On Interpretation of Hyperbolic Angle

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Abstract. Minkowski spaces have long been investigated with respect to certain properties and substructes such as hyperbolic curves, hyperbolic angles and hyperbolic arc length. In 2009, based on these properties, Chung *et al.* [3] defined the basic concepts of special relativity, and thus; they interpreted the geometry of the Minkowski spaces. Then, in 2017, E. Nesovic [6] showed the geometric meaning of pseudo angles by interpreting the angle among the unit timelike, spacelike and null vectors on the Minkowski plane. In this study, we show that hyperbolic angle depends on time, t. Moreover, using this fact, we investigate the angles between the unit timelike and spacelike vectors.

1. Introduction

In order to show a relativistic version of the Gauss-Bonnet theorem, an oriented pseudo angle between any two units or null vectors on the Minkowski plane was presented by Helzer [4]. Pseudo-angles were introduced as a generalization of the oriented hyperbolic angles between the unit vectors which were determined in [1], and [7]. Thus, it can be shown that the oriented hyperbolic angles between the unit vectors in the Minkowski spaces are equal to the oriented pseudo-angles between those vectors [6].

E. Nesovic [6] investigated if the measure of the unoriented pseudo-angles can

be represented with respect to the hyperbolic arcs of finite hyperbolic lengths. So, she defined the pseudo-perpendicular vectors in the Minkowski spaces. According to any unit or null vectors, she showed that she could associate completely eight vectors being pseudo-perpendicular on the Minkowski plane. Using the pseudo-perpendicular vectors, the geometric meaning of the oriented pseudo-angles was presented with regard to the hyperbolic arcs of finite hyperbolic lengths.

In [3], Chung *et al.* introduced the hyperbolic angle between two inertial observers. By using the hyperbolic cosine and sine functions, it was shown that the coordinate transformation rules may be obtained from the hyperbolic angles. Also, they presented that the angle connecting with the Bondi factor K by $K = e^v$ is equal to the hyperbolic angle defined by $v = \frac{s}{\rho}$, where s is the hyperbolic circumference of a hyperbolic curve and ρ is the invariant length of the curve.

In this study, by following the reference [3], we present some possible cases of the hyperbolic angle between two unit spacelike or timelike vectors in terms of the causal characters of these vectors.

2. Preliminaries

The Minkowski plane E_1^2 is an affine plane endowed with the standard flat metric given by

$$g = dx_1^2 - dx_2^2,$$

where (x_1, x_2) is a rectangular coordinate system of E_1^2. A vector $v \neq 0$ in E_1^2 can have one of the three causal characters: it can be spacelike, timelike or null (lightlike) if $g(v, v) > 0$, $g(v, v) < 0$ or $g(v, v) = 0$ and $v \neq 0$, respectively. In particular, the norm of a vector v is given by $\|v\|_L = \sqrt{|g(v, v)|}$. Two vectors v and w in E_1^2 are said to be orthogonal if $g(v, w) = 0$. A curve $\alpha : I \to E_1^2$ is defined as spacelike (resp. timelike, null) at t if $\alpha'(t)$ is a spacelike (resp. timelike, null) vector. If $e_2 = (0, 1)$ is a unit timelike vector, an arbitrary timelike vector v in E_1^2 is said to be future-pointing if $g(v, e_2) < 0$, or past-pointing if $g(v, e_2) > 0$. Two timelike vectors v and w have the same time-orientation if they are either future-pointing or past-pointing vectors. Consider that the ordered basis $\{e_2, e_1\}$ gives the positive orientation of the Minkowski plane E_1^2 with the above mentioned metric.

Let $\mathbb{R}_2^2$ be the set of matrices of two rows and two columns. Let $A = [a_{ij}]$, $B = [b_{jk}] \in \mathbb{R}_2^2$. A Lorentzian matrix multiplication denoted by "$\cdot_L$" is defined as

$$A \cdot_L B = [a_{i1}b_{1k} - a_{i2}b_{2k}].$$

The 2x2 L-identity matrix corresponding to the Lorentzian matrix multiplication denoted by I_2 is as follows:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now, the geometry of the Minkowski spaces will be studied in terms of the hyperbolic angles. Consider an inertial observer φ that presents the time and space coordinates of the events. By using a light signal, the coordinates (x_E, t_E) of an

event E can be specified and the velocity of the light signal is independent of the inertial observer.

