ON DIVERSITY OF CERTAIN $t$-INTERSECTING FAMILIES

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ABSTRACT. Let $[n] = \{1, 2, \ldots, n\}$ and $2^{[n]}$ be the set of all subsets of $[n]$. For a family $\mathcal{F} \subseteq 2^{[n]}$, its diversity, denoted by $\text{div}(\mathcal{F})$, is defined to be

$$\text{div}(\mathcal{F}) = \min_{x \in [n]} \{|\mathcal{F}(x)|\},$$

where $\mathcal{F}(x) = \{F \in \mathcal{F} : x \notin F\}$. Basically, $\text{div}(\mathcal{F})$ measures how far $\mathcal{F}$ is from a trivial intersecting family, which is called a star. In this paper, we consider a generalization of diversity for $t$-intersecting family.

1. Introduction

Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the family of all $k$-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is $t$-intersecting if $|A \cap B| \geq t$ for all $A, B \in \mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1.1 (Erdős, Ko, and Rado [8], Frankl [9], Wilson [36]). Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$ is $t$-intersecting and $n > 2k - t$. Then for $n \geq (k - t + 1)(t + 1)$, we have

$$|\mathcal{A}| \leq \binom{n - t}{k - t}.$$  

Moreover, if $n > (k - t + 1)(t + 1)$, then the equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : T \subseteq A\}$ for some $t$-set $T$.

In the celebrated paper [1], Ahlswede and Khachatrian extended the Erdős-Ko-Rado theorem by determining the structure of all $t$-intersecting set systems of maximum size for all possible $n$ (see also [3, 14–16, 22, 26, 32, 34, 35] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős-Ko-Rado theorem for the Hamming scheme is proved in [33]. A complete solution for the $t$-intersection problem in the Hamming space is given in [2]. Some recent work done on this
problem and its variants can be found in [4, 6, 7, 12, 13, 18, 24]. The Erdős-Ko-Rado type results also appear in vector spaces [5,17], set partitions [19,20,23,28–30] and weak compositions [21,25,27].

In this paper, we will consider the diversity of an intersecting family. Let \(2^n\) be the set of all subsets of \([n]\). For each \(x \in [n]\) and \(F \subseteq 2^n\), let 
\[
F(x) = \{F \in F : x \notin F\}.
\]
The diversity of a family \(F \subseteq 2^n\), denoted by \(\text{div}(F)\), is defined by 
\[
\text{div}(F) = \min_{x \in [n]} \{|F(x)|\}.
\]
We note here that an 1-intersecting family will just be called an intersecting family.

**Theorem 1.2.** Let \(k > 1\). There exists a positive integer \(n_0 = n_0(k)\) such that if \(n \geq n_0\) and \(F \subseteq \binom{[n]}{k}\) is intersecting, then 
\[
\text{div}(F) \leq \binom{n-3}{k-2}.
\]

Lemons and Palmer [31] proved Theorem 1.2 for \(n > 6k^3\). Recently, Frankl [11] improved it to \(n \geq 6k^2\). Let
\[
T = \left\{ T \in \binom{[n]}{k} : |T \cap \{1, 2, 3\}| \geq 2 \right\}.
\]
It is not hard to see that \(T\) is intersecting and \(\text{div}(T) = \binom{n-3}{k-2}\). In fact, 
\[
|T(x)| = \binom{n-3}{k-2} \text{ for all } x \in \{1, 2, 3\} \text{ and } |T(x)| > \binom{n-3}{k-2} \text{ for all } x \in [n] \setminus \{1, 2, 3\}.
\]
So, the bound in Theorem 1.2 is tight. In [10], Frankl proved the theorem for \(n > 2k\) under the additional assumption \(|F| \geq |T|\).

Two generalizations of Theorem 1.2 were given by Frankl [11], which are Theorems 1.3 and 1.4.

**Theorem 1.3.** Let \(k > t > 0\). There exists a positive integer \(n_0 = n_0(k,t)\) such that if \(n \geq n_0\) and \(F \subseteq \binom{[n]}{k}\) is \(t\)-intersecting, then 
\[
\text{div}(F) \leq \binom{n-t-2}{k-t-1}.
\]

Note that when \(t = 1\), Theorem 1.3 coincides with Theorem 1.2. The family 
\[
G = \left\{ G \in \binom{[n]}{k} : |G \cap \{t+2\}| \geq t+1 \right\},
\]
shows that the bound in Theorem 1.3 is tight.

For the other generalization, we need a new definition. The independence number of a family \(F \subseteq \binom{[n]}{k}\) is the maximum integer \(q\) such that \(F\) contains \(q\) pairwise disjoint members. The independence number of \(F\) will be denoted by \(\nu(F)\). For a subset \(Q \subseteq [n]\), let 
\[
F(Q) = \{ F \in F : F \cap Q = \emptyset \}.
\]
Theorem 1.4. Let $k, q > 0$. There exists a positive integer $n_0 = n_0(k, q)$ such that if $n \geq n_0$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\nu(\mathcal{F}) = q$, then there is a set $Q \in \binom{[n]}{k}$ with
\[ |\mathcal{F}(Q)| \leq \sum_{2 \leq i \leq q+1} \binom{q+1}{i} \binom{n-2q-1}{k-i}. \]

Note that when $q = 1$, Theorem 1.4 coincides with Theorem 1.2. The family
\[ H = \left\{ H \in \binom{[n]}{k} : |H \cap [2q+1]| \geq 2 \right\}, \]
shows that the bound in Theorem 1.4 is tight.

