KYUNGPOOK Math. J. 60(2020), 401-413
https://doi.org/10.5666/KMJ.2020.60.2.401
pISSN 1225-6951 eISSN 0454-8124

Periodic Solutions of a System of Piecewise Linear Difference Equations

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Abstract. In this article we consider the following system of piecewise linear difference equations: $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n| - 1$. We show that when the initial condition is an element of the closed second or fourth quadrant the solution to the system is either a prime period-3 solution or one of two prime period-4 solutions.

1. Introduction

Nearly ten years ago we began studying the global behavior of the following system of piecewise linear difference equations

$$(\mathcal{N}) \qquad \begin{cases} x_{n+1} = |x_n| + ay_n + b \\ \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, n = 0, 1, \ldots$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$ and the parameters $a, b, c,$ and $d \in \{-1, 0, 1\}$. Since each parameter can be one of three values, there are 81 systems. Each system is designated a number $\mathcal{N}$ which is given by

$$\mathcal{N} = 27(a + 1) + 9(b + 1) + 3(c + 1) + (d + 1) + 1.$$

* Corresponding Author.
Received June 28, 2017; revised September 2, 2019; accepted August 27, 2019.
2010 Mathematics Subject Classification: 39A10, 65Q10.
Key words and phrases: difference equation, periodic solution, stability.
This work was supported by Thailand Research Fund, National Research Council of Thailand and Pibulsongkram Rajabhat University.

Our purpose is to find patterns of behavior in order to better understand piecewise linear difference equations in general. We hope to develop methods to determine local asymptotic stability and global stability of such systems. The lack of these methods is evident by the fact that the global behavior of the Lozi equation and the Gingerbreadman Map (both can be expressed as a system of piecewise linear difference equations) are still not completely known. See [1, 2, 3, 5, 6].

After determining the global behavior of many of the 81 systems, we noticed a few trends. See [3, 4, 7, 8, 9]. Over half of the systems have exactly one equilibrium point, while some have two or three, and the remaining systems either have none or have infinitely many (which usually reside on a line). About a quarter of the systems have periodic solutions that are similar to the solutions of System(7). We were able to generalize a few systems; that is, we know their global behavior when some of the parameters are elements of $\mathbf{R}^+$, not just elements of $\{-1, 0, 1\}$. Within the next year we hope to complete a monograph that will share our results, detailed proofs, conjectures and open problems of all 81 systems.

Some of these systems are rather enigmatic. System(7) is one of them. It is the special case of System($\mathcal{N}$) where $a = b = d = -1$ and $c = 1$:

$$
(7) \qquad \begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, n = 0, 1, \dots \ .
$$

After months of brute force calculations, we only had a partial result. Wirot Tikjha shared this partial result in the 2016 International Conference on Difference Equations and Applications. See [7]. At the time of that presentation we knew the behavior of the system for only a small set of initial conditions (a section of the x-axis). Since the presentation, with the aid of computer simulations using random initial values, we were able to extend the set of initial conditions to include the closed second and fourth quadrant.

In this paper we show that when the initial condition (x_0, y_0) is an element in the closed second or fourth quadrant System(7) has one prime period-3 solution and two prime period-4 solutions given by

$$
P_3^1 = \begin{pmatrix} -\frac{1}{3}, & -1 \\ \\ \frac{1}{3}, & -\frac{1}{3} \\ \\ -\frac{1}{3}, & -\frac{1}{3} \end{pmatrix}, P_4^1 = \begin{pmatrix} -1, & -1 \\ \\ 1, & -1 \\ \\ 1, & 1 \\ \\ -1, & 1 \end{pmatrix}, \text{ and } P_4^2 = \begin{pmatrix} 1, & -3 \\ \\ 3, & 3 \\ \\ -1, & 5 \\ \\ -5, & 3 \end{pmatrix};
$$

where $\begin{pmatrix} a_1, & a_2 \\ b_1, & b_2 \\ c_1, & c_2 \end{pmatrix}$ represents the consecutive solutions (a_1, a_2), (b_1, b_2), and (c_1, c_2)

of the system. Please note that at the end of this article we share our conjecture for the global behavior of this system.

