ON CONDITIONALLY DEFINED FIBONACCI AND LUCAS SEQUENCES AND PERIODICITY

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ABSTRACT. We synthesize the recent work done on conditionally defined Lucas and Fibonacci numbers, tying together various definitions and results generalizing the linear recurrence relation. Allowing for any initial conditions, we determine the generating function and a Binet-like formula for the general sequence, in both the positive and negative directions, as well as relations among various sequence pairs. We also determine conditions for periodicity of these sequences and graph some recurrent figures in Python.

1. Introduction

Several articles have been written recently about conditionally defined Fibonacci and Lucas sequences, with various initial conditions. For example, in 2009, Edson and Yayenie [3] introduced the following definition of a generalized Fibonacci sequence, where \( a \) and \( b \) are any non-zero real numbers:

\[
q_0 = 0; \quad q_1 = 1; \quad \text{and for } m \geq 2, \quad q_m = \begin{cases} \alpha q_{m-1} + q_{m-2} & \text{if } m \text{ is even}, \\ \beta q_{m-1} + q_{m-2} & \text{if } m \text{ is odd}. \end{cases}
\]

Then, in 2012, Yayenie [10] defined the “new generalized Lucas sequence of the second kind” as

\[
V_0 = 2; \quad V_1 = ab; \quad \text{and for } m \geq 2, \quad V_m = \begin{cases} aV_{m-1} + V_{m-2} & \text{if } m \text{ is even}, \\ bV_{m-1} + V_{m-2} & \text{if } m \text{ is odd}. \end{cases}
\]

In a similar manner, in 2014 Bilgici [1] provided the following definition of a generalized Lucas sequence:

\[
l_0 = 2; \quad l_1 = a; \quad \text{and for } m \geq 2, \quad l_m = \begin{cases} bl_{m-1} + l_{m-2} & \text{if } m \text{ is even}, \\ al_{m-1} + l_{m-2} & \text{if } m \text{ is odd}, \end{cases}
\]
and demonstrated its relation to \( \{q_n\}_{n=0}^{\infty} \). Generalizing the Fibonacci further, in 2011, Yayenie [9] defined the so-called “modified generalized Fibonacci sequence”:

\[
Q_0 = 0; \quad Q_1 = 1; \quad \text{and for } m \geq 2, \quad Q_m = \begin{cases} \alpha Q_{m-1} + \beta Q_{m-2} & \text{if } m \text{ is even}, \\ \gamma Q_{m-1} + \delta Q_{m-2} & \text{if } m \text{ is odd}, \end{cases}
\]

where \( \alpha, \beta, \gamma, \delta \) are non-zero real numbers. In this paper, we take the most general situation, specifying no initial conditions, thus subsuming all previous definitions.

**Definition 1.** For any real non-zero numbers \( \alpha, \beta, \gamma, \delta \), define the most generalized conditional Fibonacci sequence \( \{\ell_n\}_{n=0}^{\infty} \subset \mathbb{R} \) recursively by

\[
\ell_0; \quad \ell_1; \quad \text{and for } m \geq 2, \quad \ell_m = \begin{cases} \alpha \ell_{m-1} + \beta \ell_{m-2} & \text{if } m \text{ is even}, \\ \gamma \ell_{m-1} + \delta \ell_{m-2} & \text{if } m \text{ is odd}, \end{cases}
\]

where \( \ell_0, \ell_1 \) are any real numbers.

By varying the assignments of \( \ell_0, \ell_1 \) and \( \alpha, \beta, \gamma, \delta \), one may obtain the sequences \( \{q_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{V_n\}_{n=0}^{\infty}, \{Q_n\}_{n=0}^{\infty}, \) among others. Likewise, the classical Fibonacci and Lucas numbers are obtained when \( \alpha = \beta = \gamma = \delta = 1 \) and \( \ell_0 = 0, \ell_1 = 1 \) and \( \ell_0 = 2, \ell_1 = 1 \), respectively.

Synthesizing these definitions and their subsequent relations, we provide the generating function and generalized Binet formula for \( \{\ell_n\}_{n=0}^{\infty} \), and tie together some of the earlier formulæ and relations. In particular, while in the classical situation of Fibonacci, \((ab + c + d)^2 - 4cd = 5\), more generally, one may have \((ab + c + d)^2 - 4cd = 0\). We consider this possibility in our computations. This material constitutes Section 2. In Section 3, we first address negative subscripted terms, and then demonstrate the relationship between conditionally defined sequences independent of the initial conditions. We apply this to obtain a generalized Catalin and Cassini formula. In addition, we investigate the periodicity of conditionally defined Fibonacci sequences. It is interesting to note that if \( \ell_0 = 2, \ell_1 = 1 \), and \( \alpha = \gamma = 1 \), whereas \( \beta = -1 \), then the resulting periodic sequence shares similarities with the Koch sequence (see Example 4.7). In Section 4, we determine necessary conditions under which the general sequence \( \{\ell_n\}_{n=0}^{\infty} \) is periodic; i.e., for which \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \setminus \{0\} \) and \( \ell_0, \ell_1 \in \mathbb{R} \) will the sequence be periodic, and which periods are possible.

2. Conditionally defined Fibonacci & Lucas sequences

**Remark 2.1.** For \( n \geq 4 \), the sequence \( \{\ell_n\}_{n=0}^{\infty} \) satisfies the linear recurrence relation \( \ell_n = (ab + c + d)\ell_{n-2} - cd\ell_{n-4} \). See e.g., [2, Theorem 2.1]. The polynomial associated to this relation is \( 1 - (ab + c + d)x^2 + cdx^4 \).

**Theorem 2.2.** The generating function of the sequence in Definition 1 is

\[
L(x) = \frac{\ell_0 + \ell_1 x + (a\ell_1 - (ab + d)\ell_0)x^2 + (b\ell_0 - c\ell_1)x^3}{1 - (ab + c + d)x^2 + cdx^4}.
\]
Theorem 2.3 (Generalized Binet Formula)

\[ \beta \}

α

the classic Fibonacci setting when

Proof. Since

Using the relation noted in the remark above, this simplifies to

Therefore, \( L(x) \) is obtained from combining \( L_{\text{even}}(x) \) and \( L_{\text{odd}}(x) \).

Next, we motivate some notation for a Binet-type formula for the general term \( \ell_n \), as well as Section 4, using Remark 2.1. Set \( v_n = \left( \frac{\ell_{n+2}}{\ell_{2n+1}} \right) \) and \( w_n = \left( \frac{\ell_{2n+3}}{\ell_{2n+1}} \right) \). Then for all \( n \geq 0 \), \( v_n, w_n \) satisfy

The characteristic equation for the matrix is \( x^2 - (ab + c + d)x + cd = 0 \). Let \( \alpha, \beta \) be the roots of the polynomial \( x^2 - (ab + c - d)x - abd \), which reduces to the classic Fibonacci setting when \( a, b, c, d \) all equal 1. Then the eigenvalues arising from the characteristic equation above can be expressed as \( \alpha + d \) and \( \beta + d \).

