DEPENDENT SUBSETS OF EMBEDDED PROJECTIVE VARIETIES

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Abstract. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim(X)$. Let $\rho(X)^{\prime\prime}$ be the maximal integer such that every zero-dimensional scheme $Z \subset X$ smoothable in $X$ is linearly independent. We prove that $X$ is linearly normal if $\rho(X)^{\prime\prime} \geq \lceil (r + 2)/2 \rceil$ and that $\rho(X)^{\prime\prime} < 2 \lceil (r + 1)/(n + 1) \rceil$, unless either $n = r$ or $X$ is a rational normal curve.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field with characteristic zero. Set $n := \dim X$. We recall that a zero-dimensional scheme $Z \subset X$ is said to be smoothable in $X$ if it is a flat limit of a family of finite subsets of $X$ with cardinality $\deg(Z)$ (see [14] for a discussion of it). If $X$ is smooth (or if $Z$ is contained in the smooth locus of $X$) $Z$ is smoothable in $X$ if and only if it is smoothable in $\mathbb{P}^r$ and the notion of smoothability in $\mathbb{P}^r$ does not depend on the choice of the embedding of $Z$ in a projective space ([14, Proposition 2.1]). Let $\rho(X)$ (resp. $\rho(X)^{\prime}$, resp. $\rho(X)^{\prime\prime}$) denote the maximal integer $t > 0$ such that each zero-dimensional scheme (resp. each finite set, resp. each zero-dimensional scheme smoothable in $X$) $Z \subset X$ with $\deg(Z) = t$ is linearly independent. Obviously $\rho(X) \leq \rho(X)^{\prime\prime} \leq \rho(X)^{\prime}$. Since $X$ is embedded in $\mathbb{P}^r$, we have $\rho(X) \geq 2$.

The integer $\rho(X)^{\prime}$ appears in very classical projective geometry papers and books (see [22, Ch. 8,9,10,12], [23, Ch. 27] and references therein for the case of a finite field). When $X$ is a finite set of a finite dimensional vector space over a finite field, this integer is related to the minimum distance of the dual of the code obtained evaluating linear forms at the points of $X$ ([18]). In the set-up of the $X$-rank described below the integer $\rho(X)^{\prime}$ gives a uniqueness result (see Remarks 2.3 for details and references).

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For algebraic varieties the integer $\rho(X)$ is very natural. Zero-dimensional schemes appeared in several papers concerning the additive decomposition of polynomials and the equations of embedded projective varieties ([7, 14, 15]) and they allowed the introduction of a name, the cactus rank (of a point, of a homogeneous polynomial or a tensor) ([11, 13, 25]); in [24] it was called the scheme-rank. Over a finite field combining zero-dimensional schemes (all of them smoothable) with [18] is a standard feature to get bounds on the minimum distance and number of minimal weight codewords for an evaluation code coming from certain projective curves ([3, 8–10]).

To justify the integer $\rho(X)^0$ one should justify the use of smoothable zero-dimensional schemes, not just of zero-dimensional schemes. Smoothable zero-dimensional subschemes of $X$ and the integer $\rho(X)^0$ arise in the study of secant varieties and border ranks described below (see [14, Proposition 2.5] for the case of additive decompositions of homogeneous polynomials). Computing $\rho(X)^0$ gives a lower bound for $\rho(X)'$ and an upper bound for $\rho(X)$. Thus one can try to compute $\rho(X)^0$, when computing $\rho(X)$ fails. At least after [14] each time a lower bound for $\rho(X)'$ is computed, it seems useful to ask oneself if the same proof works (with minimal modifications) for $\rho(X)$ or at least for $\rho(X)'$. Using only smoothable zero-dimensional schemes instead of arbitrary ones allows the check of a shorter list of schemes in several proofs ([4–7, 12, 16]).

For any $q \in \mathbb{P}^r$ the X-rank $r_X(q)$ of $q$ is the minimal positive integer $t$ such that $q \in \langle S \rangle$ for some finite subset $S \subset X$ with $\sharp(S) = t$, where $\langle \cdot \rangle$ denotes the linear span. For any positive integer $t$ the $t$-secant variety $\sigma_t(X)$ of $X$ is the closure in $\mathbb{P}^r$ of the union of all $\langle S \rangle$ with $S$ a finite subset of $X$ with cardinality $t$.