A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called *eventually periodic with prime period-p* or *eventually prime period-p solution* if there exists an integer $N > 0$ and p is the smallest positive integer such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is periodic with period-p; that is,

$$(x_{n+p}, y_{n+p}) = (x_n, y_n) \text{ for all } n \geq N.$$

2. Main Results

Set $\mathcal{L}_1 = \{(1, y)|y \geq 0\}$, $\mathcal{L}_2 = \{(1, y)|y \leq 0\}$, $\mathcal{L}_3 = \{(-1, y)|y \geq 0\}$, $\mathcal{L}_4 = \{(-1, y)|y \leq 0\}$, $\mathcal{L}_5 = \{(x, -1)|x \in \mathbf{R}\}$, $\mathcal{Q}_2 = \{(x, y)|x \leq 0, y \geq 0\}$, $\mathcal{Q}_4 = \{(x, y)|x \geq 0, y \leq 0\}$.

Theorem 2.1. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(7) with $(x_0, y_0) \in \mathcal{Q}_2 \cup \mathcal{Q}_4$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-3 solution P_3^1 or the prime period-4 solution P_4^1 or P_4^2.*

The proof of the theorem is a consequence of the following lemmas.

Lemma 2.2. *Suppose the initial condition $(x_0, y_0) \in \mathcal{L}_1 \cup \mathcal{L}_4$. Then solution of System(7), $\{(x_n, y_n)\}_{n=2}^{\infty}$ is the prime period-4 solution P_4^1.*

Proof. Let $(x_0, y_0) \in \mathcal{L}_1$. Then $(x_2, y_2) = (-1, -1) \in P_4^1$. Let $(x_0, y_0) \in \mathcal{L}_4$. Then

$$\begin{aligned} x_1 &= |x_0| - y_0 - 1 = -y_0 \geq 0 \\ y_1 &= x_0 + |y_0| - 1 = -y_0 - 2. \end{aligned}$$

If $y_1 = -y_0 - 2 < 0$, then

$$\begin{aligned} x_2 &= |x_1| - y_1 - 1 = 1 \\ y_2 &= x_1 + |y_1| - 1 = 1. \end{aligned}$$

If $y_1 = -y_0 - 2 \geq 0$, then

$$\begin{aligned} x_2 &= |x_1| - y_1 - 1 = 1 \\ y_2 &= x_1 + |y_1| - 1 = -2y_0 - 3 > 0. \end{aligned}$$

Note that $(x_2, y_2) \in \mathcal{L}_1$ and therefore $(x_4, y_4) \in P_4^1$. □

Claim 2.3. *Assume that there is a positive integer N such that $y_N = -x_N - 2 \geq 0$. Then $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is the prime period-4 solution P_4^2.*

Proof. Suppose (x_N, y_N) satisfies the hypothesis, then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 = -x_N + x_N + 2 - 1 = 1 \\ y_{N+1} &= x_N + |y_N| - 1 = x_N - x_N - 2 - 1 = -3. \end{aligned}$$

Then the proof is complete. □

Lemma 2.4. *Suppose the initial condition* $(x_0, y_0) \in \mathcal{L}_2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the prime period-4 solution* P_4^1 *or* P_4^2.

Proof. Let $(x_0, y_0) \in \mathcal{L}_2$. Then

$$\begin{aligned} x_1 &= |x_0| - y_0 - 1 = -y_0 > 0 \\ y_1 &= x_0 + |y_0| - 1 = -y_0 > 0 \\ x_2 &= |x_1| - y_1 - 1 = -1 \\ y_2 &= x_1 + |y_1| - 1 = -2y_0 - 1. \end{aligned}$$

Suppose $y_2 = -2y_0 - 1 < 0$, then $(x_2, y_2) \in \mathcal{L}_4$. We now apply Lemma 2.2. and find that $\{(x_n, y_n)\}_{n=4}^{\infty}$ is the prime period-4 solution P_4^1.

Suppose $y_2 = -2y_0 - 1 \geq 0$, that is $y_0 \leq -\dfrac{1}{2}$, then

$$\begin{aligned} x_3 &= |x_2| - y_2 - 1 = 2y_0 + 1 \leq 0 \\ y_3 &= x_2 + |y_2| - 1 = -2y_0 - 3. \end{aligned}$$

Suppose $y_3 = -2y_0 - 3 \geq 0$, that is $y_0 \leq -\dfrac{3}{2}$, then

$$\begin{aligned} x_4 &= |x_3| - y_3 - 1 = 1 \\ y_4 &= x_3 + |y_3| - 1 = -3, \end{aligned}$$

and so $(x_4, y_4) \in P_4^2$, as required.