For the sake of the argument, the light signal is sent at time t_1 towards the $+x$ direction by φ and this light signal reaches x_E at time t_E. Then, the signal is mirrored back to φ. If t_2 is the time of reception by φ, the below mentioned equalities are obtained:

(2.1)
$$t_E = \frac{t_1 + t_2}{2}$$
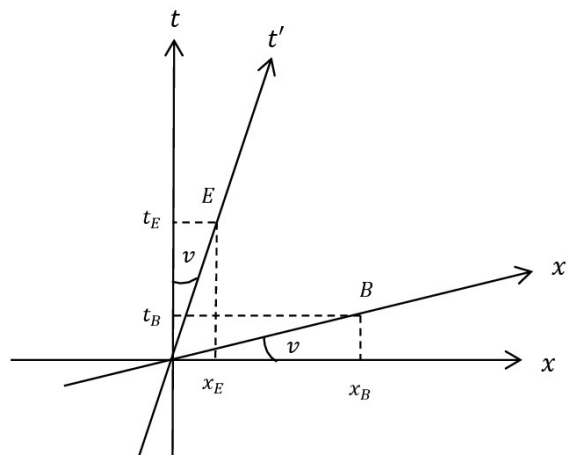
and

(2.2)
$$x_E = \frac{t_2 - t_1}{2}.$$

Fig. 1. The Coordinates of E and B Points

Once two light signals with the time duration $\triangle t$ are sent from the inertial observer to another inertial observer, the second one gets the time duration $K\triangle t$, where K is called the Bondi factor. So, when the signal is sent from φ to φ' at time t_0, the second observer φ' receives it at time Kt_0. The occurrence of signal reception by φ' is symbolized as the letter E and then, this signal returns back to φ at time $K^2 t_0$. If the parametrization $K = e^v$ is used to determine the hyperbolic angle between two temporal coordinate axes t and t', x_E and t_E are found in terms of t'_E as the below equalities:

$$t_E = t'_E \cosh v,$$
$$x_E = t'_E \sinh v,$$

where $t'_E = Kt_0$. Also, the angle between two spatial coordinate axes x and x' is equal to v. So, x_B and t_B are obtained in terms of x'_B:

$$x_B = x'_B \cosh v,$$
$$t_B = x'_B \sinh v$$

[3]. Furthermore, since the velocity of the inertial observer φ is $\frac{x_E}{t_E}$, any one of two following equalities is easily obtained:

$$\vartheta = \tanh v$$

or

$$v = \frac{1}{2} \ln \left(\frac{1 + \vartheta}{1 - \vartheta} \right).$$

For more details, we refer the readers to [2, 3].

3. Hyperbolic Angle

3.1. Hyperbolic Angles Between Two Timelike Unit Vectors

The hyperbolic circle is given by the set:

$$H_0^1 = \left\{ \overrightarrow{x} = (x_1, x_2) \in E_1^2 \mid g\left(\overrightarrow{x}, \overrightarrow{x} \right) = -1 \right\}.$$

The set H_0^1 of the unit vectors has two components, i.e.,

$$H_+^1 = \left\{ \overrightarrow{x} = (x_1, x_2) \in E_1^2 \mid g\left(\overrightarrow{x}, \overrightarrow{x} \right) = -1, \ x_2 > 0 \right\}$$
$$H_-^1 = \left\{ \overrightarrow{x} = (x_1, x_2) \in E_1^2 \mid g\left(\overrightarrow{x}, \overrightarrow{x} \right) = -1, \ x_2 < 0 \right\}.$$

