equivalent coloring matrices. Therefore, for 2-coloring matrices, we can assume $1 \leq s_{21} \leq s_{12} \leq k$. It follows that:

Lemma 2.1. *There are $\binom{k+1}{2}$ different matrices S for which S is an admissible 2-coloring matrix of some connected k-regular graph.*

We describe these matrices as follows:

$$A_{i,k-j} = \begin{bmatrix} k-i+1 & i-1 \\ j+1 & k-j-1 \end{bmatrix};\ i = 2, \ldots, k+1;\ j = 0, \ldots, i-2. \tag{2.1}$$

Lemma 2.2.([1]) *Suppose that $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$ is an admissible matrix for G. We have:*

(1) $|W| = \frac{s_{21}}{s_{12}}|B|$;

(2) $|V|$ is divisible by $\frac{(s_{12}+s_{21})}{\gcd(s_{12}, s_{21})}$.

Proof. Since every white vertex has s_{12} black neighbors, and every black vertex has s_{21} white neighbors, we have $s_{12}|W| = s_{21}|B|$. In addition, since the order of G is

$$|W| + |B| = \frac{(s_{12}+s_{21})}{s_{21}}|W| = \frac{(s_{12}+s_{21})}{s_{12}}|B|,$$

it follows that $|V|$ is divisible by $\frac{(s_{12}+s_{21})}{\gcd(s_{12}, s_{21})}$. □

Theorem 2.3. *For a graph G and with notation as in (2.1), when G is k-regular, then*

(1) *$A_{k+1,1}$ is admissible for G if and only if G is bipartite;*

(2) *if $A_{k,2}$ is admissible for G, then $|V|$ is divisible by 4;*

(3) *if G is bipartite and $A_{k-1,3}$ is admissible for G, then $|V|$ is divisible by 4.*

Proof. (1) Suppose $A_{k+1,1}$ is admissible for G. Since $s_{11} = s_{22} = 0$, no vertex has a neighbor of the same color as itself. So G is a bipartite graph with bipartition (W, B). Conversely, suppose G is a bipartite graph with bipartition (X, Y). Since G is k-regular, every vertex in X has k neighbors in Y, and every vertex in Y has k neighbors in X; and also $|X| = |Y|$. Therefore, G admits a perfect coloring with matrix $A_{k+1,1}$ by taking partite sets as W and B.
(2) Suppose $A_{k,2}$ is admissible for G. Since $s_{11} = s_{22} = 1$ and $s_{12} = s_{21}$, by Lemma 2.2, the number of white vertices is even and equals the number of black vertices. Therefore, the order of G must be a multiple of 4.
(3) Suppose $A_{k-1,3}$ is admissible for G. Since $s_{11} = s_{22} = 2$, the subgraph of G induced by the set W is a union of disjoint even cycles, as is the subgraph of G induced by the set B (note that G is bipartite). On the other hand, since $s_{12} = s_{21}$, the number of white vertices is equal to the number of black vertices. Therefore, the order of G must be a multiple of 4. □

3. Perfect 2-Colorings of Toroidal Grids

In this section we consider 4-regular graphs and obtain a characterization of perfect 2-colorings of toroidal grids. The ten possible 2-coloring matrices for 4-regular graphs are listed below.

$$A_{2,4} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A_{3,3} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad A_{3,4} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A_{4,2} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad A_{4,3} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \quad A_{4,4} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

$$A_{5,1} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \quad A_{5,2} = \begin{bmatrix} 0 & 4 \\ 3 & 1 \end{bmatrix}, \quad A_{5,3} = \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}, \quad A_{5,4} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$$

Definition 3.1. Suppose $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are simple graphs. The Cartesian product of G_1 and G_2, written $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$ in which (u, v) is adjacent to (u', v') if and only if $u = u'$ and $vv' \in E_2$, or $v = v'$ and $uu' \in E_1$. Note that the Cartesian product operation is symmetric; that is $G_1 \square G_2 \cong G_2 \square G_1$.

Let $n, m \geq 3$ be integers. The Cartesian product of two cycles, $G = C_n \square C_m$, is known as a toroidal grid graph (See Figure 1). According to the definition, the toroidal grid is a 4-regular graph.