Suppose $y_3 = -2y_0 - 3 < 0$, that is $-\dfrac{3}{2} < y_0 \leq -\dfrac{1}{2}$, then we will progress using mathematical induction. For each integer $n \geq 0$, let

$$a_n = \frac{-2^{2n+1} - 1}{2^{2n+1}}, \; b_n = \frac{-2^{2n+1} + 1}{2^{2n+1}}, \; c_n = \frac{-2^{2n} + 1}{2^{2n}}, \text{ and } \delta_n = 2^{2n} - 1.$$

Observe that

$$\begin{aligned} -\frac{3}{2} &= a_0 < a_1 < a_2 < \ldots < -1 \quad \text{and} \quad \lim_{n\to\infty} a_n = -1, \\ -\frac{1}{2} &= b_0 > b_1 > b_2 > \ldots > -1 \quad \text{and} \quad \lim_{n\to\infty} b_n = -1, \\ 0 &= c_0 > c_1 > c_2 > \ldots > -1 \quad \text{and} \quad \lim_{n\to\infty} c_n = -1. \end{aligned}$$

Furthermore for each integer $n \geq 1$, let P(n) be the following set of statements. When $y_0 \in (a_{n-1}, b_{n-1}]$, we have

$$\begin{aligned} x_{4n} &= 1 \\ y_{4n} &= 2^{2n} y_0 + \delta_n. \end{aligned}$$

When $y_0 \in [c_n, b_{n-1}]$, we have $y_{4n} \geq 0$, and so the solution is eventually the prime period-4 solution P_4^1.
When $y_0 \in (a_{n-1}, c_n)$, we have $y_{4n} < 0$. Then

$$\begin{aligned} x_{4n+1} &= -2^{2n} y_0 - \delta_n > 0 \\ y_{4n+1} &= -2^{2n} y_0 - \delta_n > 0 \\ x_{4n+2} &= -1 \\ y_{4n+2} &= -2^{2n+1} y_0 - (2\delta_n + 1). \end{aligned}$$

When $y_0 \in [b_n, c_n)$, we have $y_{4n+2} \leq 0$, and so the solution is eventually the prime period-4 solution P_4^1.
When $y_0 \in (a_{n-1}, b_n)$, we have $y_{4n+2} > 0$. Then

$$\begin{aligned} x_{4n+3} &= 2^{2n+1} y_0 + (2\delta_n + 1) < 0 \\ y_{4n+3} &= -2^{2n+1} y_0 - (2\delta_n + 3). \end{aligned}$$

When $y_0 \in (a_{n-1}, a_n]$, we have $y_{4n+3} \geq 0$, and so the solution is eventually the prime period-4 solution P_4^2.

Finally, when $y_0 \in (a_n, b_n]$, we have $y_{4n+3} < 0$.

We shall now show that P(1) is true. For $y_0 \in (a_{1-1}, b_{1-1}] = (a_0, b_0] = \left(-\frac{3}{2}, -\frac{1}{2}\right]$, recall that $x_3 = 2y_0 + 1 \leq 0$ and $y_3 = -2y_0 - 3 < 0$, then

$$\begin{aligned} x_{4(1)} &= x_4 = |x_3| - y_3 - 1 = 1 \\ y_{4(1)} &= y_4 = x_3 + |y_3| - 1 = 4y_0 + 3 = 2^{2(1)} y_0 + \delta_1. \end{aligned}$$

When $y_0 \in [c_1, b_{1-1}] = [c_1, b_0] = \left[-\frac{3}{4}, -\frac{1}{2}\right]$, then $y_4 = 4y_0 + 3 \geq 0$. We apply Lemma 2.2. and find that $\{(x_n, y_n)\}_{n=6}^{\infty}$ is the prime period-4 solution P_4^1.
When $y_0 \in (a_{1-1}, c_1) = (a_0, c_1) = \left(-\frac{3}{2}, -\frac{3}{4}\right)$, then $y_4 = 4y_0 + 3 < 0$. Thus

$$\begin{aligned} x_{4(1)+1} &= x_5 = -4y_0 - 3 = -2^{2(1)} y_0 - \delta_1 > 0 \\ y_{4(1)+1} &= y_5 = -4y_0 - 3 = -2^{2(1)} y_0 - \delta_1 > 0 \\ x_{4(1)+2} &= x_6 = -1 \\ y_{4(1)+2} &= y_6 = -8y_0 - 7 = -2^{2(1)+1} y_0 - (2\delta_1 + 1). \end{aligned}$$