Theorem 2.3 (Generalized Binet Formula). The \( n \)-th term of the sequence \( \{ \ell_n \}_{n=0}^\infty \) in Definition 1 (assuming \( \alpha \neq \beta \)) is given by the Binet-type formula:

\[ \ell_n = \frac{1}{\alpha^{\frac{n}{2}} - \beta^{\frac{n}{2}}} \left[ \left( \frac{ab + c + d}{\alpha} \right)^{\frac{n}{2}} \frac{\ell_0 (\alpha^n + \beta^n)}{\alpha - \beta} \right], \]

where \( \alpha^\alpha = \alpha^{\frac{1}{2}} (\alpha + d - c)^{\frac{n+1}{2}} \) and \( \beta^\alpha = \beta^{\frac{1}{2}} (\beta + d - c)^{\frac{n+1}{2}} \).

Proof. Since \( \frac{1}{\alpha} - \frac{(ab + c + d)}{\alpha^2} x^2 + x^4 = (x^2 - (\frac{ab + c + d}{\alpha})x^2) - (\beta + d), \) write

\[ L_{\text{even}}(x) = \frac{A_1 x + B_1}{x^2 - (\frac{ab + c + d}{\alpha})} + \frac{A_2 x + B_2}{x^2 - (\frac{\beta + d}{\alpha})}, \quad L_{\text{odd}}(x) = \frac{A'_1 x + B'_1}{x^2 - (\frac{ab + c + d}{\alpha})} + \frac{A'_2 x + B'_2}{x^2 - (\frac{\beta + d}{\alpha})}. \]
Since, for example,
\[ \frac{A_1 x + B_1}{x^2 - (\alpha + d) x + \alpha d} = -\sum_{n=0}^{\infty} \frac{A_1}{(\alpha + d)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{B_1}{(\alpha + d)^{n+1}} x^{2n}, \]
it follows that \( A_1, A_2, B_1', B_2' \) are zero. Furthermore,
\[ A'_1 = \frac{(-1)}{\alpha - \beta} \left[ \left( \frac{1}{d} \ell_1 - \frac{b}{d} \ell_0 \right) \alpha - b \ell_0 \right] \] and \( A'_2 = \frac{1}{\alpha - \beta} \left[ \left( \frac{1}{d} \ell_1 - \frac{b}{d} \ell_0 \right) \beta - b \ell_0 \right]. \]
Therefore,
\[ L_{\text{odd}}(x) = \frac{1}{d(\alpha - \beta)} \sum_{n=0}^{\infty} \left( \frac{[\ell_1 - b \ell_0] \alpha - b d \ell_0}{(\alpha + d)^{n+1}} - \frac{[\ell_1 - b \ell_0] \beta - b d \ell_0}{(\beta + d)^{n+1}} \right) x^{2n+1}, \]
which reduces to:
\[ \frac{-1}{ad(\alpha - \beta)} \sum_{n=0}^{\infty} \left( [(ab - \beta) \ell_0 - a \ell_1] \alpha (\beta + d)^n - [(ab - \alpha) \ell_0 - a \ell_1] \beta (\alpha + d)^n \right) x^{2n+1}. \]
Similarly,
\[ B_1 = \frac{(-1)}{\alpha - \beta} \left[ \left( \frac{ab}{cd} + \frac{1}{c} \right) \ell_0 - \frac{a}{c} \ell_1 \right] \alpha + \left( \frac{ab}{c} + \frac{d}{c} - 1 \right) \ell_0 - \frac{a}{c} \ell_1 \right] ; \] and \( B_2 = \frac{1}{\alpha - \beta} \left[ \left( \frac{ab}{cd} + \frac{1}{c} \right) \ell_0 - \frac{a}{c} \ell_1 \right] \beta + \left( \frac{ab}{c} + \frac{d}{c} - 1 \right) \ell_0 - \frac{a}{c} \ell_1 \right]. \]
Hence
\[ L_{\text{even}}(x) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left( [(ab - \beta) \ell_0 - a \ell_1] \beta (\alpha + d)^n - [(ab - \alpha) \ell_0 - a \ell_1] \beta (\alpha + d)^n \right) x^{2n}. \]
Next, since \( \alpha \beta = -abd \) and e.g., \( ab(\beta + d) = \beta(\beta + d - c) \), it follows that
\[ L_{\text{odd}}(x) = \frac{1}{d(\alpha - \beta)} \sum_{n=0}^{\infty} \frac{1}{(ab)^n} \left( [(ab - \beta) \ell_0 - a \ell_1] \beta^n (\beta + d - c)^n + 1 \right) x^{2n+1}, \] and
\[ L_{\text{even}}(x) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{1}{(ab)^n} \left( [(ab - \beta) \ell_0 - a \ell_1] \beta^n (\beta + d - c)^n \right) x^{2n}. \]
Therefore, noting that \( \xi(n) \) is the parity function,
\[ L(x) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{1}{\alpha \xi(n)(ab)^{\frac{n}{2}}} \left( [(ab - \beta) \ell_0 - a \ell_1] \beta^{\frac{n}{2}} (\beta + d - c)^{\frac{n}{2} + 1} \right) x^n. \]
(2.1) \[ - [(ab - \alpha) \ell_0 - a \ell_1] \alpha^{\frac{n}{2}} (\alpha + d - c)^{\frac{n}{2} + 1} \right] x^n. \]
Finally, since \( a^{(n)}(ab) \frac{1}{2} = a^{\frac{n+1}{2}} b^{\frac{n}{2}} \), using the substitutions \( \alpha \) and \( \beta \) and rearranging terms, we find

\[
L(x) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{1}{a^{\frac{n+1}{2}} b^{\frac{n}{2}}} \left[ ((a\ell_1 - \frac{1}{2}(ab - c + d)\ell_0)[\alpha^2 - \beta^2],
+ \sqrt{(ab + c - d)^2 + 4abcd} \ell_0(\alpha^2 + \beta^2) \right] x^n.
\]

The formula in the statement follows from \( \alpha - \beta = \sqrt{(ab + c - d)^2 + 4abcd} \). \( \square \)

We will use the above theorem extensively. However, for thoroughness, we provide here the expression for \( \ell_n \) when \( \alpha = \beta \); i.e., when \( (ab + c + d)^2 = 4cd \).

**Proposition 2.4.** If \( \alpha = \beta \), then the \( n \)-th term of the sequence \( \{\ell_n\}_{n=0}^{\infty} \) in Definition 1 is:

\[
\ell_n = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor (\alpha + d)^{\frac{n-1}{2}} [a\ell_1 - (ab - \alpha)\ell_0] + \ell_0(\alpha + d)^{\frac{n+1}{2}} & \text{if } n \text{ is even}, \\
\left\lfloor \frac{n}{2} \right\rfloor (\alpha + d)^{\frac{n-1}{2}} [b\ell_0 + (ab - \alpha)\ell_1] + \ell_1(\alpha + d)^{\frac{n+1}{2}} & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** Using methods similar to those above, we find that

\[
L_{\text{odd}}(x) = \frac{\frac{b}{a} \ell_0 - \frac{1}{d} \ell_1}{x^2 - (\frac{\alpha + d}{cd})^2} + \frac{1}{\alpha + d} \ell_1 + \left( \frac{\alpha + d}{cd} \right) \left[ \frac{\beta}{\alpha - \beta} \ell_0 - \frac{1}{d} \ell_1 \right],
\]

and

\[
L_{\text{even}}(x) = \frac{\frac{b}{cd} \ell_1 - (ab + c + d) \ell_0}{x^2 - (\frac{\alpha + d}{cd})^2} + \frac{1}{\alpha + d} \ell_0 + \left( \frac{\alpha + d}{cd} \right) \left[ \frac{b}{\alpha - \beta} \ell_1 - (ab + c + d) \ell_0 \right].
\]