The border X-rank $b_X(q)$ of $q \in \mathbb{P}^r$ is the minimal integer $k$ such that $q \in \sigma_k(X)$. The generic rank $r_{X,\text{gen}}$ is the minimal integer $k > 0$ such that $\sigma_k(X) = \mathbb{P}^r$. There is a non-empty open subset $U \subset \mathbb{P}^r$ such that $r_X(q) = r_{X,\text{gen}}$ for all $q \in U$.

In this paper we prove that if $\rho(X)^0$ is large, then $X$ is linearly normal and that $\rho(X)^0$ cannot be very large for $n > 1$ (Theorem 1.1).

We prove the following results.

**Proposition 1.1.** Assume that $X$ is a curve and $\rho(X)^0 \geq (r + 2)/2$. Then $X$ is linearly normal.

**Proposition 1.2.** Assume $n := \dim X \geq 2$, $r_{X,\text{gen}} = [(r + 1)/(n + 1)]$ and $\rho(X)^0 > [(r + 1)/(n + 1)]$. Then $X$ is linearly normal.

**Theorem 1.3.** Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. We have $\rho(X)^0 \geq 2[(r + 1)/(n + 1)]$ if and only if either $r = n$ (i.e., $X = \mathbb{P}^r$) or $n = 1$, $r$ is odd and $X$ is a rational normal curve.

If $n = r$ we have $\rho(X)' = \rho(X) = 2$. If $X$ is a rational normal curve we have $\rho(X) = \rho(X)' = r + 1$. This is the only case with $\rho(X)' = r + 1$ (Lemma 2.4). Theorem 1.3 implies that $\rho(X)^0 < 2[(r + 1)/(n + 1)]$ if $(r + 1)/(n + 1) \notin \mathbb{Z}$.
The example of a general linear projection in \( P^4 \) of the Veronese surface shows that in Proposition 1.2 it is not sufficient to assume that \( \rho(X)^{''} \geq \lceil (r + 2)/(n + 1) \rceil \).

We point out that to get our results we only use a small family of zero-dimensional schemes, each of them with connected components of degree 1 or 2, but that this family contains a complete family covering \( X \): each \( p \in X \) is contained in some scheme \( Z \) of the family.

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2. Preliminaries

Let \( X \subset P^r \) be an integral and non-degenerate variety. Set \( n := \dim X \).

For any \( q \in P^r \) let \( S(X, q) \) be the set of all \( S \subset X \) such that \( |S| = r_X(q) \) and \( q \in (S) \).

Remark 2.1. Fix \( q \in \sigma_k(X) \). If \( k \leq \rho(X)^{''} \) there is a zero-dimensional scheme \( Z \subset X \) smoothable in \( X \) and \( q \in \langle Z \rangle \) ([15, Lemma 2.6, Theorem 1.18] and [14, Proposition 2.5]).

Remark 2.2. Let \( X \subset P^r \) be a smooth variety with \( \dim X \leq 2 \). Every zero-dimensional subcheme of \( X \) is smoothable ([19]) and hence \( \rho(X)^{''} = \rho(X)^{'} \).

Easy examples show that we may have \( \rho(X)^{'} < \rho(X)^{''} \) for a smooth curve (Example 4.2).

Remark 2.3 ([15, Theorem 1.17]). Fix \( q \in P^r \) and \( A, B \in S(X, q) \). Set \( x := r_X(q) \) and assume \( \rho(X)^{'} \geq 2x \). Since \( |A \cup B| \leq 2x, A \cup B \) is linearly independent. Thus \( A = B \) ([4, Lemma 1]). Hence \( |S(X, q)| = 1 \).

In following extremal case we are able to use only \( \rho(X)^{'} \) instead of \( \rho(X)^{''} \).

Lemma 2.4. The following conditions are equivalent:

(1) \( X \) is a rational normal curve;
(2) \( \rho(X)^{'} = r + 2 - n \);
(3) \( \rho(X)^{'} \geq r + 2 - n \).

Proof. It is sufficient to prove that (3) implies (1).