The vectors in $H_+^1 \cup H_-^1$ are timelike [5]. If two unit timelike (spacelike) vectors do not belong to the same component of H_0^1 (S_1^1), the angle between these vectors is not determined. The angle between the unit timelike vectors on H_+^1 will be discussed. Suppose that the letter E is a point on H_+^1, that is, a vector $\overrightarrow{OE} = \overrightarrow{e}$ is the radius of the circle. Now, consider that a light signal is sent to $t = t_0$. This signal reflects from the point E and returns to the t axis. If the equalities (2.1), (2.2) and $K = e^{v_0}$ are used, the coordinates of this point are defined as follows:

$$x_E = t'_E \sinh v_0,$$
$$t_E = t'_E \cosh v_0,$$

where $t'_E = Kt_0$ [3]. Since $t'_E = \| \overrightarrow{e} \|_L$, it is possible to write the below expression

$$v_0 = \ln \left| \frac{1}{t_0} \right|$$

(See Fig. 2). It is shown that the hyperbolic angle depends on t_0, that is, $0 < t_0 \leq 1$. If $t_0 = 1$ is taken, the hyperbolic angle is equal to zero. In this case, the unit timelike vector is located on the time axis t. Besides, if the graphic of a logarithmic function is used,

$$0 \leq v_0 < \infty, \text{ i.e. }, v_0 \in \mathbb{R}^+ \cup \{0\}$$

is obtained. Thus, it is clear that the hyperbolic angle v_0 increases, reaching the infinite value while t_0 approaches to zero. Moreover, since the velocity of the vector $\overrightarrow{e}$ is $\dfrac{x_E}{t_E}$, it is possible to write

$$\vartheta = \tanh v_0$$

(See Fig. 2). By virtue of $v_0 = \ln \left| \dfrac{1}{t_0} \right|$, it is easy to see that the velocity is equal to

(3.1) $$\vartheta = \frac{1 - t_0^2}{1 + t_0^2}.$$

Hence, the velocity increases while the time decreases. If we use the equation (3.1),

$$t_0 = \sqrt{\frac{1 - \vartheta}{1 + \vartheta}}$$

is calculated. Here, because $0 < t_0 \leq 1$, the velocity ϑ can be written as the following inequality

$$0 \leq \vartheta < 1.$$

In this case, if these conventions are taken into consideration, it is certain that the below mentioned equalities are seen

$$\cosh v_0 = \frac{1}{\sqrt{1 - \vartheta^2}}$$

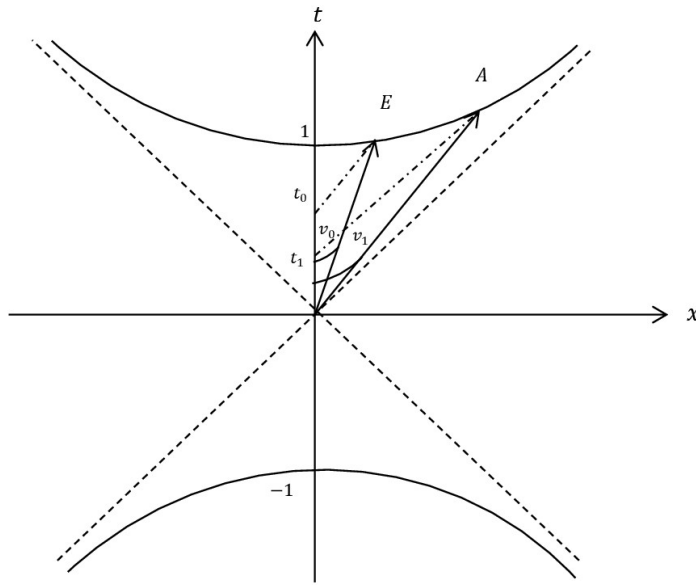$$\sinh v_0 = \frac{\vartheta}{\sqrt{1 - \vartheta^2}}.$$

Fig. 2. The Angle Between Timelike Vectors

Now, at the time $t_1 < t_0$, let us send one more light signal to this H_+^1. This signal reaches the point A and returns to the t axis. Here, the coordinates of this point are similar to the coordinates of the point E and the vector $\overrightarrow{0A} = \overrightarrow{a}$ is the radius of the hyperbolic circle again (See Fig. 2). So, the hyperbolic angle between the axis t and the unit timelike vector $\overrightarrow{a}$ is equal to

$$v_1 = \ln \left| \frac{1}{t_1} \right|,$$

where $t'_A = Kt_1$ and $K = e^{v_1}$. Similarly, the hyperbolic angle $v_1 \in \mathbb{R}^+ \cup \{0\}$ is available. It is easily seen that $v_1 > v_0$. Thus, the hyperbolic angle between the unit timelike vectors from $\overrightarrow{e}$ to $\overrightarrow{a}$ is defined as

$$(3.2) \qquad\qquad v' = v_1 - v_0 = \ln \left| \frac{t_0}{t_1} \right|,$$

where $v' \in \mathbb{R}^+ \cup \{0\}$.