In this paper, we will give another generalization of Theorem 1.2. We need a definition. The $t$-diversity of a family $\mathcal{F} \subseteq 2^{[n]}$, denoted by $\text{div}_t(\mathcal{F})$, is defined by
\[ \text{div}_t(\mathcal{F}) = \min_{X \in \binom{[n]}{t}} \{|\mathcal{F}(X)|\}. \]
Note that when $t = 1$, the $t$-diversity is the same as diversity. We will prove the following main theorem.

Theorem 1.5. Let $k \geq 2t > 0$. There exists a positive integer $n_0 = n_0(k, t)$ such that if $n \geq n_0$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is $t$-intersecting, then
\[ \text{div}_t(\mathcal{F}) \leq \binom{n-3t}{k-2t}. \]
The inequality (1) is tight.

The bound on $n$ obtained here is quadratic over $k$, that is $n_0(k, t) = \left(\frac{2t+1}{t}\right)^2 k^2$. The remaining of this paper is organised as follows: In Section 2.1, we show that the upper bound given in (1) is tight. In Section 2.2, we show that the main theorem holds by proving a refinement of Theorem 1.5.

2. Proof of the main theorem

2.1. Tightness of Theorem 1.5

We first consider the tightness of Theorem 1.5. The following lemma is obvious by the definition.

Lemma 2.1. Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \binom{[n]}{k}$. Then
\[ \text{div}_t(\mathcal{A}) \leq \text{div}_t(\mathcal{B}). \]
Let $A$ and $B$ be two subsets of $[n]$. We say $A$ avoids $B$ if $A \cap B = \emptyset$. Using this, we see that
\[ \mathcal{F}(Q) = \{F \in \mathcal{F} : F \text{ avoids } Q\}. \]
Let
\[ \mathcal{U}_t = \left\{ U \in \binom{[n]}{k} : |U \cap [3t]| \geq 2t \right\}. \]
\[ \mathcal{U}_2 = \left\{ U \in \binom{[n]}{k} : |U \cap [3t]| = 2t \right\}. \]

Note that \( \mathcal{U}_2 \subseteq \mathcal{U}_1 \). For each family \( \mathcal{F} \subseteq \binom{[n]}{k} \) with \( \mathcal{U}_2 \subseteq \mathcal{F} \subseteq \mathcal{U}_1 \) and each \( S \in \binom{[3t]}{2t} \), let
\[ \mathcal{F}[S] = \{ F : F \cap [3t] = S \}. \]

Then, \( \mathcal{F}[S] \cap \mathcal{F}[S'] = \emptyset \) for distinct \( S, S' \in \binom{[3t]}{2t} \) and
\[ \mathcal{U}_2 = \bigcup_{S \in \binom{[3t]}{2t}} \mathcal{F}[S]. \]

Furthermore, \( |\mathcal{F}[S]| = \binom{n-3t}{k-2t} \). So,
\[ |\mathcal{U}_2| = \sum_{S \in \binom{[3t]}{2t}} |\mathcal{F}[S]| = \sum_{S \in \binom{[3t]}{2t}} \binom{n-3t}{k-2t} = \binom{3t}{2t} \binom{n-3t}{k-2t}. \]

For each \( X \in \binom{[3t]}{t} \),
\[ \mathcal{F}(X) \subseteq \mathcal{U}_1(X) = \mathcal{U}_2[S], \]
where \( S = [3t] \setminus X \). The following theorem shows that the bound in Theorem 1.5 is tight.

**Theorem 2.2.** Suppose \( k \geq 2t > 0, n \geq k + t \) and \( \mathcal{F} \subseteq \binom{[n]}{k} \) with \( \mathcal{F} \subseteq \mathcal{U}_1 \). Then, the following statements hold.

(a) \( \mathcal{F} \) is \( t \)-intersecting.

(b) If \( \mathcal{U}_2 \subseteq \mathcal{F} \), then \( \text{div}_t(\mathcal{F}) = \binom{n-3t}{k-2t} \).

(c) If \( \mathcal{U}_2 \not\subseteq \mathcal{F} \), then \( \text{div}_t(\mathcal{F}) < \binom{n-3t}{k-2t} \).

**Proof.** (a) Let \( A, B \in \mathcal{F} \). Since \( (A \cap [3t]) \cup (B \cap [3t]) \subseteq [3t] \),
\[ 3t \geq |A \cap [3t]| + |B \cap [3t]| - |A \cap B \cap [3t]| \geq 4t - |A \cap B \cap [3t]|. \]

This implies that \( |A \cap B| \geq t \). Hence, \( \mathcal{F} \) is \( t \)-intersecting.