When $y_0 \in [b_1, c_1) = \left[-\frac{7}{8}, -\frac{3}{4}\right)$, then $y_6 = -8y_0 - 7 \leq 0$. We apply Lemma 2.2. and find that $\{(x_n, y_n)\}_{n=8}^{\infty}$ is the prime period-4 solution P_4^1.
When $y_0 \in (a_{1-1}, b_1) = (a_0, b_1) = \left(-\frac{3}{2}, -\frac{7}{8}\right)$, then $y_6 = -8y_0 - 7 > 0$. Thus

$$\begin{aligned} x_{4(1)+3} &= x_7 = 8y_0 + 7 = 2^{2(1)+1} y_0 + 2\delta_1 + 1 < 0 \\ y_{4(1)+3} &= y_7 = -8y_0 - 9 = -2^{2(1)+1} y_0 - (2\delta_1 + 3). \end{aligned}$$

When $y_0 \in (a_{1-1}, a_1] = (a_0, a_1] = \left(-\frac{3}{2}, -\frac{9}{8}\right]$, then $y_7 = -x_7 - 2 = -8y_0 - 9 \geq 0$. We apply Claim 2.3. and find that $\{(x_n, y_n)\}_{n=8}^{\infty}$ is the prime period-4 solution P_4^2. When $y_0 \in (a_1, b_1] = \left(-\frac{9}{8}, -\frac{7}{8}\right]$, then $y_7 = -8y_0 - 9 < 0$. Hence P(1) is true.

Next, we assume that P(N) is true. We shall show that P$(N+1)$ is true. Since P(N) is true, we know that when $y_0 \in (a_N, b_N] = \left(\frac{-2^{2N+1}-1}{2^{2N+1}}, \frac{-2^{2N+1}+1}{2^{2N+1}}\right]$, we have

$$\begin{aligned} x_{4N+3} &= 2^{2N+1}y_0 + (2\delta_N + 1) < 0 \\ y_{4N+3} &= -2^{2N+1}y_0 - (2\delta_N + 3) < 0. \end{aligned}$$

Then,

$$\begin{aligned} x_{4(N+1)} &= x_{4N+4} = 1 \\ y_{4(N+1)} &= y_{4N+4} = 2^{2(N+1)}y_0 + 4\delta_N + 3 = 2^{2(N+1)}y_0 + \delta_{N+1}. \end{aligned}$$

Note that

$$\delta_{N+1} = 2^{2(N+1)} - 1 = 2^{2N+2} - 1 = 2^{2N+2} - 4 + 3 = 4\delta_N + 3.$$

If $y_0 \in [c_{N+1}, b_{(N+1)-1}] = [c_{N+1}, b_N] = \left[\frac{-2^{2N+2}+1}{2^{2N+2}}, \frac{-2^{2N+1}+1}{2^{2N+1}}\right]$, then

$$y_{4N+4} = 2^{2(N+1)}y_0 + \delta_{N+1} = 2^{2N+2}y_0 + 2^{2N+2} - 1 \geq 0.$$

Applying Lemma 2.2., we see that $\{(x_n, y_n)\}_{n=4N+6}^{\infty}$ is the prime period-4 solution P_4^1.

If $y_0 \in (a_{(N+1)-1}, c_{N+1}) = (a_N, c_{N+1}) = \left(\frac{-2^{2N+1}-1}{2^{2N+1}}, \frac{-2^{2N+2}+1}{2^{2N+2}}\right)$, then

$$y_{4N+4} = 2^{2(N+1)}y_0 + \delta_{N+1} = 2^{2N+2}y_0 + 2^{2N+2} - 1 < 0.$$

Thus,

$$\begin{aligned} x_{4(N+1)+1} &= x_{4N+5} = -2^{2(N+1)}y_0 - \delta_{N+1} > 0 \\ y_{4(N+1)+1} &= y_{4N+5} = -2^{2(N+1)}y_0 - \delta_{N+1} > 0 \\ x_{4(N+1)+2} &= x_{4N+6} = -1 \\ y_{4(N+1)+2} &= y_{4N+6} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + 1). \end{aligned}$$

If $y_0 \in [b_{N+1}, c_{N+1}) = \left[\frac{-2^{2N+3}+1}{2^{2N+3}}, \frac{-2^{2N+2}+1}{2^{2N+2}}\right)$, then

$$y_{4N+6} = -2^{2(N+1)+1}y_0 - 2\delta_{N+1} - 1 = -2^{2N+3}y_0 - 2^{2N+3} + 1 \leq 0.$$

Applying Lemma 2.2., we see that $\{(x_n, y_n)\}_{n=4N+8}^{\infty}$ is the prime period-4 solution P_4^1.

