FLAT DIMENSIONS OF INJECTIVE MODULES
OVER DOMAINS

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Abstract. Let $R$ be a domain. It is proved that $R$ is coherent when $IFD(R) \leq 1$, and $R$ is Noetherian when $IPD(R) \leq 1$. Consequently, $R$ is a $G$-Pr"ufer domain if and only if $IFD(R) \leq 1$, if and only if $wG\text{-gldim}(R) \leq 1$; and $R$ is a $G$-Dedekind domain if and only if $IPD(R) \leq 1$.

1. Introduction

Throughout this note, all rings are commutative with identity element and all modules are unitary. A complete projective resolution is an exact sequence of projective modules, $P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$, such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$ [7]. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short) if there exists a complete projective resolution $P$ with $M = \text{Im}(P_0 \rightarrow P^0)$ [7]. A complete flat resolution is an exact sequence of flat $R$-modules, $F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$, such that $I \otimes_R F$ is exact for every injective $R$-module $I$ [7]. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M = \text{Im}(F_0 \rightarrow F^0)$ [7]. $G$-projective and $G$-flat modules are generalizations of projective and flat modules respectively. It is well known that if all $R$-modules are $G$-projective, then injective $R$-modules are projective, i.e., $R$ is a QF-ring. If all $R$-modules are $G$-flat, then injective $R$-modules are flat, i.e., $R$ is an IF-ring. From another perspective, we see that if injective modules are projective, then $R$ is Noetherian; if injective modules are flat, then $R$ is coherent. As in [3], we define $IFD(R)$ as $IFD(R) = \sup\{\text{fd}_RE \mid E \text{ is an injective } R\text{-module}\}$ and $IPD(R)$ as $IPD(R) = \sup\{\text{pd}_RE \mid E \text{ is an injective } R\text{-module}\}$ where $\text{fd}_RE$ and $\text{pd}_RE$ denote the flat and projective dimensions of $E$ respectively. Since projective modules are flat, we have $IFD(R) \leq IPD(R)$. The weak Gorenstein global dimension of $R$, which are introduced in [2], is defined as follows:

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wG-gldim(R) = sup\{Gfd_R(M) \mid M is an R-module\} where Gfd_R(M) denotes the Gorenstein flat dimension of M, as can be seen in [7]. It is proved that a domain \( R \) is coherent when wG-gldim(R) \( \leq 1 \), and is Noetherian when G-gldim(R) \( \leq 1 \). Consequently, a domain \( R \) is a G-Prüfer domain if and only if \( IFD(R) \leq 1 \), if and only if wG-gldim(R) \( \leq 1 \), and is a G-Dedekind domain if and only if \( IPD(R) \leq 1 \).

2. Main results

In order to prove the main results, we need some lemmas.

**Lemma 2.1.** Let \( R \) be a domain. If \( \frac{R}{(a)} \) is Noetherian (coherent) for any nonzero nonunit element \( a \in R \), then \( R \) is Noetherian (coherent).

**Proof.** Let \( I \) be any nonzero (finitely generated) ideal of \( R \) such that \( I \neq R \). Then there exists an element \( a \in I \) which is not a unit. Since \( \frac{R}{(a)} \) is Noetherian (coherent), \( \frac{I}{(a)} \) is finitely generated (finitely presented) as an \( \frac{R}{(a)} \)-module, and hence is also finitely generated (finitely presented) as an \( R \)-module. Because \( Ra \) is finitely generated (presented) as an \( R \)-module, it can be seen from the short exact sequence

\[
0 \rightarrow Ra \rightarrow I \rightarrow \frac{I}{(a)} \rightarrow 0
\]

that \( I \) is a finitely generated (presented) \( R \)-module. Therefore \( R \) is Noetherian (coherent). \( \square \)

We still need the following result.

**Lemma 2.2.** Let \( R \) be a ring and \( a \) be a non-zero-divisor and nonunit element of \( R \). Denote the factor ring \( \frac{R}{(a)} \) by \( T \). If \( IPD(R) < \infty \) (\( IFD(R) < \infty \)) and \( C \) is an injective \( T \)-module, then \( \text{pd}_T C < \infty \) (\( \text{fd}_T C < \infty \)).