By the Maclaurin series expansion in the previous proof, as well as the expansion

\[
\frac{Ax + B}{(x^2 - (\frac{\alpha + d}{cd})^2)^2} = \sum_{n=0}^{\infty} \frac{(n + 1)A}{(\alpha + d)^{n+1}} x^{2n+1} + \sum_{n=0}^{\infty} \frac{(n + 1)B}{(\alpha + d)^n} x^{2n},
\]

we have

\[
L_{\text{odd}}(x) = \sum_{n=0}^{\infty} \left( (n + 1)\ell_1 (\alpha + d)^n + n(\alpha + d)^{n+1} \left[ \frac{b}{d} \ell_0 - \frac{1}{d} \ell_1 \right] \right) x^{2n+1}
\]

and

\[
L_{\text{even}}(x) = \sum_{n=0}^{\infty} \left( (n + 1)\ell_0 (\alpha + d)^n + n(\alpha + d)^{n+1} \left[ \frac{b}{cd} \ell_1 - (ab + c + d) \ell_0 \right] \right) x^{2n}.
\]

To obtain the expressions for \( \ell_n \), we apply \( (\alpha + d)^2 = cd \) and simplify. \( \square \)

Because of Theorem 2.3, the \( n \)-th term of any conditionally defined Fibonacci sequence can be defined by a multivariate function.

**Definition 2.** Let \( f(n; a, b, c, d; \ell_0, \ell_1) := \ell_n \) be the multivariate function with initial conditions \( \ell_0, \ell_1 \in \mathbb{R} \), and \( a, b, c, d \in \mathbb{R} \setminus \{0\} \), with \( \alpha \) and \( \beta \) the roots of the equation \( x^2 - (ab + c + d)x - abd = 0 \). For \( n \geq 2 \), \( \ell_n \) is defined conditionally by \( \ell_n = a\ell_{n-1} + c\ell_{n-2} \) if \( n \) is even, and \( \ell_n = b\ell_{n-1} + d\ell_{n-2} \) if \( n \) is odd.
Therefore, we immediately recover results [3, Theorems 1, 2], [9, Theorems 7, 8], [1, Theorems 1, 2], and [10, Theorems 2.5, 2.6], among others, all of which were derived individually, as we need only evaluate \( f \) with the appropriate inputs.

**Corollary 2.5.** For the generalized Fibonacci definitions \( q_n, Q_n \) and the generalized Lucas definitions \( l_n, V_n \) described above, and for all \( n \geq 2 \):

1. \( q_n = f(n; a, b, c, 0, 0) \) and \( Q_n = f(n; a, b, c, d, 0, 1) \);
2. \( l_n = f(n; a, b, c, 1, 1) \) and \( V_n = f(n; a, b, 1, 1; 2, ab) \).

### 3. Generalized relations on the sequences

The function \( f \) defined in the previous section may be exploited to obtain relations and identities on the various sequences. Throughout this section, \( \alpha, \beta \) are as in Theorem 2.3. We begin by considering \( \ell_{-n} \).

#### 3.1. Negative subscripted terms

The initial conditions are \( \ell_1, \ell_0 \). Since \( \ell_1 = b\ell_0 + d\ell_{-1} \), we have \( \ell_{-1} = -\frac{b}{2}\ell_0 + \frac{1}{2}\ell_1 \), and similarly, \( \ell_{-2} = -\frac{b}{2}\ell_{-1} + \frac{1}{2}\ell_0 \). Therefore, the sequence \( \ell_1, \ell_0, \ell_{-1}, \ell_{-2}, \ldots \) is given by \( \{\ell_n\}_{n=0}^{\infty} \), where \( \ell^*_0 = \ell_1 \); \( \ell^*_1 = \ell_0 \); and for \( m \geq 1 \),

\[
\ell_{m+1}^* = \ell_{m-1} = \begin{cases} \frac{-b}{2}\ell_{-(m-1)} + \frac{1}{2}\ell_{-(m-2)} & \text{if } m \text{ is even,} \\ \frac{-b}{2}\ell_{-(m-1)} + \frac{1}{2}\ell_{-(m-2)} & \text{if } m \text{ is odd.} \end{cases}
\]

This means that the \(-n\)-th term of the sequence \( \ell_1, \ell_0, \ell_{-1}, \ell_{-2}, \ldots \) is given by \( f(n+1; -\frac{b}{2}, -\frac{a}{c}, 1, \ell_1, \ell_0) \). A Binet-like formula is provided below.

**Theorem 3.1.** For \( n \geq -1 \),

\[
f(n+1; -\frac{b}{2}, \frac{a}{c}, 1, \ell_1, \ell_0) = \ell_{-n} = (-1)^{n+1} \frac{(-bc\ell_0 - \frac{1}{2}(ab - c + d)\ell_1)(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} + \frac{1}{2} \ell_1 (\alpha^{n+1} + \beta^{n+1}) \frac{(d/\gamma)^{(n+2)/2}}{\alpha^{(n+2)/2}d^{(n+2)/2}} \left[ \frac{[(bc\ell_0 - \frac{1}{2}(ab - c + d)\ell_1)][(\alpha^{n+1} - \beta^{n+1})]}{cd(\alpha - \beta)} \right].
\]

**Proof.** The formula follows from Theorem 2.3, noting that the roles of \( \ell_0 \) and \( \ell_1 \) are interchanged. Let \( \alpha, \beta \) be the roots of the polynomial \( x^2 - (\frac{bc}{2} + \frac{1}{2})x - \frac{ab}{2} \), and \( (\alpha)^m = (\tilde{\alpha})^{1/2}(\tilde{\alpha} + \frac{1}{2} - 1/2) \) and \( (\beta)^m = (\tilde{\beta})^{1/2}(\tilde{\beta} + \frac{1}{2} - 1/2) \). Then

\[
\ell_{-n} = (-1)^{n+1} \left( \frac{d}{\gamma} \right)^{(n+2)/2} \left( \frac{\alpha}{\alpha} \right)^{(n+2)/2} \left[ \frac{[(bc\ell_0 - \frac{1}{2}(ab - c + d)\ell_1)][(\alpha^{n+1} - \beta^{n+1})]}{cd(\alpha - \beta)} \right] + \frac{1}{2} \ell_1 (\alpha^{n+1} + \beta^{n+1}).
\]

(3.1)
Now \( \alpha = \frac{1}{cd}\alpha \) and \( \beta = \frac{1}{cd}\beta \), hence, \( (\alpha l)^n = \frac{1}{(cd)^n}\alpha^\frac{n+1}{2}((\alpha + d - c)^{\frac{n+1}{2}}) \). Likewise, for \( (\beta l)^n \). Making these substitutions and simplifying \( c\frac{n+1}{2}(d)\frac{n+1}{2} \) to \( c\frac{n+1}{2}(d)\frac{n+1}{2} \), the result follows.

Theorem 3.1 may be used to provide the expression for negative subscripted terms for sequences discussed in [10]. Moreover, we may recover results for \( q_{-n} \) and \( L_{-n} \) [1, Corollary 1].