(b) Let \( X \in \binom{[3t]}{t} \). We will show that
\[ |\mathcal{F}(X)| = \binom{n-3t}{k-2t}. \]

Let \( U \in \mathcal{F}(X) \). Since \( U \) avoids \( X \) and \( U \in \mathcal{U}_1 \),
\[ |U \cap ([3t] \setminus X)| \geq 2t. \]

Therefore, \( U \cap [3t] = ([3t] \setminus X) = S_0 \). Hence, \( \mathcal{F}(X) \subseteq \mathcal{U}_2[S_0] \). On the other hand, each element in \( \mathcal{U}_2[S_0] \) avoids \( X \). Thus, \( \mathcal{U}_2[S_0] \subseteq \mathcal{F}(X) \). This implies
that \( \mathcal{F}(X) = \mathcal{U}_2[S_0] \) and

\[
|\mathcal{F}(X)| = |\mathcal{U}_2[S_0]| = \binom{n - 3t}{k - 2t}.
\]

In fact, \( |\mathcal{F}(X)| = \binom{n - 3t}{k - 2t} \) for all \( X \in \binom{[3t]}{t} \).

To complete the proof for part (b), we just need to show that

\[
|\mathcal{F}(X)| = \binom{n - 3t}{k - 2t}
\]

for all \( X \in \binom{[n]}{t} \setminus \binom{[3t]}{t} \). Note that \( |Y \cap [3t]| \leq t - 1 \). Let

\[
S = \binom{[3t] \setminus Y}{2t}.
\]

For each \( S \in \mathcal{S} \),

\[
\mathcal{F}[S] = \{ F \in \mathcal{F} : F \cap [3t] = S \} \subseteq \mathcal{F}(Y).
\]

Recall that \( \mathcal{F}[S] \cap \mathcal{F}[S'] = \emptyset \) for \( S \neq S' \). Therefore,

\[
|\mathcal{F}(Y)| \geq \sum_{S \in \mathcal{S}} |\mathcal{F}[S]| = |\mathcal{S}| \left( \binom{n - 3t}{k - 2t} \right) \left( \binom{3t - |Y|}{2t} \right) \left( \binom{n - 3t}{k - 2t} \right)
\]

\[
\geq \left( \frac{2t + 1}{2t} \right) \left( \frac{n - 3t}{k - 2t} \right) > \binom{n - 3t}{k - 2t}.
\]

(c) Since \( \mathcal{U}_2 = \bigcup_{S \in \binom{[2t]}{2t}} \mathcal{U}_2[S] \) (see (2)), there is an \( S_0 \in \binom{[3t]}{2t} \) such that \( \mathcal{U}_2[S_0] \not\subseteq \mathcal{F} \). Let \( X_0 = [3t] \setminus S_0 \). Note that \( \mathcal{F}(X_0) \subseteq \mathcal{U}_1(X_0) = \mathcal{U}_2[S_0] \). Since \( \mathcal{U}_2[S_0] \not\subseteq \mathcal{F} \),

\[
|\mathcal{F}(X_0)| < |\mathcal{U}_2[S_0]| = \binom{n - 3t}{k - 2t}.
\]

Hence, \( \text{div}_t(\mathcal{F}) < \binom{n - 3t}{k - 2t} \). \( \square \)

2.2. Proof of Theorem 1.5

We now prove Theorem 1.5.

Let \( \mathcal{F} \subseteq \binom{[n]}{k} \) be \( t \)-intersecting. A set \( T \subseteq [n] \) is said to be a transversal for \( \mathcal{F} \) if \( |T \cap \mathcal{F}| \geq t \) for all \( F \in \mathcal{F} \). Furthermore, \( T \) is said to be a minimal transversal for \( \mathcal{F} \) if \( T \) is a transversal and if \( K \subseteq T \) is a transversal, then \( K = T \). So, given a transversal \( T \), we can obtain a minimal transversal \( K \subseteq T \) by removing elements from \( T \). The minimal transversal \( K \) is said to be originated from \( T \). Since \( \mathcal{F} \) is \( t \)-intersecting, all \( F \in \mathcal{F} \) are transversal for \( \mathcal{F} \). Let \( \mathcal{K}(\mathcal{F}) \) be the set of all minimal transversals originated from some element in \( \mathcal{F} \), i.e., \( K \in \mathcal{K}(\mathcal{F}) \) if and only if

(a) \( K \) is a minimal transversal for \( \mathcal{F} \);

(b) \( K \) is originated from a \( F \in \mathcal{F} \).
A $t$-intersecting family $\mathcal{F} \subseteq \left( \binom{[n]}{k} \right)$ is said to be saturated if $\mathcal{F} \cup \{G\}$ is not $t$-intersecting for all $G \in \left( \binom{[n]}{k} \right) \setminus \mathcal{F}$.