If $y_0 \in (a_{(N+1)-1}, b_{N+1}) = (a_N, b_{N+1}) = \left(\dfrac{-2^{2N+1}-1}{2^{2N+1}}, \dfrac{-2^{2N+3}+1}{2^{2N+3}}\right)$, then

$$y_{4N+6} = -2^{2(N+1)+1}y_0 - 2\delta_{N+1} - 1 = -2^{2N+3}y_0 - 2^{2N+3} + 1 > 0,$$

thus

$$x_{4(N+1)+3} = x_{4N+7} = 2^{2(N+1)+1}y_0 + (2\delta_{N+1} + 1) < 0$$
$$y_{4(N+1)+3} = y_{4N+7} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + 3).$$

If $y_0 \in (a_{(N+1)-1}, a_{N+1}] = (a_N, a_{N+1}] = \left(\dfrac{-2^{2N+1}-1}{2^{2N+1}}, \dfrac{-2^{2N+3}-1}{2^{2N+3}}\right]$, then

$$y_{4N+7} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + 3) = -2^{2N+3}y_0 - 2^{2N+3} - 1 \geq 0.$$

We note that $y_{4N+7} = -x_{4N+7} - 2 \geq 0$. Applying Claim 2.3., we see that $\{(x_n, y_n)\}_{n=4N+8}^{\infty}$ is the prime period-4 solution P_4^2.

If $y_0 \in (a_{(N+1)}, b_{N+1}] = (a_{N+1}, b_{N+1}] = \left(\dfrac{-2^{2N+3}-1}{2^{2N+3}}, \dfrac{-2^{2N+3}+1}{2^{2N+3}}\right]$, then

$$y_{4N+7} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + 3) = -2^{2N+3} - 2^{2N+3} - 1 < 0.$$

Hence, P$(N+1)$ is true. Therefore P(n) is true for all $n \geq 1$.

Note that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = -1$ and $(1, -1) \in P_4^1$. □

Lemma 2.5. *Suppose the initial condition $(x_0, y_0) \in \mathcal{L}_3$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.*

Proof. Suppose $(x_0, y_0) \in \mathcal{L}_3$. Then by direct computations we see that $x_2 = 1$. We now apply Lemmas 2.2. and 2.4., and see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2. □

Claim 2.6. *Assume that there is a positive integer N such that $x_N = y_N \geq 0$. Then, $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.*

Proof. Suppose that (x_N, y_N) satisfies the hypothesis, then $x_{N+1} = -1$. We apply Lemmas 2.2. and 2.5., and see that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2. □

Lemma 2.7. *Suppose the initial condition $(x_0, y_0) \in \mathcal{L}_5$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-3 solution P_3^1 or the prime period-4 solution P_4^1 or P_4^2.*

Proof. Suppose that $(x_0, y_0) \in \mathcal{L}_5$ and suppose further that $x_0 \geq 0$. Then $x_1 = y_1 = x_0$. We apply Claim 2.6., and find that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

Now suppose that $(x_0, y_0) \in \mathcal{L}_5$ but $x_0 \leq 0$. Then

$$\begin{aligned} x_1 &= |x_0| - y_0 - 1 = -x_0 + 1 - 1 = -x_0 > 0 \\ y_1 &= x_0 + |y_0| - 1 = x_0 + 1 - 1 = x_0 < 0 \\ x_2 &= |x_1| - y_1 - 1 = -x_0 - x_0 - 1 = -2x_0 - 1 \\ y_2 &= x_1 + |y_1| - 1 = -x_0 - x_0 - 1 = -2x_0 - 1. \end{aligned}$$

If $x_2 = y_2 = -2x_0 - 1 \geq 0$, that is $x_0 \leq -\frac{1}{2}$, then we can apply Claim 2.6. and see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

If $x_2 = y_2 = -2x_0 - 1 < 0$, that is $-\dfrac{1}{2} < x_0 < 0$. Then

$$\begin{aligned} x_3 &= |x_2| - y_2 - 1 = 2x_0 + 1 + 2x_0 + 1 - 1 = 4x_0 + 1 \\ y_3 &= x_2 + |y_2| - 1 = -2x_0 - 1 + 2x_0 + 1 - 1 = -1. \end{aligned}$$

If $x_3 = 4x_0 + 1 \geq 0$, then since $(x_3, y_3) \in \mathcal{L}_5$ by the above case, we see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

If $x_3 = 4x_0 + 1 < 0$, that is $-\dfrac{1}{2} < x_0 < -\dfrac{1}{4}$, then we will progress by using mathematical induction.