**Proof.** By [9, Theorem 2.4.22], there exists an exact sequence: \( 0 \rightarrow C \rightarrow E \rightarrow E \rightarrow 0 \) where \( E \) is an injective \( R \)-module. Since \( \text{pd}_R E < \infty \) (\( \text{fd}_R E < \infty \)), we have \( \text{pd}_R C < \infty \) (\( \text{fd}_R C < \infty \)). Assume \( \text{pd}_R C = k \) (\( \text{fd}_R C = k \)), we have the following exact sequence:

\[
0 \rightarrow P_k \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow C \rightarrow 0
\]

where \( P_i \)'s are projective (flat) \( R \)-modules. Denote the image of the homomorphism \( P_1 \rightarrow P_0 \) by \( K \). Then \( K \) is torsion-free and we have the following exact sequence:

\[
0 \rightarrow \frac{P_k}{aP_k} \rightarrow \cdots \rightarrow \frac{P_1}{aP_1} \rightarrow \frac{K}{aK} \rightarrow 0.
\]

Since \( \frac{P_i}{aP_i} \)'s are projective (flat) \( T \)-modules, we get that \( \text{pd}_T \left( \frac{K}{aK} \right) < \infty \) (\( \text{fd}_T \left( \frac{K}{aK} \right) < \infty \)). On the other hand, we also have the following commutative diagram

\[
\text{Diagram}
\]
with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & P_0 & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K & \rightarrow & P_0 & \rightarrow & C & \rightarrow & 0
\end{array}
\]

where three vertical arrows are the homomorphism of multiplying \(a\). Since \(C\) is a \(T\)-module, the kernel of the right vertical arrow is just \(C\). So, by the Snake Lemma, we have the following short exact sequence

\[
0 \rightarrow C \rightarrow \frac{K}{aK} \rightarrow \frac{P_0}{aP_0} \rightarrow C \rightarrow 0.
\]

Since this exact sequence is of \(T\)-modules and \(C\) is an injective \(T\)-module, we get that \(C\) is isomorphic to a direct summand of \(\frac{K}{aK}\). Because \(\text{pd}_T(\frac{K}{aK}) < \infty\) (\(\text{fd}_T(\frac{K}{aK}) < \infty\)), we get that \(\text{pd}_T C < \infty\) (\(\text{fd}_T C < \infty\)). □

The following result about this kind of global dimension is interesting:

**Lemma 2.3.** Let \(R\) be a ring and \(a\) be a non-zero-divisor and nonunit element of \(R\). Denote the factor ring \(\frac{R}{a}\) by \(T\). If \(\text{IPD}(R) \leq k\) \((\text{IFD}(R) \leq k)\) where \(k\) is a positive integer, then \(\text{IPD}(T) \leq k - 1\) \((\text{IFD}(T) \leq k - 1)\).

**Proof.** Let \(C\) be an injective \(T\)-module. As in the proof of the above lemma, we have a short exact sequence: \(0 \rightarrow C \rightarrow \frac{K}{aK} \rightarrow \frac{P_0}{aP_0} \rightarrow C \rightarrow 0\) where \(E\) is an injective \(R\)-module. Because \(\text{pd}_R E \leq k\) \((\text{fd}_R E \leq k)\), we get that \(\text{pd}_R C \leq k\) \((\text{fd}_R C \leq k)\) by the Horseshoe Lemma. Since \(\text{IPD}(R) \leq \infty\) \((\text{IFD}(R) \leq \infty)\), we have \(\text{pd}_T C < \infty\) \((\text{fd}_T C < \infty)\) by Lemma 2.2. So, by [9, Theorem 3.8.13] \((\text{by [9, Theorem 3.8.15]}))\), \(\text{pd}_T C = \text{pd}_T C + 1 \leq k\) \((\text{fd}_T C = \text{fd}_T C + 1 \leq k)\). Thus \(\text{pd}_T C \leq k - 1\) \((\text{fd}_T C \leq k - 1)\) and so \(\text{IPD}(T) \leq k - 1\) \((\text{IFD}(T) \leq k - 1)\). □

Recall that a ring \(R\) is called Gorenstein hereditary (\(G\)-hereditary) if every submodule of a projective \(R\)-module is \(G\)-projective; Recall that a ring \(R\) is called Gorenstein semihereditary (\(G\)-semihereditary) if it is coherent and every submodule of a flat \(R\)-module is \(G\)-flat, i.e., \(wG\text{-gldim}(R) \leq 1\). A ring is called Gorenstein Dedekind (\(G\)-Dedekind) if it is a \(G\)-hereditary domain. Similarly, we call a ring Gorenstein Pr"ufer domain (\(G\)-Pr"ufer domain) if it is a \(G\)-semihereditary domain. Of course a \(G\)-Dedekind domain is a \(G\)-Pr"ufer domain. Bennis proved that if \(R\) is a coherent ring, then \(wG\text{-gldim}(R) \leq 1\) if and only if \(\text{IFD}(R) \leq 1\) ([1]). The following result shows that Bennis’ result still holds without the coherent condition when \(R\) is a domain. It was also proved in [10, Corollary 2.9] that a domain \(R\) is a \(G\)-Pr"ufer domain if and only if \(R\) is coherent and every finitely generated torsion-free \(R\)-module is \(G\)-flat. The following result tells us that the condition that \(R\) is coherent is also superfluous.