**Corollary 3.2.** When \( c = d = 1 \), for \( n \geq -1 \),

\[
L_{-n} = f\left(n + 1; -\frac{b}{d}, -\frac{a}{c}, 1, 1; \ell_1, \ell_0\right) = \frac{(-1)^{n+1}}{a\frac{n+1}{2}(b)\frac{n+1}{2}}\left[\frac{\left|-b\ell_0 - \frac{1}{2}ab\ell_1\right|\left((\alpha^{n+1} + \beta^{n+1})\right)}{\alpha - \beta}\right] + \frac{1}{2}\frac{1}{f_1(1 + 1)}\frac{1}{2}\frac{1}{2}f_1\left(\alpha^{n+1} + \beta^{n+1}\right).
\]

Next, we consider how terms of the sequence interact with one another. For example, the relation \( L_n = q_{n-1} + q_{n+1} \) in [1, Theorem 3], i.e., \( f(n; b, a, 1; 1; 2; a) = f(n - 1; a, b, 1; 0, 1) + f(n + 1; a, b, 1; 0, 1) \), generalizes the classic fact that \( L_n = F_{n-1} + F_{n+1} \). Likewise, using \( Q_n \) and \( U_n \) in [1, Theorem 20], we have:

\[f\left(n; b, a, d, c; \frac{d+1}{d}, a\right) = f(n - 1; a, b, c, d; 0, 1) + f(n + 1; a, b, c, d; 0, 1).\]

The result below expresses these relations in full generality, no matter the initial values \( \ell_0, \ell_1 \).

**Theorem 3.3.** For \( n \geq 0 \), the following holds:

\[f\left(n; b, a, d, c; \left(\frac{d+1}{d}\right)\ell_1 - \left(\frac{b}{d}\right)\ell_0, a\ell_1 + (c + 1)\ell_0\right) = \ell_{n-1} + \ell_{n+1}.
\]

**Proof.** Set \( L_n = f(n; b, a, d, c; \left(\frac{d+1}{d}\right)\ell_1 - \left(\frac{b}{d}\right)\ell_0, a\ell_1 + (c + 1)\ell_0) \); i.e., \( L_0 = \frac{d+1}{d}\ell_1 - \left(\frac{b}{d}\right)\ell_0 \) and \( L_1 = a\ell_1 + (c + 1)\ell_0 \); then the statement above is \( L_n = \ell_{n-1} + \ell_{n+1} \).

As per Definition 2, let \( A \) and \( B \) be the roots of the equation \( x^2 - (ab - c + d)x - abc = 0 \), where for \( n \geq 2 \), \( L_n \) is defined conditionally by \( L_n = bL_{n-1} + cL_{n-2} \) if \( n \) is even, and \( L_n = aL_{n-1} + cL_{n-2} \) if \( n \) is odd. The following identities hold:

1. \( A = \alpha - c + d \) and \( B = \beta - c + d \);
2. \( A = ab - \alpha \) and \( B = ab - \beta \);
3. \( ab(A + c) = A(A + c - d) \) and \( ab(B + c) = B(B + c - d) \).

Using the expression for a general term of \( \ell_n \) in the equation (2.1), we have

\[
\ell_{n+1} - \ell_{n+1} = \frac{1}{a\ell(n-1)(\alpha l)^{\frac{n+1}{2}}(\alpha - \beta)}\left(a\ell_1 - (ab - \alpha)\ell_0\right)(\alpha^{\frac{n+1}{2}}(\alpha + d - c)^{\frac{n+1}{2}})(1 + \alpha + d)
\]
\[ -[a\ell_1 - (ab - \beta)\ell_0]b^{\frac{c + 1}{c}}(b + d - c)^{\frac{c + 1}{2}(1 + \beta + d)} \]

\[ = \frac{b}{b^{(n)}(ab)^{\frac{c}{2}}(A - B)} \left( [a\ell_1 - (ab - \alpha)\ell_0](A - d + c)^{\frac{c + 1}{2}}(1 + \alpha + d) \right. \]

\[ - [a\ell_1 - (ab - \beta)\ell_0](B - d + c)^{\frac{c + 1}{2}}(1 + \beta + d) \]

\[ = \frac{1}{b^{(n)}(ab)^{\frac{c}{2}}(A - B)} \left( [ab\ell_1 - (ab^2 - ba)\ell_0](A - d + c)^{\frac{c + 1}{2}} \right. \]

\[ - [ab\ell_1 - (ab - \beta)\ell_0](B - d + c)^{\frac{c + 1}{2}} \left. \left( \frac{1 + \alpha + d}{\alpha} \right) \right) \]

\[ = \frac{1}{b^{(n)}(ab)^{\frac{c}{2}}(A - B)} \left( [bL_1 - (ab - A)L_0](A - d + c)^{\frac{c + 1}{2}} \right. \]

\[ - [bL_1 - (ab - B)L_0](B - d + c)^{\frac{c + 1}{2}} \left. \right) = L_n. \]

Reversing the operation in Theorem 2.3, we can split a general term \( \ell_n \) into the sum of two other terms:

**Corollary 3.4.** For \( \ell_n = f(n; a, b, c, d; \ell_0, \ell_1) \) and \( n \geq 0 \):

\[ \ell_n = f(n - 1; b, a, d, c; \frac{(c + 1)\ell_1 - b\ell_0}{ab + (c + 1)(d + 1)}, \frac{a\ell_1 + c(d + 1)\ell_0}{ab + (c + 1)(d + 1)}) \]

\[ + f(n + 1; b, a, d, c; \frac{(c + 1)\ell_1 - b\ell_0}{ab + (c + 1)(d + 1)}, \frac{a\ell_1 + c(d + 1)\ell_0}{ab + (c + 1)(d + 1)}) \].

**Proof.** The solution to the system

\[ \left( \begin{array}{c} c + 1 \\ \beta \end{array} \right) y - a \cdot x = \ell_0 \quad \text{and} \quad by + (d + 1)x = \ell_1 \]

is \( x = \frac{(c + 1)\ell_1 - b\ell_0}{ab + (c + 1)(d + 1)} \) and \( y = \frac{a\ell_1 + c(d + 1)\ell_0}{ab + (c + 1)(d + 1)} \). Apply Theorem 3.3. \hfill \Box

Finally, we may exploit \( \{Q_n\}_{n=0}^\infty \) as in [9] to obtain further results, including a generalized Cassini and Catalin relation.

**Proposition 3.5.** For all \( n \geq 0 \), with \( \{\ell_n\}_{n=0}^\infty \) as in Definition 1,

\[ \ell_n = (\ell_1 - \frac{1}{2}b\ell_0)Q_n + \frac{1}{2}\ell_0 \cdot \left( \frac{b}{a} \right)^{\ell(n)} \cdot f(n; b, a, d, c; 2, a). \]

**Proof.** Recall that \( Q_n = f(n; a, b, c, d; 0, 1) \) (see page 1), hence,

\[ Q_n = \frac{1}{a^{(n)}(b)^{\frac{c}{2}}(b)^{\frac{d}{2}}} \cdot \frac{(a^2 - b^2)}{(a - b)} = \frac{a}{a^{(n)}(b)^{\frac{c}{2}}(b)^{\frac{d}{2}}} \cdot \frac{(a^2 - b^2)}{(a - b)}. \]