For each integer $n \geq 0$, let

$$a_n = \frac{-2^{2n+1} - 1}{3 \times 2^{2n+1}}, \ b_n = \frac{-2^{2n+2} + 1}{3 \times 2^{2n+2}} \text{ and } \delta_n = \frac{2^{2n} - 1}{3}.$$

Observe that

$$\begin{aligned} -\frac{1}{2} = a_0 < a_1 < a_2 < \ldots < -\frac{1}{3} \quad &\text{and} \quad \lim_{n \to \infty} a_n = -\frac{1}{3}, \\ -\frac{1}{4} = b_0 > b_1 > b_2 > \ldots > -\frac{1}{3} \quad &\text{and} \quad \lim_{n \to \infty} b_n = -\frac{1}{3}. \end{aligned}$$

Furthermore for each integer $n \geq 1$, let $P(n)$ be the following set of statements. When $x_0 \in (a_{n-1}, b_{n-1})$, we have

$$\begin{aligned} x_{3n+1} &= -2^{2n} x_0 - \delta_n > 0 \\ y_{3n+1} &= 2^{2n} x_0 + \delta_n < 0 \\ x_{3n+2} &= -2^{2n+1} x_0 - (2\delta_n + 1) \\ y_{3n+2} &= -2^{2n+1} x_0 - (2\delta_n + 1). \end{aligned}$$

When $x_0 \in (a_{n-1}, a_n]$, we have $x_{3n+2} = y_{3n+2} \geq 0$, and so by Claim 2.6. $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2. When $x_0 \in (a_n, b_{n-1})$, we have $x_{3n+2} = y_{3n+2} < 0$, and so

$$\begin{aligned} x_{3n+3} &= 2^{2n+2} x_0 + 4\delta_n + 1 \\ y_{3n+3} &= -1. \end{aligned}$$

When $x_0 \in [b_n, b_{n-1})$, we have $x_{3n+3} \geq 0$, and so by Claim 2.6. $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.
Finally, when $x_0 \in (a_n, b_n)$, we have $x_{3n+3} < 0$.

We shall show that P(1) is true.

For $x_0 \in (a_{n-1}, b_{n-1}) = (a_0, b_0) = \left(-\frac{1}{2}, -\frac{1}{4}\right)$, recall that $x_3 = 4x_0 + 1 < 0$ and $y_3 = -1$. Then,

$$\begin{aligned}
x_{3(1)+1} &= x_4 = -4x_0 - 1 = -2^{2(1)}x_0 - \delta_1 > 0\\
y_{3(1)+1} &= y_4 = 4x_0 + 1 = 2^{2(1)}x_0 + \delta_1 < 0\\
x_{3(1)+2} &= x_5 = -8x_0 - 3 = -2^{2(1)+1}x_0 - (2\delta_1 + 1)\\
y_{3(1)+2} &= y_5 = -8x_0 - 3 = -2^{2(1)+1}x_0 - (2\delta_1 + 1).
\end{aligned}$$

If $x_0 \in (a_{1-1}, a_1] = (a_0, a_1] = \left(-\frac{1}{2}, -\frac{3}{8}\right]$, then $x_5 = y_5 = -8x_0 - 3 \geq 0$, and so we apply Claim 2.6. and see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

If $x_0 \in (a_1, b_{1-1}) = (a_1, b_0) = \left(-\frac{3}{8}, -\frac{1}{4}\right)$, then $x_5 = y_5 = -8x_0 - 3 < 0$, and so

$$\begin{aligned}
x_{3(1)+3} = x_6 = 16x_0 + 5 &= 2^{2(1)+2}x_0 + 4\delta_1 + 1\\
y_{3(1)+3} &= y_6 = -1.
\end{aligned}$$

If $x_0 \in [b_1, b_{1-1}) = [b_1, b_0) = \left[-\frac{5}{16}, -\frac{1}{4}\right)$, then $x_6 = 16x_0 + 5 \geq 0$, and $(x_6, y_6) \in \mathcal{L}_5$. Applying earlier work in this proof we see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

If $x_0 \in (a_1, b_1) = \left(-\frac{3}{8}, -\frac{5}{16}\right)$, then $x_6 = 16x_0 + 5 < 0$. Hence P(1) is true.

Suppose that P(N) is true. We shall show that P($N+1$) is true.