Theorem 3.1. *Suppose that two points 'E and A' are located on H_+^1 and the coordinates of these points are $x = (\sinh v_0, \cosh v_0)$ and $X' = (\sinh v_1, \cosh v_1)$, respectively. Here, the hyperbolic angles are $v_0 = \ln \left| \dfrac{1}{t_0} \right|$ and $v_1 = \ln \left| \dfrac{1}{t_1} \right|$, where $t_1 < t_0$. When the point E is rotated to the hyperbolic angle $v' = \ln \left| \dfrac{t_0}{t_1} \right|$, it reaches*

the point A.

Proof. It is enough to show the existence of the equality $X' = \mathcal{F} \cdot_L x$, where the matrix $\mathcal{F}$ is defined as follows:

$$[\mathcal{F}] = \begin{bmatrix} \cosh v' & -\sinh v' \\ \sinh v' & -\cosh v' \end{bmatrix}.$$

If $\cosh v'$, $\sinh v'$ $\cosh v_0$ and $\sinh v_0$ are calculated, the above mentioned equality is obtained

$$\mathcal{F} \cdot_L x = \begin{bmatrix} \cosh v' & -\sinh v' \\ \sinh v' & -\cosh v' \end{bmatrix} \cdot_L \begin{bmatrix} \sinh v_0 \\ \cosh v_0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+t_1^2}{2t_1} \\ \frac{1-t_1^2}{2t_1} \end{bmatrix}$$

$$= \begin{bmatrix} \cosh v_1 \\ \sinh v_1 \end{bmatrix}$$

$$= X'.$$

Thus, the proof is completed.   □

3.2. Hyperbolic Angles Between Two Spacelike Unit Vectors

The Lorentz circle is expressed by

$$S_1^1 = \left\{ \overrightarrow{x} = (x_1, x_2) \in E_1^2 \mid g(\overrightarrow{x}, \overrightarrow{x}) = 1 \right\}.$$

Two components

$$S_1^{1+} = \left\{ \overrightarrow{x} = (x_1, x_2) \in E_1^2 \mid g(\overrightarrow{x}, \overrightarrow{x}) = 1, \ x_1 > 0 \right\}$$
$$S_1^{1-} = \left\{ \overrightarrow{x} = (x_1, x_2) \in E_1^2 \mid g(\overrightarrow{x}, \overrightarrow{x}) = 1, \ x_1 < 0 \right\}$$

belong to the set of S_1^1. The vectors in $S_1^{1+} \cup S_1^{1-}$ are spacelike. The angle between unit spacelike vectors on S_1^{1+} will be discussed. Consider that a light signal is sent at time $t = -1$. This signal is reflected from a point F and then, returns back to $t = 1$. It is easily seen that

$$t_F = 0, \ x_F = 1.$$

On the other hand, since $t'_a = K^{-1}$, $t'_b = -K$, the following equalities are obtained