First assume \( n = 1 \). Let \( H \subset P^r \) be a general linear hyperplane. Since \( X \cap H \) is formed by \( \deg(X) \) points, if \( \rho(X)^{'} > r \) we have \( \deg(X) = r \) and hence \( X \) is a rational normal curve.

Now assume \( n > 1 \). Take a general linear subspace \( V \subset P^r \) with codimension \( n - 1 \). The scheme \( X \cap V \) is an integral curve spanning \( V \). If \( X \cap V \) is not a rational normal curve we have \( \rho(X)^{'} \leq \rho(X \cap V)^{'} \leq r - n + 1 \) by the case \( n = 1 \) just proved. Now assume that \( X \cap V \) is a rational normal curve, i.e., assume \( \deg(X) = r + 1 - n \). The classification of minimal degree subvarieties of projective spaces ([21, Proposition 3.10]) show that \( X \) contains lines (and hence \( \rho(X)^{'} = 3 \)) unless \( r = 5, n = 2 \) and \( X \) is the Veronese surface. If \( X \) is the Veronese surface of \( P^5 \) we have \( \rho(X)^{'} = 3 \), because no 3 points of \( X \) are coplanar, but \( X \) contains plane conics.

\[ \square \]
3. The proofs

Proof of Proposition 1.1. Assume that $X$ is not linearly normal. Thus there is a non-degenerate curve $Y \subset \mathbb{P}^{r+1}$ such that $X$ is an isomorphic linear projection of $Y$ from some $o \in \mathbb{P}^{r+1} \setminus Y$. Set $b := b_Y(o)$. Each secant variety of a curve has the expected dimension ([1, Remark 1.6]). Thus $r_{Y,\text{gen}} = \lceil (r + 2)/2 \rceil$. Hence $b \leq \lceil (r + 2)/2 \rceil$. Let $\ell : \mathbb{P}^{r+1} \setminus \{o\} \to \mathbb{P}^r$ denote the linear projection from $o$. By assumption $o \notin Y$ and $\ell|_Y$ is an embedding with $\ell(Y) = X$. Let $W \subset Y$ be any zero-dimensional scheme. Since $\ell|_Y : Y \to X$ is an isomorphism, $W$ is smoothable in $Y$ if and only if $\ell(W)$ is smoothable in $X$ and any degree $b$ smoothable zero-dimensional subscheme of $X$ is the image of a unique degree $b$ zero-dimensional subscheme of $Y$. Thus $\rho(Y)^{b} \geq \rho(X)^{b}$. The image in $\mathbb{P}^r$ of a linear subspace $V \subset \mathbb{P}^{r+1}$ has either dimension $\dim V$ (case $o \notin V$) or dimension $\dim V - 1$ (case $o \in V$). Since $\rho(Y)^{b} \geq \rho(X)^{b} \geq \lceil (r + 2)/2 \rceil = r_{Y,\text{gen}}$ and $b \leq r_{Y,\text{gen}}$, there is a smoothable zero-dimensional scheme $W \subset Y$ such that $o \in \langle W \rangle$ and $\deg(W) = b$ (Remark 2.1). Since $\ell(W)$ is not linearly independent, we have $\rho(X)^{b} \leq b - 1$, a contradiction. □

Proof of Proposition 1.2. Assume that $X$ is not linearly normal. Thus there is a non-degenerate variety $Y \subset \mathbb{P}^{r+1}$ such that $X$ is an isomorphic linear projection of $Y$ from some $o \in \mathbb{P}^{r+1} \setminus Y$. Set $b := b_Y(o)$ and $a := r_{X,\text{gen}} = \lceil (r + 1)/(n + 1) \rceil$. Let $\ell : \mathbb{P}^{r+1} \setminus \{o\} \to \mathbb{P}^r$ denote the linear projection from $o$. By assumption $o \notin Y$ and $\ell|_Y$ is an embedding with $\ell(Y) = X$. As in the proof of Proposition 1.1 we have $\rho(Y)^{a} \geq \rho(X)^{a}$ and to get a contradiction it is sufficient to prove that $b \leq \rho(X)^{a}$. Assume $b > \rho(X)^{a}$, i.e., assume $b \geq a + 2$. Since $b > a$, we have $o \notin \sigma_a(Y)$. Hence $\ell|_{\sigma_a(Y)} : \sigma_a(Y) \to \mathbb{P}^r$ is a finite map. Since $\ell(\sigma_a(Y)) = \sigma_a(X)$, we get $\dim \sigma_a(Y) = r$. Since $\dim \sigma_a(Y) > \dim \sigma_a(Y)$ ([1, Proposition 1.3]), we get $\sigma_a(Y) = \mathbb{P}^{r+1}$. Thus $b \leq a + 1$, a contradiction. □