**Lemma 2.3.** Suppose $k \geq t > 0$ and $n \geq 2k$. If $\mathcal{F} \subseteq \left( \binom{[n]}{k} \right)$ is $t$-intersecting and saturated, then $\mathcal{K}(\mathcal{F})$ is $t$-intersecting.

**Proof.** Suppose $\mathcal{K}(\mathcal{F})$ is not $t$-intersecting. Then, there are $A, B \in \mathcal{K}(\mathcal{F})$ such that $|A \cap B| \leq t - 1$. Since $A$ originated from some element in $\mathcal{F}$, $|A| \leq k$. Similarly, $|B| \leq k$. Now, choose $2k - (|A| + |B|)$ elements from $\binom{[n]}{(A \cup B)}$. Put $k - |A|$ of them in a set, say $C$ and the remaining in a set, say $D$. Note that $C \cap D = \emptyset$ and both $(A \cup C)$ and $(B \cup D)$ are in $\left( \binom{[n]}{k} \right)$. Since $A$ is a transversal, $|(A \cup C) \cap F| \geq |A \cap F| \geq t$ for all $F \in \mathcal{F}$. This means $(A \cup C) \in \mathcal{F}$, for $\mathcal{F}$ is saturated. Similarly, $(B \cup D) \in \mathcal{F}$. Thus,

$$t \leq |(A \cup C) \cap (B \cup D)| = |A \cap B|,$$

a contradiction. Hence, $\mathcal{K}(\mathcal{F})$ is $t$-intersecting.

Let

$$\tau(\mathcal{F}) = \min_{K \in \mathcal{K}(\mathcal{F})} \{|K|\}$$

be called the transversal number of a $t$-intersecting family $\mathcal{F}$. Note that

$$\tau(\mathcal{F}) \geq t.$$

**Lemma 2.4.** Suppose $n \geq k \geq t > 0$ and $\mathcal{F} \subseteq \left( \binom{[n]}{k} \right)$ is $t$-intersecting. If $\tau(\mathcal{F}) \leq 2t - 1$, then $\text{div}_t(\mathcal{F}) = 0$.

**Proof.** Suppose $\tau(\mathcal{F}) = l \leq 2t - 1$. Let $K_0 \in \mathcal{K}(\mathcal{F})$ with $|K_0| = l$. Choose a subset $X \subseteq K_0$ of size $t$. Let $Y = K_0 \setminus X$. Then, $K_0 = X \cup Y$ and $X \cap Y = \emptyset$. We claim that $\mathcal{F}(\mathcal{X}) = \emptyset$. Suppose $\mathcal{F}(\mathcal{X}) \neq \emptyset$. Then, there is a $U \in \mathcal{F}(\mathcal{X})$. Since $K_0$ is a transversal for $\mathcal{F}$, $|U \cap K_0| \geq t$. Now, $U$ avoids $X$ implies that $U \cap K_0 = U \cap Y$. Thus, $|Y| \geq t$ and $l = |K_0| = |X| + |Y| \geq t + t = 2t$, a contradiction. Hence, $\mathcal{F}(\mathcal{X}) = \emptyset$ and $\text{div}_t(\mathcal{F}) = 0$.

Given a $t$-intersecting family $\mathcal{F} \subseteq \left( \binom{[n]}{k} \right)$ and $1 \leq i \leq k$, let

$$\mathcal{K}_i(\mathcal{F}) = \{K \in \mathcal{K}(\mathcal{F}) : |K| = i\}.$$

We consider two cases based on the definition of $\mathcal{K}_i(\mathcal{F})$ separately: the first case is when

$$\mathcal{K}_{2t}(\mathcal{F})(\mathcal{X}) \neq \emptyset$$

for all $X \in \left( \binom{[n]}{t} \right)$.

and the second case is when

$$\mathcal{K}_{2t}(\mathcal{F})(\mathcal{X}) = \emptyset$$

for some $X \in \left( \binom{[n]}{t} \right)$.

We first deal with the first case.
Lemma 2.5. Let $k \geq 2t > 0$, $n \geq 2k$ and $F \subseteq {[n]\choose k}$ be $t$-intersecting and saturated. If $K_{2t}(F)(X) \neq \emptyset$ for all $X \in {[n]\choose t}$, then
\[
\text{div}_t(F) = \binom{n-3t}{k-2t}.
\]