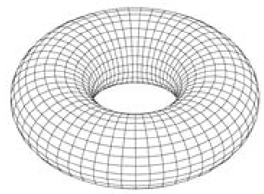

Figure 1: Toroidal grid graph

Suppose $V(G) = \{(i,j) |\, 0 \leq i \leq n-1; 0 \leq j \leq m-1\}$. To show perfect 2-colorings of G in the next theorems, we consider a part of G as an orthogonal grid, as shown in Figure 2. We assume that its horizontal paths are of length n such that the leftmost vertex of each path is adjacent to rightmost vertex of it, and its vertical paths are of length m such that the topmost vertex of each path is adjacent to the bottom-most vertex of it. Assume that the vertex in the box is (0, 0). Index i increases with left-right orientation of the horizontal cycles, and index j increases with down-up orientation of the vertical cycles.

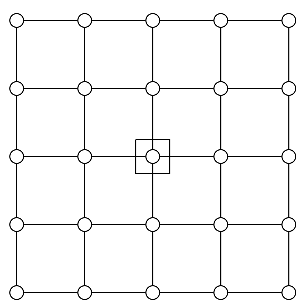

Figure 2: A part of toroidal grid $C_n \square C_m$

In the following, we investigate necessary and sufficient conditions for the admissibility of each of ten coloring matrices for this class of graphs. Note that since $C_n \square C_m \cong C_m \square C_n$, in the following results, we can switch conditions from m to n and vice versa.

Theorem 3.2. *Let $3 \leq n, m < \infty$. The toroidal grid $G = C_n \square C_m$ admits a perfect 2-coloring with matrix $A_{2,4}$ if and only if $m \equiv 0 \pmod 4$.*

Proof. Suppose a perfect 2-coloring with matrix $A_{2,4}$ exists. Then, since $s_{11} =$

$s_{22} = 3$, the subgraph of G induced by the set of white vertices is 3-regular, as is the subgraph of G induced by the set of black vertices. On the other hand, $s_{12} = s_{21} = 1$. Thus the number of white vertices is even and equals the number of black vertices. Therefore, the order of G must be a multiple of 4.

We now show that $A_{2,4}$ is admissible only for $G = C_n \Box C_m$ with $m \equiv 0 \pmod 4$. Since G is vertex-transitive and $C_n \Box C_m \cong C_m \Box C_n$, without loss of generality, vertex $(0, 0)$ is black and vertex $(0, 1)$ is white, as shown in Figure 3. Every white vertex has one black neighbor and every black vertex has one white neighbor. So the vertices $(1, 0), (n-1, 0), (0, m-1)$ are black and the vertices $(1, 1), (n-1, 1), (0, 2)$ are white. Continuing in this way, the vertices on cycles $\{(i, 0)|0 \le i \le n-1\}$ and $\{(i, m-1)|0 \le i \le n-1\}$ are black, and vertices on cycles $\{(i, 1)|0 \le i \le n-1\}$ and $\{(i, 2)|0 \le i \le n-1\}$ are white. Assuming $m \equiv 0 \pmod 4$, this coloring can uniquely be extended to other vertices. Let $W = \{(i, j)|\, j \equiv 1 \,\text{or}\, 2 \pmod 4\}$ and $B = V(G) \backslash W$. Then, every vertex in W has three neighbors in W and one neighbor in B, and every vertex in B has one neighbor in W and three neighbors in B. This is a perfect 2-coloring with matrix $A_{2,4}$. □

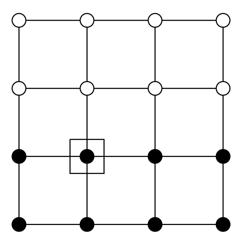

Figure 3: Perfect 2-coloring of $C_n \Box C_m$ with matrix $A_{2,4}$

Theorem 3.3. *Let $3 \le n, m < \infty$. The toroidal grid $G = C_n \Box C_m$ admits a perfect 2-coloring with matrix $A_{3,3}$ if and only if $m \equiv 0 \pmod 2$.*