Lemma 3.1. Assume $\rho(X)^{a} \geq 2[(r + 1)/(n + 1)]$. Then $X$ is not defective, $r + 1 \equiv 0 \pmod{n + 1}$ and for a general $q \in \mathbb{P}^r$ we have $|\mathcal{S}(X, q)| = 1$.

Proof. Set $a := \lceil (r + 1)/(n + 1) \rceil$. Fix any $q \in \mathbb{P}^r$ such that $r_X(q) \leq a$. Remark 2.3 gives $|\mathcal{S}(X, q)| = 1$ if $\rho(X)^{a} \geq 2r_X(q)$. In particular $|\mathcal{S}(X, q)| = 1$ for a general $q \in \sigma_a(X)$. Thus a dimensional count shows that $\dim \sigma_a(X) = a(n + 1) - 1$. Since $\dim \sigma_a(X) \leq r$, we get $a \in \mathbb{Z}$, $\mathbb{P}^r = \sigma_a(X)$ and that $X$ is not defective. Since $\mathbb{P}^r = \sigma_a(X)$, we have $r_X(q) = a$ for a general $q \in \mathbb{P}^r$. Hence $|\mathcal{S}(X, q)| = 1$ for a general $q \in \mathbb{P}^r$. □

The (smooth) $n$-dimensional varieties $X \subset \mathbb{P}^{2n+1}$ such that $\sigma_2(X) = \mathbb{P}^{2n+1}$ and $|\mathcal{S}(X, q)| = 1$ are classically called OADP (or varieties with only one apparent double point), because projecting them from a general point of $X$ one gets a variety with a unique singular point ([17]). They are always linearly normal ([17, Remark 1.2]). In [17] there are also older references and the classification of the smooth ones with dimension up to 3 ([26], [17, Theorem 7.1]). Thus the thesis of Lemma 3.1 is a generalization of this concept to
the case in which \((r+1)/(n+1)\) is an integer > 2. But the assumption "\(\rho(X) > 2[(r+1)/(n+1)]\)" of the lemma is too strong to be interesting for the classification of extremal varieties. Just assuming \(\rho(X) > 2\) excludes all \(X\) containing lines and hence all smooth OADP’s of dimension 2 and 3.

**Corollary 3.2.** Assume \(n := \dim X \geq 2\), \(\rho(X)^{\prime\prime} \geq [(r+1)/(n+1)]\) and \(\rho(X)^{\prime} \geq 2[(r+1)/(n+1)]\). Then \(X\) is linearly normal, non-defective, \(r+1 \equiv 0\) (mod \(n+1\)) and \(|S(X,q)| = 1\) for a general \(q \in \mathbb{P}^r\).

**Proof of Corollary 3.2.** Apply Lemma 3.1 and Proposition 1.2. \(\square\)

**Proof of Theorem 1.3.** Assume the existence of \(X\) with \(\rho(X)^{\prime\prime} \geq 2[(r+1)/(n+1)]\). We may assume \(n < r\), i.e., \(X \neq \mathbb{P}^r\).

First assume \(n = 1\). Lemma 2.4 gives that \(X\) is a rational normal curve, that \(r\) is odd and that \(\rho(X) = \rho(X)^{\prime} = r+1\).

Now assume \(n \geq 2\). By Lemma 3.1, \(a := (r+1)/(n+1)\) is an integer. Since \(r > n\), we have \(a \geq 2\). Fix a general \(S \subseteq X\) such that \(|S| = a - 1\). Since \(S\) is general, each \(p \in S\) is a smooth point of \(X\). We saw in the proof of Proposition 1.2 that \(V \coloneqq (\cup_{p \in S T_p X})\) has dimension \((a-1)(n+1) - 1\). Fix \(o \in X \setminus S\).