Proof. Clearly, $K_{2t}(F) \neq \emptyset$. Relabelling if necessary, we may assume that
\[
A_0 = \{1, 2, 3, \ldots, 2t\} \in K_{2t}(F).
\]
Let
\[
X_1 = \{1, 2, 3, \ldots, t\};
\]
\[
X_2 = \{t+1, t+2, t+3, \ldots, 2t\}.
\]
Since $K_{2t}(F)(X_1) \neq \emptyset$, there is a $B_0 \in K_{2t}(F)$ that avoids $X_1$. By Lemma 2.3, $K_{2t}(F)$ is $t$-intersecting. Therefore $|A_0 \cap B_0| \geq t$. This implies that $A_0 \cap B_0 = X_2$. Relabelling if necessary, we may assume that
\[
B_0 = X_2 \cup X_3,
\]
where $X_3 = \{2t+1, 2t+2, 2t+3, \ldots, 3t\}$. Next, $K_{2t}(F)(X_2) \neq \emptyset$ implies that there is a $C_0 \in K_{2t}(F)$ that avoids $X_2$. Since $K_{2t}(F)$ is $t$-intersecting, we must have $|C_0 \cap A_0| \geq t$ and $|C_0 \cap B_0| \geq t$. Thus,
\[
C_0 = X_1 \cup X_3.
\]
Let $U \in F(X_3)$. Since $B_0$ is a transversal for $F$, $|U \cap B_0| \geq t$. So, $X_2 \subseteq U$ for $U$ avoids $X_3$. Similarly, $C_0$ is a transversal for $F$ implies that $X_1 \subseteq U$. Thus, $A_0 \subseteq U$, and
\[
F(X_3) \subseteq \mathcal{Y} = \left\{ Y \in {[n]\choose k} : A_0 \subseteq Y \text{ and } Y \text{ avoids } X_3 \right\}.
\]
Since $F$ is saturated and $A_0$ is a transversal for $F$, we have $F(X_3) = \mathcal{Y}$. So
\[
|F(X_3)| = |\mathcal{Y}| = \binom{n-3t}{k-2t}.
\]
It is left to show that $|F(X)| \geq \binom{n-3t}{k-2t}$ for all $X \in {[n]\choose t}$. Since $K_{2t}(F)(X) \neq \emptyset$, there is a $D \in K_{2t}(F)$ that avoids $X$. Let
\[
Z = \left\{ Z \in {[n]\choose k} : D \subseteq Y \text{ and } D \text{ avoids } X \right\}.
\]
Since $F$ is saturated and $D$ is a transversal for $F$, $Z \subseteq F(X)$. Hence,
\[
|F(X)| \geq |\mathcal{Z}| = \binom{n-3t}{k-2t}.
\]
This completes the proof of the lemma. \qed

Next we consider the second case. We will use the following computation in the proof of Lemma 2.7.
Claim 2.6. Let \( k \geq 2t > 0 \) and \( n \geq \binom{2t+1}{t}^2 k^2 \). If \( 2t + 1 \leq u_0 \leq i \leq k \), then
\[
\frac{(n-i-t)}{(k-i)} < \frac{1}{k^{i-2t}(u_0)^2}.
\]

Proof. Note that
\[
k^{i-2t} \left( \frac{(n-i-t)}{(k-i)} \right) = k^{i-2t} \left( \frac{(k-2t)(k-2t-1) \cdots (k-i+1)}{(n-3t)(n-3t-1) \cdots (n-i-t+1)} \right)
= \prod_{1 \leq j \leq i-2t} \frac{k(k-i+j)}{n-i-t+j}.
\]

Next,
\[
\left( \frac{2t+1}{t} \right)^2 (k(k-i+j)) + (i-j+t)
\leq \left( \frac{2t+1}{t} \right)^2 k^2 - \left( \frac{2t+1}{t} \right)^2 k-1 (i-j+t)
\leq \left( \frac{2t+1}{t} \right)^2 k^2 - \left( \frac{2t+1}{t} \right)^2 k-1 (2t+1)
\leq \left( \frac{2t+1}{t} \right)^2 (2t+1)
\leq \left( \frac{2t+1}{t} \right)^2 k^2 \leq n.
\]

Therefore, \( \frac{1}{n-i-t+j} < \frac{1}{\left( \frac{2t+1}{t} \right)^2} \) and
\[
k^{i-2t} \left( \frac{(n-i-t)}{(k-i)} \right) < \prod_{1 \leq j \leq i-2t} \frac{1}{(2t+1)}^2
= \frac{1}{(2t+1)^{2(t-2t)}} \leq \frac{1}{(2t+1)^{2(u_0-2t)}},
\]

Now, we shall show that
\[
\left( \frac{2t+1}{t} \right)^{2(u_0-2t)} \geq (u_0)^2.
\]

Clearly, it is true for \( u_0 = 2t+1 \). Assume that it is true for some \( u_0 \geq 2t+1 \).