Note that \( \ell_n = f(n; a, b, c, d; \ell_0, \ell_1) \), where \( \ell_0, \ell_1 \) are any real numbers. Since

\[ f(n; b, a, d, c; 2, a) = \frac{1}{a^{(n)}(b)^{\frac{c}{2}}(b)^{\frac{d}{2}}} \cdot \left[ \frac{[(d - c)](a^2 - b^2)}{(a - b)} + (a^2 + b^2) \right], \]

the result follows by the expression for \( \ell_n \) in Theorem 2.3. \hfill \Box
Corollary 3.6 (General Cassini & Catalin). Suppose that \( c = d \). Then \( \alpha, \beta = \frac{1}{2}(ab \pm \sqrt{(ab)^2 + 4abcd}) \) and the sequence \( \{\ell_n\}_{n=0}^{\infty} \) satisfies:

(i) \[
\left(\frac{a}{b}\right) \xi(n+1) \left(\ell_{n-1} + \frac{1}{2} \ell_0 \frac{a^n + \beta^n}{\alpha^{n+1} + \beta^{n+1}}\right) \left(\ell_{n+1} + \frac{1}{2} \ell_0 \frac{a^n + \beta^n}{\alpha^{n+1} + \beta^{n+1}}\right) = \left(\frac{a}{b}\right) \xi(n) \left(\ell_n + \frac{1}{2} \ell_0 \frac{a^n + \beta^n}{\alpha^{n+1} + \beta^{n+1}}\right)^2 - \frac{a}{b} \left(\ell_1 - \frac{1}{2} \ell_0 \right)^2 (-d)^{n-1}.
\]

(ii) \[
\left(\frac{a}{b}\right) \xi(n) \xi(r) \left(\ell_{n-r} + \frac{1}{2} \ell_0 \frac{a^n + \beta^n}{\alpha^{n+1} + \beta^{n+1}}\right) = \left(\frac{a}{b}\right) \xi(n) \xi(r) \left(\ell_n + \frac{1}{2} \ell_0 \frac{a^n + \beta^n}{\alpha^{n+1} + \beta^{n+1}}\right)^2 - \left(\ell_1 - \frac{1}{2} \ell_0 \right)^2 (-1)^{n-r} a^{n-r} \left(\ell_r + \frac{1}{2} \ell_0 \frac{a^r + \beta^r}{\alpha^{r+1} + \beta^{r+1}}\right)^2.
\]

Proof. Since \( c = d \), one has \( \alpha^n = \alpha \) and \( \beta^n = \beta \), resulting in a simplified expression for \( f(n; b, a, d, c; 2, a) \) in the equation (3.2). As per Proposition 3.5, \( (\ell_1 - \frac{1}{2} \ell_0) P_n = \ell_n - \frac{1}{2} \ell_0 \cdot \frac{1}{a^{n+1} + \beta^{n+1}}(\alpha^n + \beta^n) \). Apply [9, Theorem 9]. \( \square \)

Since Theorem 2.3 expresses conditionally defined Fibonacci and Lucas sequences in full generality, it is appropriate to ask what other identities exist between terms of various definitions of \( \{\ell_n\} \), as in [1], [9], [10], not only for the aforementioned sequences with specific initial conditions, but by varying the initial condition \( \ell_0, \ell_1 \) and conditional coefficients \( a, b, c, d \in \mathbb{R} \setminus \{0\} \).

4. Periodicity of conditionally defined Fibonacci sequences

In [9] it was noted that when \( c = 1 - a \) and \( d = 1 - b \), then the sequence \( \{Q_n\}_{n=0}^{\infty} \) is eventually constant. For the general sequence \( \{\ell_n\}_{n=0}^{\infty} \), we focus on periodic behavior. Recalling the vectors \( v_n, w_n \) from Section 2, if \( \{\ell_n\}_{n=0}^{\infty} \) is \( 2p \)-periodic, with \( p \in \mathbb{N} \), then

\[
v_n = \left(\begin{array}{c} ab + c + d \\ 1 \end{array}\right)^p v_n \quad \text{and} \quad w_n = \left(\begin{array}{c} ab + c + d \\ 1 \end{array}\right)^p w_n \quad \text{for all} \quad n \geq 0.
\]

Thus, \( v_n, w_n \) are eigenvectors associated to \( p \)-powers of the matrix arising from the relation \( \ell_{k+1} = (ab + c + d) \ell_k + c d \ell_k \) for \( n \geq 4 \). In particular, with \( \alpha, \beta \) as in Sections 2, 3, the eigenvalues must satisfy \( (\alpha + d)^p = 1 \) and \( (\beta + d)^p = 1 \). However, the relation from which this analysis stems, while necessary, is not sufficient. The fact that there are only two initial conditions in Definition 1 means that sequences satisfying the relation are a subset of the family of sequences \( \{\ell_n\}_{n=0}^{\infty} \).
In general, if \( \ell_n = \ell_{n+m} \) for all \( n \geq 0 \) for a fixed \( m \geq 1 \), then using the expression in the equation (2.1),

\[
(4.1) \quad \frac{1}{a^l + \lfloor b/m \rfloor} \left[ (a\ell_1 - (ab - \alpha)\ell_0)\mathbf{a}^n - [a\ell_1 - (ab - \beta)\ell_0]\mathbf{b}^n \right] = \frac{1}{a^\ell + \lfloor b/m \rfloor} \left[ (a\ell_1 - (ab - \alpha)\ell_0)\mathbf{a}^{n+m} - [a\ell_1 - (ab - \beta)\ell_0]\mathbf{b}^{n+m} \right].
\]

Set

\[
X_{n,m} = \frac{1}{a^\ell + \lfloor b/m \rfloor} = \frac{1}{(ab)^\ell \prod_{j=1}^{\lfloor \frac{m}{2} \rfloor} b^{\ell_j}} (\alpha + d - c)^{\ell(n+1)};
\]

Since \( \mathbf{a}^{n+m} = \mathbf{a}^n \mathbf{a}^m \), the equation (4.1) simplifies to

\[
(4.2) \quad \mathbf{a}^\ell (a\ell_1 - (ab - \alpha)\ell_0)(1 - X_{n,m} \mathbf{a})^{\frac{\ell}{2}} \alpha \mathbf{a}^{\ell(n)}(\alpha + d - c)^{\ell(n+1)}
\]

\[
- \mathbf{b}^\ell (a\ell_1 - (ab - \beta)\ell_0)(1 - X_{n,m} \mathbf{b})^{\frac{\ell}{2}} \beta \mathbf{b}^{\ell(n)}(\beta + d - c)^{\ell(n+1)} = 0.
\]

Likewise, \( \ell_{n+2} = \ell_{n+2+m} \), and

\[
(4.3) \quad \mathbf{a}^{n+2} (a\ell_1 - (ab - \alpha)\ell_0)(1 - X_{n,m} \mathbf{a})^{\frac{\ell}{2}} \alpha \mathbf{a}^{\ell(n)}(\alpha + d - c)^{\ell(n+1)}
\]

\[
- \mathbf{b}^{n+2} (a\ell_1 - (ab - \beta)\ell_0)(1 - X_{n,m} \mathbf{b})^{\frac{\ell}{2}} \beta \mathbf{b}^{\ell(n)}(\beta + d - c)^{\ell(n+1)} = 0.
\]