**Corollary 2.4.** Let \(R\) be a domain with quotient field \(K\). The following statements are equivalent:

\((1)\) \(wG\text{-gldim}(R) \leq 1;\)
(2) every torsion-free module is $G$-flat;
(3) every finitely generated torsion-free module is $G$-flat;
(4) every finitely generated ideal is $G$-flat;
(5) $IFD(R) \leq 1$;
(6) $R$ is a $G$-Prüfer domain.

Proof. (1)⇒(2) Let $M$ be a torsion-free module. Then $M \to M \otimes K$ is an injective homomorphism. Since $M \otimes K$ is isomorphic to a direct sum of some $K$, $M \otimes K$ is flat. Now, as a submodule of a flat module, $M$ is $G$-flat.

(2)⇒(3) and (3)⇒(4) They are trivial.

(4)⇒(5) Let $E$ be an injective $R$-module. For any finitely generated ideal $I$ of $R$, we will prove that $\text{Tor}_2^R(R, E) = 0$. Considering the short exact sequence

$$0 \to I \to R \to R/I \to 0,$$

we have $\text{Tor}_2^R(R, E) \cong \text{Tor}_1^R(I, E) = 0$ since $I$ is $G$-flat. So, by [9, Theorem 3.6.4], we have $\text{fd}_R E \leq 1$.

(5)⇒(6) Let $a$ be any nonzero nonunit element in $R$. Denote the factor ring $R/(a)$ by $T$. Since $IFD(R) \leq 1$, $IFD(T) = 0$ by Lemma 2.3. This shows that $T$ is a $QF$-ring, and hence $T$ is Noetherian. Now an application of Lemma 2.1 shows that $R$ is coherent. So, by [1, Theorem 2.8], we get that $\text{wG-gl.dim}(R) \leq 1$ and $R$ is a $G$-Prüfer domain.

(6)⇒(1) This is obvious. □

From the proof of Corollary 2.4, we get that a domain $R$ is coherent when $IFD(R) \leq 1$. Enochs and Jenda introduce the concepts of copure injective modules and strongly copure injective modules in [4]. For an $R$-module $M$, $M$ is called copure injective if $\text{Ext}_i^R(E, M) = 0$ for any injective $R$-module $E$, and $M$ is called strongly copure injective if $\text{Ext}_i^R(E, M) = 0$ for any injective $R$-module $E$ and for all $i \geq 1$. They also defined the copure injective dimension $\text{cid}_R M$ of an $R$-module $M$ to be the largest integer $n \geq 0$ such that $\text{Ext}_n^R(E, M) \neq 0$ for some injective $R$-module $E$. Of course, if no such $n$ exists, write $\text{cid}_R(M) = \infty$. The copure injective dimension of a ring $R$ was defined in [5] as $\text{cid}(R) = \sup\{\text{cid}_R(M) \mid M \text{ is an } R\text{-module}\}$. Using this concept, we have the following characterizations of $G$-Dedekind domains.

**Corollary 2.5.** Let $R$ be a domain. The following statements are equivalent:

(1) $G$-gldim$(R) \leq 1$, i.e., $R$ is a $G$-Dedekind domain;
(2) $IPD(R) \leq 1$;
(3) $\text{cid}(R) \leq 1$.

Proof. (1)⇒(2) Let $E$ be an injective $R$-module. By [6, Theorem 2.2], $\text{pd}_R E = \text{Gpd}_R E$. So $\text{pd}_R E \leq 1$ and $IPD(R) \leq 1$ holds;

(2)⇒(1) Let $a$ be any nonzero nonunit element in $R$. Denote the factor ring $R/(a)$ by $T$. Since $IPD(R) \leq 1$, we have $IPD(T) = 0$ by Lemma 2.3. This shows that $T$ is a $QF$-ring, and hence $T$ is Noetherian. Now an application of
Lemma 2.1 shows that \( R \) is Noetherian. Since \( IPD(R) \leq IFD(R) \), we surely have \( IFD(R) \leq 1 \) and, by Corollary 2.4, \( R \) is a G-Prüfer domain. Now, as a Noetherian G-Prüfer domain, \( R \) is a G-Dedekind domain by [8, Corollary 4.3].

\( (2) \Rightarrow (3) \) Let \( M \) be any \( R \)-module. Pick an exact sequence \( 0 \rightarrow M \rightarrow E \rightarrow C \rightarrow 0 \) where \( E \) is injective. It suffices to prove that \( C \) is copure injective. Let \( I \) be any injective module. Since \( IPD(R) \leq 1 \), we have the following short exact sequence:

\[
0 \rightarrow P_1 \rightarrow P_0 \rightarrow I \rightarrow 0
\]

where \( P_1, P_0 \) are projective. So, we have

\[
\text{Ext}^1_R(I, C) \cong \text{Ext}^2_R(I, M) \cong \text{Ext}^1_R(P_1, M) = 0,
\]

and hence \( C \) is copure injective.