Note that
\[
\left( \frac{u_0+1}{t} \right)^2 = \left( \left( \frac{u_0}{t} \right) + \left( \frac{u_0}{t-1} \right) \right)^2
= \left( \frac{u_0}{t} \right)^2 + 2 \left( \frac{u_0}{t} \right) \left( \frac{u_0}{t-1} \right) + \left( \frac{u_0}{t-1} \right)^2.
\]
By Lemma 2.4, we may assume that

Proof. Let \( A \) be saturated. If \( u \geq 2t + 1 \), then we may assume that \( F \in Y \). For each \( X \in \binom{[n]}{t} \) and \( K \in \mathcal{K}(F)(\overline{X}) \), there is a

integer such that \( \text{div}_i(F) = \binom{n-3i}{k-2t} \). Since \( F \in Y \), let

\( \mathcal{Y}(A) = \{ U \in Y : A \subseteq U \} ; \)

\( \mathcal{Y}(A)(\overline{X}) = \{ U \in \mathcal{Y}(A) : U \text{ avoids } X \} . \)

For each \( F \in \mathcal{F} \), there is a \( K \in \mathcal{K}(F) \) such that \( K \) is originated from \( F \). So, \( F \in \mathcal{Y}(K) \). Since \( F \) is saturated, \( \mathcal{Y}(K) \subseteq \mathcal{F} \). Therefore,

\[ \mathcal{F} = \bigcup_{2t \leq i \leq k} \bigcup_{K \in \mathcal{K}_i(F)} \mathcal{Y}(K), \]

and thus,

\[ \mathcal{F}(\overline{X}) = \bigcup_{2t \leq i \leq k} \bigcup_{K \in \mathcal{K}_i(F)(\overline{X})} \mathcal{Y}(K)(\overline{X}) \]

for all \( X \in \binom{[n]}{t} \). Note that if \( K \in \mathcal{K}_i(F)(\overline{X}) \) and \( X \in \binom{[n]}{t} \), then

\[ |\mathcal{Y}(K)(\overline{X})| = \binom{n - |K| - |X|}{k - i} = \binom{n - i - t}{k - i}. \]

For each \( X \in \binom{[n]}{t} \) with \( \mathcal{K}_i(F)(\overline{X}) = \emptyset \), let \( u_X \) be the smallest positive integer such that

\[ \mathcal{K}_i(F)(\overline{X}) = \emptyset \text{ for all } 2t \leq i \leq u_X - 1 \text{ and } \mathcal{K}_{u_X}(F)(\overline{X}) \neq \emptyset. \]

Since \( \mathcal{F}(\overline{X}) \neq \emptyset \), by (5), \( \mathcal{K}_i(F)(\overline{X}) \neq \emptyset \) for some \( i \geq 2t + 1 \). So, \( 2t + 1 \leq u_X \leq k \). Let

\[ u_0 = \max_{X \in \binom{[n]}{t}} \{ u_X \}. \]
Then, $2t + 1 \leq u_0 \leq k$ and there is an $X_0 \in \binom{[n]}{t}$ such that
\[ K_i(\mathcal{F})(X_0) = \emptyset \]
for all $2t \leq i \leq u_0 - 1$ and $K_{u_0}(\mathcal{F})(X_0) \neq \emptyset$.

By (5) and (6),
\[ |\mathcal{F}(X_0)| \leq \sum_{u_0 \leq i \leq k} \sum_{K \in K_i(\mathcal{F})(X_0)} |\mathcal{Y}(K)(X_0)| = \sum_{u_0 \leq i \leq k} w_i \binom{n - i - t}{k - i}, \]
where $w_i = |K_i(\mathcal{F})(X_0)|$. By using the ‘branching algorithm’ of Frankl [11, Proposition 6.1], we will show that
\[ \sum_{u_0 \leq i \leq k} k^{i - 1} w_i \leq \binom{u_0}{t} k^{k - 2t}. \]

Before we proceed to prove (7), let us use it to show that
\[ \text{div}_t(\mathcal{F}) < \binom{n - 3t}{k - 2t}. \]

By Lemma 2.6,
\[ \frac{|\mathcal{F}(X_0)|}{\binom{n - 3t}{k - 2t}} \leq \sum_{u_0 \leq i \leq k} w_i \frac{\binom{n - i - 1}{k - i}}{\binom{n - 3t}{k - 2t}} \leq \sum_{u_0 \leq i \leq k} w_i \frac{1}{k^{i - 1} t^2} \leq 1, \]
where the last inequality follows from (7). Hence, $|\mathcal{F}(X_0)| < \binom{n - 3t}{k - 2t}$ and $\text{div}_t(\mathcal{F}) < \binom{n - 3t}{k - 2t}$.

Now, it is left to prove (7) holds. A sequence of length $j$ over $[n]$ will be denoted by
\[(a_1, a_2, \ldots, a_j),\]
where $a_i \in [n]$. For each $j$ ($2t \leq j \leq k$), we will construct a family $\mathcal{L}_j$ of sequences of length $j$ over $[n]$ such that
\[ \begin{align*}
  (a) \quad & |\mathcal{L}_j| \leq \binom{u_0}{t} k^{j - 2t}, \\
  (b) \quad & \text{for each } (a_1, a_2, \ldots, a_j, a_{j+1}) \in \mathcal{L}_{j+1}, \text{ we have } (a_1, a_2, \ldots, a_j) \in \mathcal{L}_j.
\end{align*} \]