$$t'_F = \frac{t'_a + t'_b}{2}$$

and

$$x'_F = \frac{t'_a - t'_b}{2}.$$

Because the expression $t'_F = 0$, $v = 0$ is easily obtained. Moreover, due to the equation $x'_F = \cosh v$, $x'_F = 1$ is calculated. Now, let us send the light signal at time $-\infty < -t'_1 < -1$, i.e., $1 < t'_1 < \infty$. This signal reflects from a point B. After that, another signal is sent at time $-1 < -t'_2 < 0$, i.e., $0 < t'_2 < 1$ and it reflects from a point C, where $x_C = x_B$ (See Fig. 3).
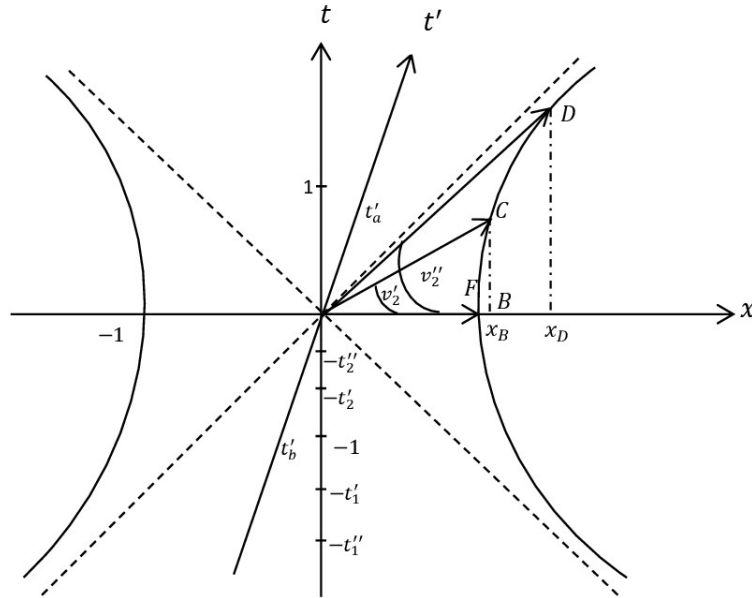

Fig. 3. The Angle Between Spacelike Vectors

If this point C is on S_1^{1+}, the vector $\overrightarrow{OC} = \overrightarrow{c}$ is the radius of this Lorentz circle. If the equalities (2.1), (2.2) and $K = e^{v'_2}$ are used, there exist the below mentioned expressions

$$x'_C = Kt'_2,$$
$$x_C = x'_C \cosh v'_2,$$
$$t_C = x'_C \sinh v'_2$$

[3]. Because $x'_C = \|\overrightarrow{c}\|_L$,

(3.3) $$v'_2 = \ln \left| \frac{1}{t'_2} \right|$$

is easily obtained. If the calculations and geometric comments mentioned above are taken into consideration, it becomes obvious that the hyperbolic angle depends on

time parameters. Chung *et al.* [3] show that $t'_1 = K \cosh{(v'_2)} t'_2$. In this case, the following expressions can be written

$$(3.4) \qquad\qquad v'_2 = \ln \sqrt{2 \left(\frac{t'_1}{t'_2} - \frac{1}{2} \right)}.$$

If we think about the equalities (3.3) and (3.4), it is clear that the below expression can be easily obtained

$$(3.5) \qquad\qquad t'_2 \left(2t'_1 - t'_2 \right) = 1.$$

In this case, when the above mentioned equalities are taken into consideration, it is easily seen that

$$(3.6) \qquad\qquad t'_1 = \cosh{\left(\ln{(t'_2)} \right)}.$$

The equality (3.6) shows the relationship between t'_1 and t'_2. Here, while $t'_2 = 1$, $t'_1 = 1$ is obtained. Moreover, the equation (3.5) indicates that while t'_2 approaches zero, t'_1 approaches the infinite value.

   At the present time, let us send two light signals at times $t''_1 > t'_1$ and $t''_2 < t'_2$. These signals reflect from two points x_D and D, respectively (See Fig. 3). Here, the vector $\overrightarrow{OD} = \overrightarrow{d}$ is the radius of the Lorentz circle, and the coordinates of the point D are obtained according to the coordinates of the point C. In this case, it is easily seen that

$$v''_2 = \ln \left| \frac{1}{t''_2} \right|,$$

where $K = e^{v''_2}$. Similarly, the equality $t''_1 = \cosh{\left(\ln{(t''_2)} \right)}$ can be available. Thus, it is presented that the hyperbolic angle between the unit spacelike vectors from $\overrightarrow{c}$ to $\overrightarrow{d}$ is given by

$$(3.7) \qquad\qquad v'' = v''_2 - v'_2 = \ln \left| \frac{t'_2}{t''_2} \right|,$$

where $v'', v'_2, v''_2 \in \mathbb{R}^+ \cup \{0\}$.