Next, \( \mathbf{b}^{n+2} = \beta(\beta + d - c)\mathbf{b}^n \), hence multiplying the equation (4.2) by \(-\beta(\beta + d - c)\) and adding to the equation (4.3), we get

\[
ab(\alpha - \beta)\mathbf{a}^\ell (a\ell_1 - (ab - \alpha)\ell_0)(1 - X_{n,m} \mathbf{a})^{\frac{\ell}{2}} \alpha \mathbf{a}^{\ell(n)}(\alpha + d - c)^{\ell(n+1)} = 0.
\]

Since the working assumption is that \( \alpha \neq \beta \), if \( a\ell_1 - (ab - \alpha)\ell_0 \neq 0 \), then it must hold that \( 1 - X_{n,m} \mathbf{a}^{\frac{\ell}{2}} \alpha \mathbf{a}^{\ell(n+1)} = 0 \); i.e.,

\[
\mathbf{a}^{\frac{\ell}{2}} \alpha \mathbf{a}^{\ell(n+1)} = (ab)^\ell \alpha \mathbf{a}^{\ell(n+1)} \beta \mathbf{b}^{\ell(n)}.
\]

Moreover, the same holds true for the \( \beta \) term. Since \( \alpha(\alpha + d - c) = ab(\alpha + d) \) and \( \beta(\beta + d - c) = ab(\beta + d) \), the equation above produces the system (s) below, depending upon whether \( m \) is even or odd. Thus, we have established the following:

**Theorem 4.1.** If the sequence in Definition 1 is periodic of period \( m \geq 1 \), with \( \ell_0, \ell_1 \) not both zero, and \( \alpha \neq \beta \), then one of each pair of conditions must hold:

(1A) either \( a\ell_1 - (ab - \alpha)\ell_0 = 0 \) or

(2A) \( (\alpha + d)^\frac{\ell}{2} \alpha \mathbf{a}^{\ell(n+1)}(\alpha + d - c)\mathbf{a}^{\ell(n+1)} = a\mathbf{a}^{\ell(n+1)} \beta \mathbf{b}^{\ell(n)} \) for all \( n \);

and

(1B) either \( a\ell_1 - (ab - \beta)\ell_0 = 0 \) or

(2B) \( (\beta + d)^\frac{\ell}{2} \beta \mathbf{b}^{\ell(n+1)}(\alpha + d - c)\mathbf{b}^{\ell(n+1)} = a\mathbf{a}^{\ell(n+1)} \beta \mathbf{b}^{\ell(n)} \) for all \( n \).

In particular, under the assumption that neither \( a\ell_1 - (ab - \alpha)\ell_0 \) nor \( a\ell_1 - (ab - \beta)\ell_0 \) is zero, the following must hold:

(3E) If \( m \) is even, then \( (\alpha + d)^\frac{\ell}{2} = 1 \) and \( (\beta + d)^\frac{\ell}{2} = 1 \).
Remark

Lemma 4.3. If $m$ is odd, then:

\[
\begin{cases}
(a+d)^{n+1} + (a-d-n) = a & \text{for even } n, \text{ and;}
\end{cases}
\]

\[
(b + d) = \alpha
\]

for odd $n$.

Remark 4.2. From Case (3.O) in Theorem 4.1 one can deduce that $a = b$ if and only if $c = d$. In fact, this result holds more generally and simply when $m \geq 3$.

Lemma 4.3. For $p \geq 1$, if $\{\ell_n\}_{n=0}^{\infty}$ is periodic of period $2p + 1$, then $a = b$ if and only if $c = d$. In particular, if exactly one of $\ell_0, \ell_1$ is zero, then $a = b$ and $c = d$.

Proof. Fix $p \geq 1$ and suppose $\ell_{2p+1} = \ell_0$. Then $\ell_{2p+2} = \ell_1$, and $\ell_{2p+3} := b + d + d + d = \ell_1 + d + \ell_0$. On the other hand, $\ell_{2p+3} = \ell_2 := a + c + \ell_0$; hence, (i) $(a-b) = (d-c)\ell_0$, establishing the first statement.

The result below provides the input for the function $f$ to obtain $p$-periodic sequences, for small $p$. For example, when $a, b, c, d \in \mathbb{R} \setminus \{0\}$ are chosen such that $1 - (ab + c + d)x^2 + cdx^4 = \Phi_12(x)$, the twelfth cyclotomic polynomial, then twelve is the shortest period of the solutions of the recurrence relation $\ell_{n+k} - \ell_k = 0$ (see e.g., [4, Theorem 1] or [7]).

Theorem 4.4. There exist $a, b, c, d \in \mathbb{R} \setminus \{0\}$ such that the shortest period of the sequence $\{\ell_n\}_{n=0}^{\infty}$ is $p$, for $p \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$. In particular, with $\ell_0, \ell_1 \in \mathbb{R}$, excluding $\ell_0 = \ell_1 = 0$, and assuming all $a, b, c, d \in \mathbb{R} \setminus \{0\}$:

\begin{itemize}
  \item [p = 1] if and only if $\ell_n = f(n; a, b, 1 - a - b; \ell_0, \ell_1)$;
  \item [p = 2] if and only if $\ell_n = f(n; a, b, 1 - a + b; \ell_0, \ell_1)$ for any distinct non-zero $\ell_0, \ell_1$;
  \item [p = 3] if $\ell_n = f(n; -\ell_0 + \ell_1, \ell_0 + \ell_1, c, \ell_0 + \ell_1; \ell_0, \ell_1)$, where $\ell_1 \neq -\frac{1}{c} \ell_0$;
  \item [p = 4] if $\ell_n$ satisfies the following conditions:
    \begin{enumerate}
      \item[(a)] $\ell_n = f(n; a, b, -\frac{-ab \sqrt{(ab)^2 + 4} - \frac{-ab + \sqrt{(ab)^2 + 4}}{2}}{2}; \ell_0, \ell_1)$;
      \item[(b)] if neither $\ell_0$ nor $\ell_1$ is zero, and $\ell_n = f(n; a, b, 1 - a \ell_0, b \ell_0 - 1; \ell_0, \ell_1)$, or $\ell_n = f(n; 2 \ell_0, b, 1 - a \ell_0 - 1; \ell_0, \ell_1)$;
    \end{enumerate}
  \item [p = 5] if $\ell_n = f(n; 0, 0, -\frac{1}{c} \ell_0, -\frac{1}{c} \ell_1, -1, 0; \ell_0, \ell_1)$;
  \item [p = 6] if $\ell_n = f(n; a, b, \ell_0, \ell_1, 0; \ell_0, \ell_1)$, with $c = -\frac{-1 + \sqrt{q^2 + 4q - 3}}{2}$ for $q \in (-\infty, -3]$ \cup \{1, \infty\};
  \item [p = 8] implies that $d = \frac{1}{c}$ and $a, b, c$ satisfy the equation $ab + c + \frac{1}{c} = 0$.
\end{itemize}

In particular, there exist $a, b, c$, with $c = -\frac{ab + \sqrt{a^2 + 1}}{2}$ such that the shortest period of the sequence is period eight.

\footnote{If $\ell_0 = \ell_1 = 0$, then the zero sequence $\ell_n = f(n; a, b, c; d; 0,0)$ is also constant.}
implies that \( d = \frac{1}{2} \) and \( a, b, c \) satisfy the equation \( ab + c + \frac{1}{c} = -\frac{1+\sqrt{5}}{2} \). In particular, there exist \( a, b, c \) such that the shortest period of the sequence is period ten;

\[ \text{if } \ell_0 = f(n; a, \frac{2}{3}, c, \frac{1}{c}; \ell_0, \ell_1), \text{ with } c := \frac{1-q+\sqrt{q^2-2q-3}}{2} \text{ for } q \in (-\infty, -1) \cup [3, \infty). \]

Proof of Theorem 4.4. Cases \([p = 1, 2]\). The constant sequence is obvious. Next, if either of the initial conditions are zero, then it is easy to see that \( a \) or \( b \) must be zero. Contradiction. Thus, \( \ell_0, \ell_1 \) must be non-zero and distinct.