**Claim 1.** \(o \notin V\).

**Proof of Claim 1.** Assume \(o \in V\). We saw in the proof of Proposition 1.2 that there are connected degree 2 zero-dimensional schemes \(v_p \subset X\) such that \((v_p)_{\text{red}} = \{p\}\) and \(o \in (Z), \text{where } Z := \cup_{p \in S v_p}. \text{Since } o \notin S, \text{we have } o \notin Z. \text{Thus the scheme } Z \cup \{o\} \text{is linearly dependent. Since } \deg(Z \cup \{o\}) = 2a - 1 < \rho(X)^{\prime\prime} \text{and } Z \cup \{o\} \text{is smoothable, we got a contradiction.} \square\)

Let \(\ell : \mathbb{P}^r \setminus V \rightarrow \mathbb{P}^n\) denote the linear projection from \(V\). By Claim 1 we have \(S = X \cap V\) and hence \(\mu = \ell_{|X \setminus S} : X \setminus S \rightarrow \mathbb{P}^n\) is a morphism. Fix \(o \in X \setminus S\) and assume the existence of \(o' \in X \setminus S\) such that \(o \neq o'\) and \(\mu(o) = \mu(o')\). Thus \(o' \in (\{o\} \cup Z)\). Hence \(o, o' \cup Z\) is linearly dependent. The zero-dimensional scheme \(\{o, o'\} \cup Z\) is smoothable and it has degree \(2a \leq \rho(X)^{\prime\prime}\), a contradiction.

Thus \(\mu : X \setminus S \rightarrow \mathbb{P}^n\) is an injective morphism between two quasi-projective varieties. Since \(\mathbb{P}^n\) is smooth (it would be sufficient to assume that the target, \(X\), is normal or even less (weakly normal)) and we are in characteristic zero, \(\mu\) is an open map which is an isomorphism onto its image \(\langle\mathbb{P}^r\rangle\). Since \(X\) is smooth at each point of \(S\), \(X\) is smooth. Since \(S\) is finite, the \(\acute{e}tale\) cohomology of \(X \setminus S\) in dimension \(n-1\) shows that \(\mathbb{P}^n \setminus \mu(X \setminus S)\) is finite with cardinality \(\mathbb{Z}(S)\). Thus \(\mu\) extends to an isomorphism \(u : X \rightarrow \mathbb{P}^n\). However, by the definition of linear projection from \(V = (\cup_{p \in S T_p X})\), the linear independence of the linear spaces \(T_p, x \in S, \text{and the smoothness of } X, \mu\) extends to some non-empty open subset of the exceptional divisor of the blowing-up of \(X\) at the points of \(S\). Since \(S \neq \emptyset\), this is absurd. \(\square\)
4. Elementary examples

By Proposition 1.1 to complete the picture for curves we need to describe the linearly normal curves with very high $\rho(X)'$, $\rho(X)$ and $\rho(X)''$. We also give examples of smooth curves $X$ with prescribed $\rho(X)$ or prescribed $\rho(X)'$.

Remark 4.1. Let $X$ be an integral projective curve. To compute $\rho(X)''$ we recall that every Cartier divisor of $X$ is smoothable. Let $F$ be any torsion free sheaf of $X$. Duality gives $h^1(F) = \dim \text{Hom}(F, \omega_X)$, ([2, 1.1 at p. 5]). Thus for any zero-dimensional scheme $Z \subset X$ we have $h^1(I_Z(1)) = \dim \text{Hom}(I_Z, \omega_X(1))$. If $d \geq 3g - 2$ we have $\rho(X) = d - 2g + 2$. We have $\rho(X)' = d - 2g + 2$ if and only if $X$ is Gorenstein, i.e., $\omega_X$ is locally free. For lower $d$ the integers $\rho(X)$, $\rho(X)'$ and $\rho(X)''$ depends both from the Brill-Noether theory of the special line bundles on $X$ and the choice of the very ample line bundle $O_X(1)$, not just the integers $d$ and $g$.