Since P(N) is true, we know that $x_0 \in (a_{(N+1)-1}, b_{(N+1)-1}) = (a_N, b_N) = \left(\frac{-2^{2N+1} - 1}{3 \times 2^{2N+1}}, \frac{-2^{2N+2} + 1}{3 \times 2^{2N+2}}\right)$, and

$$\begin{aligned}
x_{3N+3} &= 2^{2N+2}x_0 + 4\delta_N + 1 < 0\\
y_{3N+3} &= -1.
\end{aligned}$$

Note that

$$\delta_{N+1} = \frac{2^{2(N+1)} - 1}{3} = \frac{2^{2N+2} - 4}{3} + \frac{3}{3} = 4\left(\frac{2^{2N} - 1}{3}\right) + 1 = 4\delta_N + 1.$$

Then

$$\begin{aligned}
x_{3(N+1)+1} &= x_{3N+4} = -2^{2N+2} - 4\delta_N - 1 = -2^{2(N+1)}x_0 - \delta_{N+1} > 0\\
y_{3(N+1)+1} &= y_{3N+4} = 2^{2N+2} + 4\delta_N + 1 = 2^{2(N+1)}x_0 + \delta_{N+1} < 0\\
x_{3(N+1)+2} &= x_{3N+5} = -2^{2(N+1)+1}x_0 - (2\delta_{N+1} + 1)\\
y_{3(N+1)+2} &= y_{3N+5} = -2^{2(N+1)+1}x_0 - (2\delta_{N+1} + 1).
\end{aligned}$$

If $x_0 \in (a_{(N+1)-1}, a_{N+1}] = (a_N, a_{N+1}] = \left(\dfrac{-2^{2N+1} - 1}{3 \times 2^{2N+1}}, \dfrac{-2^{2N+3} - 1}{3 \times 2^{2N+3}}\right]$, then

$$x_{3N+5} = y_{3N+5} = -2^{2(N+1)+1}x_0 - (2\delta_{N+1} + 1) = -2^{2N+3}x_0 + \left(\frac{-2^{2N+3} - 1}{3}\right) \geq 0,$$

and so we apply Claim 2.6. and see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

If $x_0 \in (a_{N+1}, b_{(N+1)-1}) = (a_{N+1}, b_N) = \left(\dfrac{-2^{2N+3} - 1}{3 \times 2^{2N+3}}, \dfrac{-2^{2N+2} + 1}{3 \times 2^{2N+2}}\right)$, then

$$x_{3N+5} = y_{3N+5} = -2^{2(N+1)+1}x_0 - (2\delta_{N+1} + 1) = -2^{2N+3}x_0 + \left(\frac{-2^{2N+3} - 1}{3}\right) < 0,$$

and so

$$\begin{aligned}
x_{3(N+1)+3} &= x_{3N+6} = 2^{2(N+1)+2}x_0 + 4\delta_{N+1} + 1\\
y_{3(N+1)+3} &= y_{3N+6} = -1.
\end{aligned}$$

If $x_0 \in [b_{N+1}, b_{(N+1)-1}) = [b_{N+1}, b_N) = \left[\dfrac{-2^{2N+4} + 1}{3 \times 2^{2N+4}}, \dfrac{-2^{2N+2} + 1}{3 \times 2^{2N+2}}\right)$, then

$$x_{3N+6} = 2^{2(N+1)+2}x_0 + 4\delta_{N+1} + 1 = 2^{2N+4}x_0 + \frac{2^{2N+4} - 1}{3} \geq 0,$$

and $(x_{3N+6}, y_{3N+6}) \in \mathcal{L}_5$ so by previous work in this proof $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

If $x_0 \in (a_{N+1}, b_{N+1}) = \left(\dfrac{-2^{2N+3} - 1}{3 \times 2^{2N+3}}, \dfrac{-2^{2N+4} + 1}{3 \times 2^{2N+4}}\right)$, then

$$x_{3N+6} = 2^{2(N+1)+2}x_0 + 4\delta_{N+1} + 1 = 2^{2N+4}x_0 + \frac{2^{2N+4} - 1}{3} < 0.$$

Hence, $\mathrm{P}(N+1)$ is true. Therefore $\mathrm{P}(n)$ is true for all $n \geq 1$.
Please note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = -\frac{1}{3}$$

and $(x_0, y_0) = \left(-\frac{1}{3}, -1\right) \in P_3^1$. □

Lemma 2.8. *Suppose the initial condition* $(x_0, y_0) \in \mathcal{Q}_2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the prime period-4 solution* P_4^1 *or* P_4^2.