Let $K_0 \in K_{u_0}(\mathcal{F})(X_0)$. Note that such an element exists for $K_{u_0}(\mathcal{F})(X_0) \neq \emptyset$. For each $Y \subseteq K_0$ with $|Y| = t$, we have $\mathcal{F}(Y) \neq \emptyset$. So, by (5), $K_i(Y) \neq \emptyset$ for some $2t \leq i \leq k$. If $K_{2t}(Y) \neq \emptyset$, then choose a $K_Y \in K_{2t}(Y)$. If $K_{2t-1}(Y) = \emptyset$, then choose a $K_Y \in K_i(Y)$ where $i$ is the smallest positive integer such that $K_i(Y) \neq \emptyset$ and $K_j(Y) = \emptyset$ for $2t \leq j \leq i - 1$. By the choice of $u_0$, $i \leq u_0$. Thus, $|K_Y| \leq u_0$. Let $\mathcal{L}_{2t}$ be the family of all sequences of length $2t$ over $[n]$ of the following form
\[ L = (a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t), \]
where $a_1 < a_2 < \cdots < a_t$, $b_1 < b_2 < \cdots < b_t$, $\{a_1, a_2, \ldots, a_t\} = Y \subseteq K_0$ and $\{b_1, b_2, \ldots, b_t\} = Z \subseteq K_Y$. Therefore, $|\mathcal{L}_{2t}| \leq \binom{u_0}{t}^2$. 

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Choose an arbitrary set $V_0 \in \binom{[n]}{k}$. We are ready to construct the family $L_{2t+1}$.

Algorithm:
(i) Partition $L_{2t}$ into two parts, say $L_{2t}(1)$ and $L_{2t}(2) = L_{2t} \setminus L_{2t}(1)$ where

$$(c_1, c_2, \ldots, c_{2t}) \in L_{2t}(1) \text{ if and only if } |\{c_1, c_2, \ldots, c_{2t}\} \cap F| \geq t \quad \text{ for all } F \in \mathcal{F}.$$ 

(ii) For each sequence $c = (c_1, c_2, \ldots, c_{2t}) \in L_{2t}(1)$ and all $v \in V_0$, put

$$(c, v) = (c_1, c_2, \ldots, c_{2t}, v) \in L_{2t+1}.$$ 

(iii) For each sequence $c = (c_1, c_2, \ldots, c_{2t}) \in L_{2t}(2)$, we have

$$(c, v) \in L_{2t+1}.$$ 

So, for each $v \in F_c \setminus \{c_1, c_2, \ldots, c_{2t}\}$, put $(c, v) \in L_{2t+1}$.

Since $|V_0| = |F_c| = k$, $|L_{2t+1}| \leq |L_{2t}| k \leq \left(\frac{\binom{t}{1}}{t}\right)^2 k$.

Suppose we have constructed the families $L_j$ for $2t \leq j \leq k$ such that (a) and (b) hold. Now, we shall construct the family $L_{j+1}$.

Algorithm:
(i) Partition $L_j$ into two parts, say $L_j(1)$ and $L_j(2) = L_j \setminus L_j(1)$ where

$$(c_1, c_2, \ldots, c_j) \in L_j(1) \text{ if and only if } |\{c_1, c_2, \ldots, c_j\} \cap F| \geq t \quad \text{ for all } F \in \mathcal{F}.$$ 

(ii) For each sequence $c = (c_1, c_2, \ldots, c_j) \in L_j(1)$ and all $v \in V_0$, put

$$(c, v) = (c_1, c_2, \ldots, c_j, v) \in L_{j+1}.$$ 

(iii) For each sequence $c = (c_1, c_2, \ldots, c_j) \in L_j(2)$, we have

$$(c, v) \in L_{j+1}.$$ 

So, for each $v \in F_c \setminus \{c_1, c_2, \ldots, c_j\}$, put $(c, v) \in L_{j+1}$.

Since $|V_0| = |F_c| = k$, $|L_{j+1}| \leq |L_{j}| k \leq \left(\frac{\binom{t}{1}}{t}\right)^2 k^{j+1-2t}$. Note that $|L_k| \leq \left(\frac{\binom{t}{1}}{t}\right)^2 k^{k-2t}$.

Let $K \in \mathcal{K}_j(F)(\overline{X}_0)$ ($a_0 \leq j \leq k$). We claim that there is a sequence $d = (d_1, d_2, \ldots, d_j) \in L_j$ such that

$$(d_1, d_2, \ldots, d_j) = K.$$ 

By Lemma 2.3, $K(F)$ is $t$-intersecting. Therefore, $|K \cap K_0| \geq t$, and there is a $Y = \{a_1, a_2, \ldots, a_t\} \subseteq K \cap K_0$. We may assume that $a_1 < a_2 < \cdots < a_t$. Recall that $K_Y$ avoids $Y$. So, $|K \cap K_Y| \geq t$ implies that there is a $\{b_1, b_2, \ldots, b_t\} \subseteq K \cap K_Y$. We may assume that $b_1 < b_2 < \cdots < b_t$. Thus, $(a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t) \in L_{2t}$ and