Proof. Suppose a perfect 2-coloring with matrix $A_{3,3}$ exists. Then, since $s_{12} = s_{21}$, we have by Lemma 2.2 that the number of white vertices is equal to the number of black vertices. So the order of G must be even. Therefore m (or n) must be even. Let $m \equiv 0 \pmod 2$. The sets $W = \{(i, j)|\, j \equiv 0 \pmod 2\}$ and $B = V(G) \setminus W$ give us a perfect 2-coloring with matrix $A_{3,3}$. The coloring of a part of graph is shown in Figure 4. □

Theorem 3.4. *Let $3 \le n, m < \infty$. The toroidal grid $G = C_n \Box C_m$ admits a perfect 2-coloring with matrix $A_{3,4}$ if and only if $m \equiv 0 \pmod 3$.*

Proof. Suppose a perfect 2-coloring with matrix $A_{3,4}$ exists. Then, since $s_{12} = 2$ and $s_{21} = 1$, we have by Lemma 2.2 that $2w = b$ where w is the number of white vertices and b is the number of black vertices. So the order of G must be a multiple of 3. Therefore m (or n) must be a multiple of 3. Let $m \equiv 0 \pmod 3$. The sets

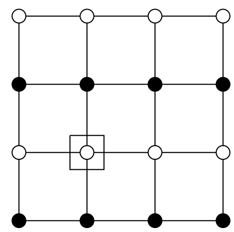

Figure 4: Perfect 2-coloring of $C_n \Box C_m$ with matrix $A_{3,3}$

$W = \{(i,j) | j \equiv 1 \pmod 3\}$ and $B = V(G) \setminus W$ give us a perfect 2-coloring with matrix $A_{3,4}$. The coloring of a part of graph is shown in Figure 5. □

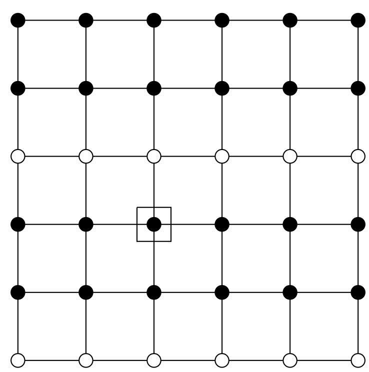

Figure 5: Perfect 2-coloring of $C_n \Box C_m$ with matrix $A_{3,4}$

Theorem 3.5. *Let $3 \leq n, m < \infty$. The toroidal grid $G = C_n \Box C_m$ admits a perfect 2-coloring with matrix $A_{4,2}$ if and only if $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 2$.*

Proof. Suppose $A_{4,2}$ is admissible for G. By Lemma 2.3, the order of G must be a multiple of 4. But this condition is not sufficient. We will show that the sufficient condition for the existence of this coloring is that $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 2$.

Every white vertex has one white neighbor. According to the structure of graph, without loss of generality the vertices $(0,0)$ and $(1,0)$ are white, as shown in Figure 6. So the vertices $(0,1), (1,1), (2,0), (0,m-1), (1,m-1), (n-1,0)$ are all black. Since $(2,1)$ is adjacent to two black vertices $(1,1), (2,0)$, it must be white. Similarly, $(2,m-1)$ is also white. Now consider $(3,0)$. The vertex $(2,0)$ has three white neighbors, $(2,1), (1,0), (2,m-1)$, so $(3,0)$ is black. This vertex has one black neighbor, so its other neighbors, $(4,0), (3,1), (3,m-1)$ are white. Continuing we find that in each cycle of length n, every vertex has one neighbor of the same color, and in each cycle of length m, the color of verticies changes alternately. Therefore to complete the coloring we must have $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 2$. $\square$

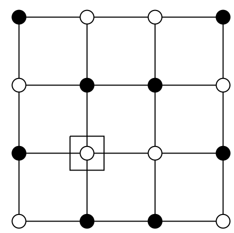

Figure 6: Perfect 2-coloring of $C_n \square C_m$ with matrix $A_{4,2}$

Theorem 3.6. *Let $3 \leq n, m < \infty$. The toroidal grid $G = C_n \square C_m$ admits a perfect 2-coloring with matrix $A_{4,3}$ if and only if $m, n \equiv 0 \pmod 5$.*

Proof. We specify the colors of the vertices of a 5×5 grid and show that this pattern repeats in each consecutive 5×5 grid on n-cycles and m-cycles.