\( (3) \Rightarrow (2) \) Let \( I \) be an injective module. Pick an exact sequence \( 0 \rightarrow K \rightarrow P \rightarrow I \rightarrow 0 \) where \( P \) is projective. It suffices to prove that \( K \) is projective. Let \( X \) be any \( R \)-module. Since \( \text{ciD}(R) \leq 1 \), we have an exact sequence \( 0 \rightarrow X \rightarrow E \rightarrow G \rightarrow 0 \) where \( E \) is injective and \( G \) is copure injective. So, we have that \( \text{Ext}^1_R(K, X) \cong \text{Ext}^2_R(I, X) \cong \text{Ext}^1_R(I, G) = 0 \), and hence \( K \) is projective.

From the proof of Corollary 2.5, we can get that a domain \( R \) is Noetherian when \( IPD(R) \leq 1 \). Next we prove that \( IPD(R[x]) = IPD(R) + 1 \) (\( IFD(R[x]) = IFD(R) + 1 \)) for any ring \( R \).

**Theorem 2.6.** Let \( R \) be a ring. Then \( IPD(R) < \infty \) (\( IFD(R) < \infty \)) if and only if \( IPD(R[x]) < \infty \) (\( IFD(R[x]) < \infty \)). When they are finite, we have \( IPD(R[x]) = IPD(R) + 1 \) (\( IFD(R[x]) = IFD(R) + 1 \)).

**Proof.** When \( IPD(R[x]) < \infty \) (\( IFD(R[x]) < \infty \)), we have \( IPD(R) < \infty \) (\( IFD(R) < \infty \)) by Lemma 2.2. On the other hand, if \( IPD(R) < \infty \) (\( IFD(R) < \infty \)) and \( E \) be an injective \( R[x] \)-module, then \( E \) is also an injective \( R \)-module and there exists an exact sequence \( 0 \rightarrow F_m \rightarrow \cdots \rightarrow F_0 \rightarrow E \rightarrow 0 \) for some integer \( m \) where \( F_i \)'s are projective (flat) \( R \)-modules. Since \( \text{pd}_R F_i \leq \text{pd}_R F_{i-1} + \text{pd}_R F \) (\( \leq \text{fd}_R F \), \( \text{fd}_R F \leq \text{fd}_R F \), \( R = 1 \) for \( 0 \leq i \leq m \)), we have \( \text{pd}_R E \leq \infty \) (\( \text{fd}_R E \), \( \infty \)). So, we have \( IPD(R[x]) < \infty \) (\( IFD(R[x]) < \infty \)).

Now we prove the equation. It follows from Lemma 2.3 that if \( IPD(R[x]) \leq n \) (\( IFD(R[x]) \leq n \)) for some positive integer \( n \), then \( IPD(R) \leq n-1 \) (\( IFD(R) \leq n-1 \)). Reversely, if \( IPD(R) \leq n-1 \) (\( IFD(R) \leq n-1 \)) for some positive integer \( n \) and \( E \) is an injective \( R[x] \)-module, then \( E \) is also an injective \( R \)-module and there exists the following exact sequence of \( R \)-modules \( 0 \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow E \rightarrow 0 \) where \( (F_i)'s \) are projective (flat). Pick short exact sequences \( 0 \rightarrow F_{n-1} \rightarrow P_{n-1} \rightarrow F_{n-1} \rightarrow 0 \) and \( 0 \rightarrow N \rightarrow P \rightarrow E \rightarrow 0 \) of \( R[x] \)-modules where \( P_{n-1} \) and \( P \) are projective, by iterated use of the Horse-shoe Lemma, we can get the following commutative diagram with exact rows...
and columns:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 \\
0 & F'_{n-1} & F'_{n-2} & \cdots & F'_{0} & N & 0 \\
0 & P_{n-1} & P_{n-2} & \cdots & P_{0} & P & 0 \\
0 & F'_{n-1} & F'_{n-2} & \cdots & F'_{0} & E & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

where \(P'_i\)’s are projective \(R[x]\)-modules. Since for every \(F_i\), \(\text{pd}_{R[x]}F_i \leq 1\) (\(\text{fd}_{R[x]}F_i \leq 1\)), we have every \(F'_i\) (\(0 \leq i \leq n-1\)) is projective (flat) \(R[x]\)-module. Now, combining the top horizontal exact sequence and the right vertical exact sequence, we will see that \(\text{pd}_{R[x]}E \leq n\) (\(\text{fd}_{R[x]}E \leq n\)). So, \(\text{IPD}(R[x]) \leq n\) (\(\text{IFD}(R[x]) \leq n\)).

\[\square\]

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