Example 4.2. Fix integers $r, a$ such that $2 \leq a \leq r + 1$. Here we prove the existence of a smooth and non-degenerate curve $X \subset \mathbb{P}^r$ such that $\rho(X)' = \rho(X) = a$ and, if $a \leq r - 1$ and $2a \geq r + 2$, another example $\tilde{X}$ with $\rho(\tilde{X})' > \rho(\tilde{X}) = a$. If $a = r + 1$ we know that $X$ is a rational normal curve. The case $g = 1, d = r + 1$ covers the the case $a = r$. Now assume $2 \leq a \leq r - 1$.

(a) We first cover the case $2a \leq r + 1$. In this range we construct a smooth rational curve $X$ with $\rho(X) = \rho(X)' = a$, but of course $X$ is not linearly normal. Let $Y \subset \mathbb{P}^{r+1}$ be a rational normal curve. Fix a set $S \subset X$ such that $|S| = a + 1$ and take any $o \in \langle S \rangle$ such that $o \notin \langle S' \rangle$ for any $S' \subsetneq S$. Let $\ell : \mathbb{P}^{r+1} \setminus \{o\} \to \mathbb{P}^r$ denote the linear projection from $o$. Since $a \geq 2$ and $\rho(Y) = r + 2$, we have $o \notin Y$. Hence $\ell|_Y$ is a morphism. Set $X := \ell(Y)$.

Claim 1. $\ell|_Y$ is an embedding.

Proof of Claim 1. It is sufficient to prove that for any zero-dimensional scheme $A \subset Y$ with $\deg(A) \leq 2$ we have $o \notin \langle A \rangle$. Assume the existence of a zero-dimensional scheme $A \subset Y$ with $\deg(A) \leq 2$ and $o \in \langle A \rangle$. Since $o \notin Y$, we have $\deg(A) = 2$. Since $o \notin \langle S' \rangle$ for any $S' \subsetneq S$ and $|S| = a + 1 > 2$, we have $A \notin S$. Since $o \in \langle A \rangle \cap \langle S \rangle$, $A \cup S$ is linearly dependent. Since $\deg(A \cup S) \leq a + 3$ and $\rho(Y) = r + 1$, we get a contradiction.

By Claim 1 $X$ is a smooth rational curve and $\deg(X) = r + 1$.

Claim 2. We have $\rho(X) = \rho(X)' = a$.

Proof of Claim 2. Since $\ell|_X$ is an embedding, we have $|\ell(S)| = a + 1$. Since $o \in \langle S \rangle$, $\ell(S)$ is linear dependent and hence $\rho(X)' \leq a$. Assume $\rho(X)' < a$ and take a zero-dimensional scheme $Z \subset X$ such that $\deg(Z) \leq a$ and $Z$ is linearly dependent. Let $W \subset Y$ be the only scheme such that $\ell(W) = Z$. Since $Z$ is linearly dependent, we have $o \in \langle W \rangle$. Since $\deg(W) \leq a$ and $o \notin \langle S' \rangle$ for any $S' \subsetneq S$, we have $W \notin S$. Thus $S \cup W$ is linearly dependent. Hence $2a + 1 \geq r + 2$, a contradiction. □
(b) Now assume $2 \leq a \leq r - 1$ and $2a \geq r + 2$. Set $g := r + 1 - a$. Fix a smooth curve $C$ of genus $g$ and a zero-dimensional scheme $A \subset X$ such that $\deg(A) = a + 1$. Since $\deg(\omega_C(A)) = 2g + a + 1 - 2 \geq 2g + 1$, $\omega_C(A)$ is very ample. By Riemann-Roch we have $h^0(\omega_C(A)) = g + a = r + 1$. Let $f : C \to \mathbb{P}^r$, be the embedding induced by $|\omega_X(A)|$. Set $X := f(C)$ and $Z := f(A)$. By Riemann-Roch $Z$ is linearly dependent and $Z$ is the only linearly dependent zero-dimensional scheme $W \subset X$ such that $\deg(W) \leq a + 1$. Thus $\rho(X) = a$. We have $\rho(X)' = a$ if and only if $A$ is a reduced set. If $A$ is not reduced we get the promised curve $\tilde{X}$.

References


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