Since G is vertex-transitive, without loss of generality $(0,0)$ and $(1,0)$ are white, as shown in Figure 7. So since $s_{11} = 1$, their neighbors, $(n-1,0), (0,m-1), (0,1), (1,m-1), (1,1), (2,0)$ are all black. The vertex $(1,1)$ is black and must have two black neighbors. So one of the vertices $(2,1)$ and $(1,2)$ is black. Without loss of generality, $(2,1)$ is black. The vertices $(1,1)$ and $(2,1)$ have two black neighbors, so the vertices $(1,2)$ and $(2,2)$ are white and their neighbors, $(0,2), (1,3), (2,3), (3,2)$ are all black. Also, the vertex $(3,1)$ is white. Thus, the black vertex $(3,2)$ has two white neighbors and so $(3,3)$ is black. Similarly, the vertices $(2,3)$ and $(3,3)$ have two black neighbors, so the vertices $(2,4)$ and $(3,4)$ are white and $(1,4)$ is black. Now consider $(0,3)$. Since the black vertex $(1,3)$ has two black neighbors, the vertex $(0,3)$ is white and so $(n-1,2)$ is black. Since $(n-1,1)$ has three black neighbors, this vertex and its fourth neighbor, $(n-2,1)$, are white. Therefore, $(n-2,2)$ is black and $(n-1,3)$ is white. Also, the vertex $(n-1,m-1)$ is black. Arguing in the same way, the vertices $(2,m-1)$ and $(3,m-1)$ are white and $(3,0)$ is black.

Continuing we specify the color of other vertices and find that the pattern of 5×5 grid with vertices $\{(i,j) | i = n-1, 0, 1, 2, 3;, \ j = m-1, 0, 1, 2, 3\}$ repeats in each consecutive 5×5 grid on n-cycles and m-cycles. So to complete the coloring we must have $m, n \equiv 0 \pmod 5$. $\square$

Definition 3.7. A set $S \subseteq V(G)$ is called a total dominating set if each vertex $v \in V(G)$ is adjacent to a vertex in S. A total dominating set S is called efficient, if every vertex $v \in V(G)$ is adjacent to exactly one vertex in S.

The above definition immediately implies that G admits a perfect 2-coloring with matrix $A_{4,4}$ if and only if G has an efficient total dominating set. Dejter [4] referred to a efficient total dominating set as a total perfect code. If the induced

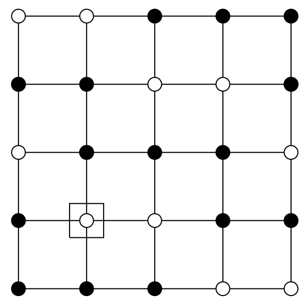

Figure 7: Perfect 2-coloring of $C_n \square C_m$ with matrix $A_{4,3}$

components of a total perfect code in a grid graph are pairwise parallel edges, then the code is called parallel. Similarly, we call a perfect 2-coloring with matrix $A_{4,4}$ in toroidal grid parallel, if the edges with white ends are parallel. So we have the following theorem:

Theorem 3.8. *Let $3 \leq n, m < \infty$. The toroidal grid $G = C_n \square C_m$ admits a parallel perfect 2-coloring with matrix $A_{4,4}$ if and only if $m, n \equiv 0 \pmod 4$.*

The coloring of a part of graph with matrix $A_{4,4}$ is shown in Figure 8.

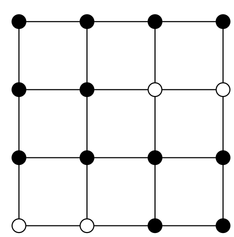

Figure 8: Parallel perfect 2-coloring of $C_n \square C_m$ with matrix $A_{4,4}$

Theorem 3.9. *Let $3 \leq n, m < \infty$. The toroidal grid $G = C_n \square C_m$ admits a perfect 2-coloring with matrix $A_{5,1}$ if and only if $m, n \equiv 0 \pmod 2$.*