Case \([p = 3]\). From the definition of the elements \( \ell_2, \ell_3, \ell_4, \ell_5, \) one obtains

\[
\begin{align*}
(ab + d)\ell_1 &= (1 - be)\ell_0, \\
(1 - ac)\ell_1 &= (a + c^2)\ell_0, \\
(a - b)\ell_1 &= (d - c)\ell_0.
\end{align*}
\]

If one of \( \ell_0, \ell_1 \) is zero, then \( a = b \) and \( c = d \) by Lemma 4.3. It follows that \( a = b = c = d = -1 \), and the sequence is either \( 0, \ell_1, -\ell_1, 0, \ell_1, \ldots \) or \( \ell_0, 0, -\ell_0, 0, 0, \ldots \). Similarly, if \( \ell_0, \ell_1 \) are both non-zero and \( a = b \) or \( c = d \), then \( a = b = c = d = -1 \), and the sequence is either \( \ell_0, -\ell_0, 0, \ell_0, -\ell_0, 0, \ell_0, \ell_1, \ldots \). Thus, let \( a \neq b \) and \( \ell_0, \ell_1 \) non-zero. We solve the resulting system in terms of \( c = d + (b - a)\frac{\ell_2}{\ell_0} \), obtaining the sequence

\[ \ell_0, \ell_1, \frac{\ell^2 + c^2}{c^2 + d^2}, \ell_0, \ell_1, \ldots \]

Case \([p = 4]\). Suppose \( \ell_0 = 0 \) and \( \ell_1 \neq 0 \). Then \( 0 = \ell_4 = a(ab + d)\ell_1 + ac\ell_1 \). Since \( a \neq 0 \), this implies that \( ab + c + d = 0 \), hence, \( d = -ab - c \). Thus, \( \ell_3 = -c\ell_1 \). Likewise, \( \ell_1 = \ell_5 \) implies \( c^2 + abc - 1 = 0 \). Thus, \( c = \frac{-ab\pm\sqrt{(ab)^2+4}}{2} \), and hence, \( d = -ab - c = 7 \). It follows that \( \ell_6 = a\ell_1 = \ell_2 \) and \( \ell_7 = -c\ell_1 = \ell_3 \).

More generally, let \( \ell_0, \ell_1 \) be any real numbers as long as both are not zero. With \( a, b, c \) defined as in the previous paragraph, the sequence is \( \ell_0, \ell_1, a\ell_1 + c\ell_0, -c\ell_1 + b\ell_0, \ell_0, \ell_1, \ldots \). Under the conditions of (b), the sequence is \( \ell_0, \ell_1, -\ell_0, -\ell_1, \ell_0, \ell_1, \ldots \)

Case \([p = 5]\). Suppose \( \ell_0 = 0 \) and \( \ell_1 \neq 0 \). Then \( 0 = \ell_5 = (a^2b^2 + 2abd + abc + d^2)\ell_1 \), so by Lemma 4.3, \( a^4 + 3a^2c + c^2 = 0 \). Also, \( \ell_1 = \ell_6 = (a^2b^2 + 2a^3bd + 2a^2bc + acd + a^2 + ad^2)\ell_1 \), hence \( 1 = a(a^2 + 4a^2c + 3c^2) = a^3 + 2ac^2 \). It follows that \( c = \frac{a^3 + 1}{2a} \), hence, \( a^3 + 11c^2 = 1 = 0 \). The real roots are \( a = \frac{-1\pm\sqrt{5}}{2} \), and thus, \( c = -1 \).

Let \( \ell_0, \ell_1 \) be any real numbers as long as both are not zero and \( a = b = c = d \) as described above. The sequences below display the periodic behavior.

\[ \ell_0, \ell_1, \left( \frac{-1 \pm \sqrt{5}}{2} \right) \ell_1, -\ell_0, \left( \frac{1 \mp \sqrt{5}}{2} \right) (\ell_1 + \ell_0), -\ell_1 + \left( \frac{-1 \pm \sqrt{5}}{2} \right) \ell_0, \ell_0, \ell_1, \ldots \]

Case \([p = 6]\). Suppose \( \ell_0 = 0 \) and \( \ell_1 \neq 0 \). Then \( 0 = \ell_6 = (a^3b^2 + 2a^2bd + 2a^2bc + acd + a^2 + ad^2)\ell_1 \), and hence, \( a^2b^2 + 2abd + 2abc + cd + c^2 + d^2 = 0 \). This can be rewritten as \( (ab + c + d)^2 - cd = 0 \). Next, \( \ell_1 = \ell_7 = (a^2b^2d + 2abd^2 + a^2bd + acd + a^2 + ad^2)\ell_1 \).
abcd + d^3)\ell_1$, hence, \(cd(ab + c + d) = -1\). Thus, \(ab + c + d = -\frac{1}{c}\). Solving the system, \(cd = 1\) is the only real solution. Thus, \(d = \frac{1}{c}\), and moreover, \(ab + c + \frac{1}{c} = -1\), hence, \(c = \frac{-ab - 1 + \sqrt{(ab + 1)^2 - 4}}{2}\). For \(c \in \mathbb{R}\), necessarily \(ab \geq 1\) or \(ab \leq -3\). Set \(q = ab\)

In general, for \(\ell_0, \ell_1\) not both zero and \(a, q, c\) as above, the sequence

\[
\ell_0, \ell_1, -(c + 1)\ell_1 + \frac{q}{a}c\ell_0, -a\ell_1 - (c + 1)\ell_0, c\ell_1 - \frac{q}{a}c\ell_0, \ell_0, \ell_1, \ldots,
\]
is clearly periodic with shortest period six.

Case \([p = 8]\). Suppose \(\ell_0 = 0\) and \(\ell_1 \neq 0\). Then \(0 = \ell_0 = (a^3b^3 + 3a^3b^2d + 3a^3b^2c + 4a^2bcd + 3a^2b^2c^2 + 3a^2bc^2d + ac^3 + acd^2)\ell_1\), from which we obtain \((ab + c + d)[(ab + c + d)^2 - 2cd] = 0\). Thus, either \(ab + c + d = 0\) or \((ab + c + d)^2 = 2cd\).

Next, using the fact that \(\ell_1 = \ell_0\), it follows that \(cd[(ab + c + d)^2 - cd] = -1\). If \((ab + c + d)^2 = 2cd\), then \(cd(2cd) = -1\), resulting in no real solution. On the other hand, if \(ab + c + d = 0\), then \((cd)^2 = 1\). Consequently, \(d = \pm \frac{1}{\sqrt{2}}\).