$$\{a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t\} \subseteq K.$$ 

So, there is a $d_0 = (d_1, d_2, \ldots, d_{j_0}) \in L_{j_0}$ such that

$$(d_1, d_2, \ldots, d_{j_0}) \subseteq K.$$
We may assume that \( j_0 \) is the largest with such property. If \( \{d_1, d_2, \ldots, d_{j_0}\} = K \), we are done. Suppose \( \{d_1, d_2, \ldots, d_{j_0}\} \subseteq K \). Since \( K \) is a minimal transversal, \( \{d_1, d_2, \ldots, d_{j_0}\} \) is not a transversal. So, \( |\{d_1, d_2, \ldots, d_{j_0}\} \cap F| \leq t - 1 \) for some \( F \in \mathcal{F} \). This means \( d_0 \in L_{j_0}(2) \) and
\[
|\{d_1, d_2, \ldots, d_{j_0}\} \cap F_{d_0}| \leq t - 1.
\]
Now, \( |K \cap F_{d_0}| \geq t \) implies that there is a \( d_{j_0+1} \in (K \cap F_{d_0}) \setminus \{d_1, d_2, \ldots, d_{j_0}\} \).
By construction (‘branching algorithm’), \( (d_0, d_{j_0+1}) \in L_{j_0+1} \) and
\[
\{d_1, d_2, \ldots, d_{j_0}, d_{j_0+1}\} \subseteq K.
\]
This contradicts the choice of \( j_0 \). Hence, for each \( K \in \mathcal{K}_j(\mathcal{F}(X_0)) \), there is a sequence \( d_K = (d_1, d_2, \ldots, d_j) \in L_j \) such that
\[
\{d_1, d_2, \ldots, d_j\} = K.
\]
The sequence \( d_K \) is said to be associated to \( K \).

A sequence \( c = (c_1, c_2, \ldots, c_k) \in L_k \) is said to be an extension of a sequence \( d = (d_1, d_2, \ldots, d_j) \in L_j \) if \( c_i = d_i \) for all \( 1 \leq i \leq j \), i.e., \( c = (d, c_{j+1}, c_{j+2}, \ldots, c_k) \). For each \( K \in \mathcal{K}_j(\mathcal{F}(X_0)) \), let
\[
Z_K = \{c \in L_k : c \text{ is an extension of } d_K\}.
\]
By construction, \( |Z_K| = k^{k-j} \). Furthermore,
\[
\bigcup_{u_0 \leq j \leq k} \bigcup_{K \in \mathcal{K}_j(\mathcal{F}(X_0))} Z_K \subseteq L_k.
\]
Now, we shall show that \( Z_{K_1} \cap Z_{K_2} = \emptyset \) for all \( K_1 \in \mathcal{K}_{j_1}(\mathcal{F}(X_0)) \) and \( K_2 \in \mathcal{K}_{j_2}(\mathcal{F}(X_0)) \) where \( j_1 \neq j_2 \). Let \( c \in Z_{K_1} \cap Z_{K_2} \). Then,
\[
c = (d_{K_1}, c_{j_1+1}, c_{j_1+2}, \ldots, c_k);
\]
\[
c = (d_{K_2}, c_{j_2+1}, c_{j_2+2}, \ldots, c_k).
\]
Without loss of generality, we may assume that \( j_2 \geq j_1 \). So, \( d_{K_2} \) is an extension of \( d_{K_1} \), and \( K_1 \subseteq K_2 \). Since \( K_1 \) and \( K_2 \) are minimal transversal, we must have \( j_2 = j_1 \) and \( K_1 = K_2 \). Thus, \( Z_{K_1} \cap Z_{K_2} = \emptyset \).

So,
\[
\binom{u_0}{t}^2 k^{2t-2} \geq |L_k| \geq \bigg| \bigcup_{u_0 \leq j \leq k} \bigcup_{K \in \mathcal{K}_j(\mathcal{F}(X_0))} Z_K \bigg| \geq \sum_{u_0 \leq j \leq k} \sum_{K \in \mathcal{K}_j(\mathcal{F}(X_0))} |Z_K| \geq \sum_{u_0 \leq j \leq k} \sum_{K \in \mathcal{K}_j(\mathcal{F}(X_0))} k^{k-j} = \sum_{u_0 \leq j \leq k} k^{k-j} w_j.
\]
This proves (7).
\[\square\]

Finally, Theorem 1.5 follows from Lemmas 2.1, 2.4, 2.5 and 2.7. In fact by Lemmas 2.1, 2.4, 2.5 and 2.7, we have the following refinement of Theorem 1.5.
Theorem 2.8. Let $k \geq 2t > 0$ and $n \geq \binom{2t+1}{t}^2k^2$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is $t$-intersecting, then
\[
\text{div}_t(\mathcal{F}) \leq \frac{(n-3t)(k-2t)}{k^2}.
\]
Moreover, if $\mathcal{F}$ is saturated, then equality holds if and only if $K_{2t}(\mathcal{F})(X) \neq \emptyset$ for all $X \in \binom{[n]}{t}$.

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References


ON DIVERSITY OF CERTAIN $t$-INTERSECTING FAMILIES

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