Proof. A graph admits a perfect 2-coloring with matrix $A_{5,1}$ if and only if it is bipartite. A toroidal grid $G = C_n \square C_m$ is bipartite if and only if $m, n \equiv 0 \pmod 2$. Let $W = \{(i,j) | \, i, j \equiv 0 \pmod 2\} \cup \{(i,j) | \, i, j \equiv 1 \pmod 2\}$ and $B = V(G) \setminus W$. Clearly (W, B) is a bipartition of G, and therefore gives us a perfect 2-coloring with matrix $A_{5,1}$. The coloring of a part of the graph is shown in Figure 9. □

Theorem 3.10. *Let $3 \leq n, m < \infty$. The toroidal grid $G = C_n \square C_m$ does not admit a perfect 2-coloring with matrix $A_{5,2}$.*

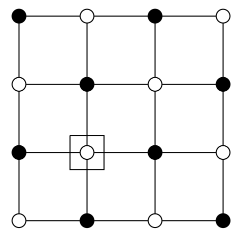

Figure 9: Perfect 2-coloring of $C_n \square C_m$ with matrix $A_{5,1}$

Proof. Suppose a perfect 2-coloring with matrix $A_{5,2}$ exists. Then, since $s_{11} = 0$ and $s_{22} = 1$, the neighbors of each white vertex are all black, and every black vertex has exactly one black neighbor. Since G is vertex-transitive and $C_n \square C_m \cong C_m \square C_n$, without loss of generality, $(0,0)$ and $(1,0)$ are black. So their neighbors, $(0,1),(1,1),(n-1,0),(2,0),(0,m-1),(1,m-1)$, are all white. According to the structure of the graph, this contradicts that the set of white vertices is independent. Because, $(0,1)$ is adjacent to $(1,1)$, and also $(0,m-1)$ is adjacent to $(1,m-1)$. □

Theorem 3.11. *Let* $3 \leq n,m < \infty$. *The toroidal grid* $G = C_n \square C_m$ *admits a perfect* 2*-coloring with matrix* $A_{5,3}$ *if and only if* $m,n \equiv 0 \pmod 3$.

Proof. As shown in Figure 10, we specify the colors of the vertices of a 3×3 grid with vertices $\{(i,j)|i = n-1,0,1;\ j = m-1,0,1\}$ (the details are similar to the proof of Theorem 3.6). Continuing we can specify the color of other vertices and find that this pattern repeats in each consecutive 3×3 grid on n-cycles and m-cycles. So the necessary and sufficient condition for the existence of this coloring is that $m,n \equiv 0 \pmod 3$. □

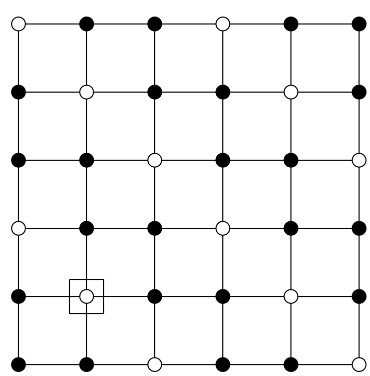

Figure 10: Perfect 2-coloring of $C_n \square C_m$ with matrix $A_{5,3}$

Theorem 3.12. *Let* $3 \leq n,m < \infty$. *The toroidal grid* $G = C_n \square C_m$ *admits a* *perfect* 2*-coloring with matrix* $A_{5,4}$ *if and only if* $m, n \equiv 0 \pmod 5$.

Proof. As in the proof of Theorem 3.11, we can specify the colors of the vertices of a 5×5 grid and show that this pattern repeats in each consecutive 5×5 grid on n-cycles and m-cycles. So the necessary and sufficient condition for the existence of this coloring is that $m, n \equiv 0 \pmod 5$. The coloring of a part of graph is shown in Figure 11. □

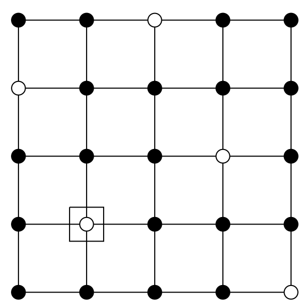

Figure 11: Perfect 2-coloring of $C_n \Box C_m$ with matrix $A_{5,4}$

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