Theorem 3.2. *Suppose that two points 'C and D' are located on S_1^{1+} and the coordinates of these points are $y = (\cosh{v'_2}, \sinh{v'_2})$ and $Y' = (\cosh{v''_2}, \sinh{v''_2})$, respectively. Here, the hyperbolic angles are $v'_2 = \ln \left| \dfrac{1}{t'_2} \right|$ and $v''_2 = \ln \left| \dfrac{1}{t''_2} \right|$, where $t''_2 < t'_2$. When the point C is rotated to the hyperbolic angle $v'' = \ln \left| \dfrac{t'_2}{t''_2} \right|$, it reaches the point D.*

3.3. Hyperbolic Angles Between Timelike Unit Vector and Spacelike Unit Vector

Suppose that the points A and C are located on H_+^1 and S_1^{1+}, respectively. So, $\overrightarrow{OA} = \overrightarrow{a}$ is the unit timelike vector and $\overrightarrow{OC} = \overrightarrow{c}$ is the unit spacelike vector. In sections (3.1) and (3.2), it was shown that the angle between the vector $\overrightarrow{a}$ and the axis t is defined as $v_1 = \ln \left| \dfrac{1}{t_1} \right|$ and similarly, the angle between the vector $\overrightarrow{c}$ and the axis x is determined by $v_2' = \ln \left| \dfrac{1}{t_2'} \right|$, where $|t_2'| > |t_1|$ (See Fig. 4).
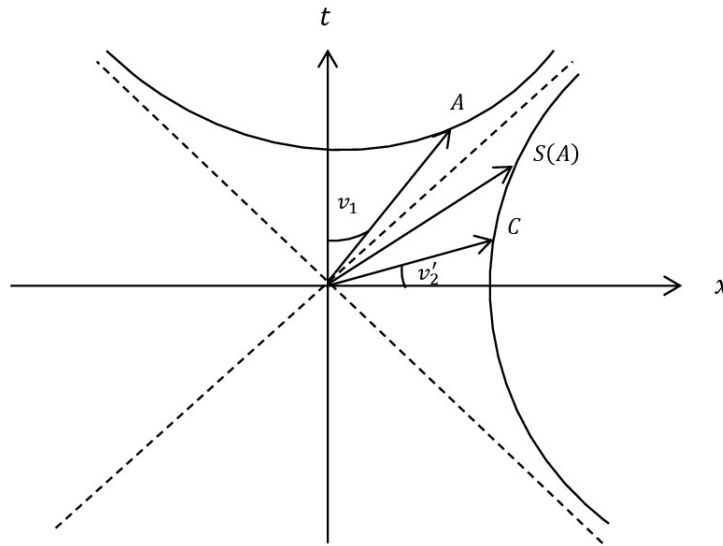

Fig. 4. The Angle Between Timelike and Spacelike Vectors

Let us choose a point $S(A)$ which is reflection with respect to the line $x = t$ of the point A. The point $S(A)$ is located on the S_1^{1+} circle and $\overrightarrow{OS(A)} = \overrightarrow{s}$ is the unit spacelike vector. Thus, the angle between the vector $\overrightarrow{a}$ and the axis t is equal to the angle between the vector $\overrightarrow{s}$ and the axis x. Consequently, the hyperbolic angle between two unit spacelike vectors $\overrightarrow{c}$ and $\overrightarrow{s}$ is shown by

$$(3.8) \qquad\qquad v = v_1 - v_2' = \ln \left| \frac{t_2'}{t_1} \right|.$$

Conclusion

This paper points out that the angle between the unit timelike vector $\overrightarrow{e}$ and the time axis t is $v_0 = \ln\left|\dfrac{1}{t_0}\right|$, where $0 < t_0 \leq 1$, and the angle between the unit spacelike vector $\overrightarrow{c}$ and the space axis x is $v_2' = \ln\left|\dfrac{1}{t_2'}\right|$, where $0 < t_2' < 1$. It is seen that the hyperbolic angle between the unit timelike or spacelike vectors changes according to the time axis t. If the time parameter t decreases, then both the hyperbolic angle v and the velocity ϑ increase.

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