Proof. Let $(x_0, y_0) \in Q_2$. Then

$$\begin{aligned} x_1 &= |x_0| - y_0 - 1 = -x_0 - y_0 - 1 \\ y_1 &= x_0 + |y_0| - 1 = x_0 + y_0 - 1. \end{aligned}$$

Case 1: Suppose $-x_0 = y_0$, then $(x_1, y_1) = (-1, -1) \in P_4^1$.
Case 2: Suppose $-x_0 > y_0$, then $y_1 = x_0 + y_0 - 1 < 0$.
Suppose further that $x_1 = -x_0 - y_0 - 1 < 0$, then

$$\begin{aligned} x_2 &= |x_1| - y_1 - 1 = 1 \\ y_2 &= x_1 + |y_1| - 1 = -2x_0 - 2y_0 - 1. \end{aligned}$$

We see that $(x_2, y_2) \in \mathcal{L}_1 \cup \mathcal{L}_2$. Applying Lemmas 2.2. and 2.4., we see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

Now suppose that $x_1 = -x_0 - y_0 - 1 \geq 0$, then

$$\begin{aligned} x_2 &= |x_1| - y_1 - 1 = -2x_0 - 2y_0 - 1 > 0 \\ y_2 &= x_1 + |y_1| - 1 = -2x_0 - 2y_0 - 1 > 0 \\ x_3 &= |x_2| - y_2 - 1 = -1 \\ y_3 &= x_2 + |y_2| - 1 = -4x_0 - 4y_0 - 3 > 0. \end{aligned}$$

We apply Lemma 2.5., and see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.
Case 3: Suppose $-x_0 < y_0$, then

$$\begin{aligned} x_1 &= -x_0 - y_0 - 1 < 0 \\ x_2 &= |x_1| - y_1 - 1 = 1. \end{aligned}$$

We apply Lemmas 2.2. and 2.4., and the proof is complete. □

Lemma 2.9. *Suppose the initial condition* $(x_0, y_0) \in \mathcal{Q}_4$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the prime period-3 solution* P_3^1 *or the prime period-4 solution* P_4^1 *or* P_4^2.

Proof. Let $(x_0, y_0) \in Q_4$. Then,

$$\begin{aligned} x_1 &= |x_0| - y_0 - 1 = x_0 - y_0 - 1 \\ y_1 &= x_0 + |y_0| - 1 = x_0 - y_0 - 1. \end{aligned}$$

Suppose $x_0 - y_0 - 1 \geq 0$, then we apply Claim 2.6. and see that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period-4 solution P_4^1 or P_4^2.

Suppose that $x_0 - y_0 - 1 < 0$, then

$$y_2 = x_1 + |y_1| - 1 = -1.$$

We see that $(x_2, y_2) \in \mathcal{L}_5$. By Lemma 2.7. $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the prime period-3 solution P_3^1 or the prime period-4 solution P_4^1 or P_4^2. □

3. Discussion and Conclusion

Returning our attention to the original family of System($\mathcal{N}$), number 7 of this group is one of the most interesting systems. Initially, when we only understood its behavior for a small set of initial conditions (a segments on x-axis), we were only able to prove that every solution was eventually prime period-4. See Ref. [7]. Now that we are able to include the closed second and fourth quadrant in the set of initial conditions we see that this is one of the few systems that exhibit solutions of varying periodicity. Although we have not yet proved the global behavior of System(7) we have a conjecture.

Conjecture 3.1. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(7) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $\left(-\frac{1}{5}, -\frac{3}{5}\right)$, or eventually the prime period-3 solution P_3^1 or P_3^2, or the prime period-4 solution P_4^1 or P_4^2* where

$$P_3^1 = \begin{pmatrix} -\frac{1}{3}, & -1 \\ \frac{1}{3}, & -\frac{1}{3} \\ -\frac{1}{3}, & -\frac{1}{3} \end{pmatrix}, P_3^2 = \begin{pmatrix} \frac{3}{5}, & \frac{1}{5} \\ -\frac{3}{5}, & -\frac{1}{5} \\ -\frac{1}{5}, & -\frac{7}{5} \end{pmatrix}, P_4^1 = \begin{pmatrix} -1, & -1 \\ 1, & -1 \\ 1, & 1 \\ -1, & 1 \end{pmatrix}, \text{ and } P_4^2 = \begin{pmatrix} 1, & -3 \\ 3, & 3 \\ -1, & 5 \\ -5, & 3 \end{pmatrix}.$$

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