In fact, when \(\ell_0, \ell_1\) are not both zero and \(a, b, c \in \mathbb{R}\) \(\{0\}\) satisfy the equation \(ab + c + \frac{1}{c} = 0\), the shortest period of the sequence below is eight.

\[
\ell_0, \ell_1, a\ell_1 + c\ell_0, -c\ell_1 + b\ell_0, -\ell_0, -a\ell_1 - \ell_0, c\ell_1 - b\ell_0, \ell_0, \ell_1, \ldots
\]

Case \([p = 10]\). Assuming \(0 = \ell_{10}\) and \(\ell_1 = \ell_{11}\), one obtains the system

\[
(cd)^2 - 3cd(ab + c + d)^2 + (ab + c + d)^4 = 0,
\]

\[
cd(ab + c + d)[(ab + c + d)^2 - 2cd] = -1.
\]

The real solutions are \(cd = 1\) and \(ab + c + d = -\frac{1 + \sqrt{5}}{2}\). However, when \(c = d = -1\) and \(a = b = -\frac{1 + \sqrt{5}}{2}\), then the resulting sequence is period five, as per the previous case. However, other choices will result in a sequence non-trivially of period ten: let \(c = d = 1\), \(a = -5 + \sqrt{5}\), and \(b = \frac{1}{2}\). As long as \(\ell_0, \ell_1\) are not both zero, the sequence below displays the periodic behavior

\[
\ell_0, \ell_1, (-5 + \sqrt{5})\ell_1 + \ell_0, \left(-\frac{3 + \sqrt{5}}{2}\right)\ell_1 + \frac{1}{2}\ell_0,
\]

\[
(5 - 3\sqrt{5})\ell_1 + \left(-\frac{3 + \sqrt{5}}{2}\right)\ell_0, (1 - \sqrt{5})\ell_1 + \left(-\frac{1 + \sqrt{5}}{4}\right)\ell_0,
\]

\[
(-5 + 3\sqrt{5})\ell_1 + (1 - \sqrt{5})\ell_0, \left(-\frac{3 + \sqrt{5}}{2}\right)\ell_1 + \left(\frac{1 - \sqrt{5}}{4}\right)\ell_0,
\]

\[
(5 - \sqrt{5})\ell_1 + \left(-\frac{3 + \sqrt{5}}{2}\right)\ell_0, \ell_1 - \frac{1}{2}\ell_0, \ell_0, \ell_1, \ldots
\]

Case \([p = 12]\). To obtain \(1 - (ab + c + d)x^2 + cdx^4 = \Phi_{12}(x)\), the twelfth cyclotomic polynomial, take \(d = \frac{1}{c}\) and consider \(ab + c + \frac{1}{c} = 1\). It follows
that $c = \frac{1 - \sqrt{\frac{1}{2}b(4b^2 - 2ab - 3)}}{2}$. Set $q = ab$ for $q \in (-\infty, -1] \cup [3, \infty)$, and let $b = \frac{a}{2}$. Then for any real numbers $\ell_0, \ell_1$, excepting $\ell_0 = \ell_1 = 0$, the terms of the sequence are: $\ell_0, \ell_1, a\ell_1 + c\ell_0, (q + \frac{1}{c})\ell_1 + \frac{q}{a}\ell_0, \ell_1 + (qc + c^2)\ell_0, (q^2 + \frac{2a}{c} + q)c + \frac{1}{c}\ell_1 + \frac{qc}{a}\ell_0, -\ell_0, -\ell_1, -a\ell_1 - c\ell_0, -(q + \frac{1}{c})\ell_1 - \frac{q}{a}\ell_0, -\ell_1 - (qc + c^2)\ell_0,$ and $-(q^2 + \frac{2a}{c} + q)c + \frac{1}{c}\ell_1 - \frac{qc}{a}\ell_0$, before it begins repeating. □

**Example 4.5.** In the case $[p = 4]$, consider $f(n; -2 \frac{b}{a}, 0, 1, \frac{b}{a} - 1; \ell_0, \ell_1)$, with neither $\ell_0, \ell_1$ equal to zero. Then $\alpha = \beta$ if and only if $b\frac{b}{a} = 2$. Moreover, if $b\frac{b}{a} \neq 2$, then $\alpha\ell_1 - (ab - \beta)\ell_0 = 0$, but $a\ell_1 - (ab - \alpha)\ell_0 = b\frac{b}{a} - 2$, and $(\alpha + d)^2 = 1$; i.e., in Theorem 4.1, (2A) and (1B) hold. On the other hand, in the case that $[p = 3]$, consider $f(n; -1, -1, -1, -1; \ell_0, \ell_1)$, with neither $\ell_0, \ell_1$ equal to zero. Then $\alpha = \zeta_3, \beta = \zeta_3^2$, and in Theorem 4.1, (3.O) holds.

**Proposition 4.6.** If either $\ell_0 = 0$ or $\ell_1 = 0$, and $\alpha \neq \beta$, then the sequence $\{\ell_n\}_{n=0}^\infty$ can not have period seven or eleven for any $a, b, c, d \in \mathbb{R} \setminus \{0\}$, where $\ell_n = f(n; a, b, c, d; \ell_0, \ell_1)$. Similarly, the shortest period can not be nine when either of $\ell_0, \ell_1$ is zero.

**Proof.** Using methods similar to those above, the resulting systems for $p = 7, 11$ are shown below:

\[
c^2 + 6a^2 c^2 + 5a^4 c + a^6 = 0, \quad c^5 + 15a^2 c^4 + 35a^4 c^3 + 28a^6 c^2 + 9a^8 c + a^{10} = 0, \\
(3a)c^3 + 4a^3 c^2 + a^5 c - 1 = 0, \quad (5a)c^5 + 20a^3 c^4 + 21a^5 c^3 + 8a^7 c^2 + a^9 c = 1.
\]

Neither of these systems have real solutions. Next, if either $\ell_0 = 0$ or $\ell_1 = 0$ and $p = 9$, then the system below results.

\[
c^4 + 10a^2 c^3 + 15a^4 c^2 + 7a^6 c + a^8 = 0, \\
(4a)c^4 + 10a^3 c^3 + 6a^5 c^2 + a^7 c = 1.
\]

However, the only real solution is $a = c = -1$, which produces, via Lemma 4.3, the sequences $f(n; -1, -1, -1, -1; \ell_0, \ell_1)$ of period three. □

**Question.** If $\ell_0, \ell_1$ are both non-zero, then do there exist values of $a, b, c, d \in \mathbb{R} \setminus \{0\}$ such that the sequence $\{\ell_n\}_{n=0}^\infty$ has (shortest) period seven, nine, or eleven?

Finally, as an interesting application, take $f(n; -1, -1, 1, 1; \ell_0, \ell_1)$, the sequence mentioned in the Introduction. As long as both initial conditions are not zero, the sequence is periodic of period twelve since $1 - (ab + c + d)x^2 + cdx^4 = \Phi_{12}(x)$. Starting the indexing at 1, with $\ell_1 = 1$ and $\ell_2 = 3$ as in the Lucas sequence, the result $1, 3, -2, 1, -3, -2, -1, -3, 2, -1, 3, 2$ evokes the sequence representing the Koch curve:

**Example 4.7.** Using the assignments $P(1) = 1, -3, 2, 1; P(2) = 2, 1, 3, 2; P(3) = 3, 2, -1, 3$ from the Koch curve [6, A229216], with $P(-x) = -P(x)$, on the sequence $S = \{1, 3, -2, \ldots, 3, 2\}$, we obtain the figures in Figure 1.
The curves were configured using turtle geometry commands in Python [8], and other figures may be obtained by varying the inputs and their assignments. See e.g., [5].

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