COMMUTATIVITY WITH ALGEBRAIC IDENTITIES INVOLVING PRIME IDEALS

Hajar El Mir, Abdellah Mamouni, and Lahcen Oukhtite

Abstract. The purpose of this paper is to study the structure of quotient rings $R/P$ where $R$ is an arbitrary ring and $P$ is a prime ideal of $R$. Especially, we will establish a relationship between the structure of this class of rings and the behavior of derivations satisfying algebraic identities involving prime ideals. Furthermore, the characteristic of the quotient ring $R/P$ has been determined in some situations.

1. Introduction

Throughout this article, $R$ will represent an associative ring with center $Z(R)$. Recall that an ideal $P$ of $R$ is said to be prime if $P \neq R$ and for $x, y \in R$, $xPy \subseteq P$ implies that $x \in P$ or $y \in P$. Therefore, $R$ is called a prime ring if and only if $(0)$ is the prime ideal of $R$. $R$ is 2-torsion free if whenever $2x = 0$, with $x \in R$ implies $x = 0$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. A map $d : R \to R$ is a derivation of a ring $R$ if $d$ is additive and satisfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Recently many authors have obtained commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings. We first recall that for a subset $S$ of $R$, a mapping $f : S \to R$ is called centralizing if $[f(x), x] \in Z(R)$ for all $x \in S$, in the special case where $[f(x), x] = 0$ for all $x \in S$, the mapping $f$ is said to be commuting on $S$. In [9], Posner proved that if a prime ring $R$ admits a nonzero derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. Over the last few decades, several authors have subsequently refined and extended these result in various directions (see [1], [6], [7] and [8] where further references can be found). Recall that a mapping $f : R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory.

Received September 26, 2019; Accepted February 18, 2020.

2010 Mathematics Subject Classification. 16N60, 16W10, 16W25.

Key words and phrases. Derivations, prime ideals, commutativity.

©2020 Korean Mathematical Society

723
and ring theory (see [3], [10] for references). A mapping \( f : R \rightarrow R \) is said to be strong commutativity preserving (SCP) on a subset \( S \) of \( R \) if \( [f(x), f(y)] = [x, y] \) for all \( x, y \in S \). In [2], Bell and Daif investigated the commutativity of rings admitting a derivation that is SCP on a nonzero right ideal. Indeed, they proved that if a semiprime ring \( R \) admits a derivation \( d \) satisfying \( [d(x), d(y)] = [x, y] \) for all \( x, y \) in a right ideal \( I \) of \( R \), then \( I \subseteq Z(R) \). In particular, \( R \) is commutative if \( I = R \).

Later, Deng and Ashraf [5] proved that if there exist a derivation \( d \) of a semiprime ring \( R \) and a mapping \( f : I \rightarrow R \) defined on a nonzero ideal \( I \) such that \( [f(x), d(y)] = [x, y] \) for all \( x, y \in I \), then \( R \) contains a nonzero central ideal. In particular, they showed that \( R \) is commutative if \( I = R \). Many related generalizations of these results can be found in the literature (see for instance [4]).

The present paper is motivated by the previous results and we here continue this line of investigation by considering a more general concept rather than SCP derivation. Moreover, we will establish a relationship between the structure of quotient rings \( R/P \) and the behavior of derivations satisfying algebraic identities involving prime ideals.

2. Main results

We will use frequently the following Lemma which is very crucial for developing the proofs of our main results.

**Lemma 1.** Let \( R \) be a ring, \( P \) a prime ideal, \( d_1 \) and \( d_2 \) derivations of \( R \). Then \( d_1(x)x - xd_2(x) \in P \) if and only if \( (d_1(R) \subseteq P \) and \( d_2(R) \subseteq P) \) or \( R/P \) is a commutative integral domain.

**Proof.** We are given that
\[
d_1(x)x - xd_2(x) \in P \quad \text{for all} \quad x \in R.
\]
Linearizing (2.1) we get
\[
d_1(x)y + d_1(y)x - xd_2(y) - yd_2(x) \in P \quad \text{for all} \quad x, y \in R.
\]
Replacing \( y \) by \( yx \), we obtain
\[
d_1(x)yx + d_1(y)x^2 + yd_1(x)x - xd_2(y)x - xyd_2(x) - yxd_2(x) \in P.
\]
Right multiplying (2.2) by \( x \) and using (2.3), we find that
\[
yd_1(x)x - yxd_2(x) + yd_2(x)x \in P \quad \text{for all} \quad x, y \in R.
\]
Once again, multiplying (2.1) by \( y \) one can see that
\[
yd_1(x)x - yxd_2(x) \in P \quad \text{for all} \quad x, y \in R.
\]
Using (2.4) together with (2.5) we arrive at
\[
[yd_2(x), x] \in P \quad \text{for all} \quad x, y \in R.
\]
Substituting \( yr \) for \( y \) one can verify
\[
[y, x]rd_2(x) \in P \quad \text{for all} \quad r, x, y \in R.
\]
Since $P$ is a prime ideal, then for all $x, y \in R$ either $[x,y] \in P$ or $d_2(x) \in P$ which implies that $R = R_1 \cup R_2$ with $R_1 = \{ x \in R \mid [x,y] \in P \text{ for all } y \in R \}$ and $R_2 = \{ x \in R \mid d_2(x) \in P \}$. Since a group cannot be union of its subgroups then $R = R_1$ in which case $R/P$ is a commutative integral domain or $R = R_2$. In the last case $d_2(R) \subseteq P$; using the hypothesis one can easily verify that $d_1(R) \subseteq P$. \hfill \square

In [2], Bell and Daif investigated the commutativity of rings admitting an SCP derivation. Indeed, they proved that if a semiprime ring $R$ admits a derivation $d$ satisfying $[d(x), d(y)] - [x,y] = 0$ for all $x, y$ in a right ideal $I$ of $R$, then $I \subseteq Z(R)$. Our aim in the following theorem is to generalize this result of Bell and Daif in two directions. First of all, we will assume that for all $x, y \in R$, “$[d(x), d(y)] - [x,y]$” belongs to a prime ideal $P$ rather than “$[d(x), d(y)] = [x,y]$” and the ring is not necessarily semiprime. Secondly, we will treat a more general differential identity involving two derivations.

**Theorem 1.** Let $R$ be a ring, $P$ a prime ideal, $d_1$ and $d_2$ derivations of $R$. Then:

1. $[d_1(x), d_2(y)] - [x,y] \in P$ for all $x, y \in R$ if and only if $R/P$ is a commutative integral domain.
2. $d_1(x)d_2(y) - [x,y] \in P$ for all $x, y \in R$ implies that $R/P$ is a commutative integral domain.

**Proof.** (1) For the nontrivial implication suppose that

\begin{equation}
[d_1(x), d_2(y)] - [x,y] \in P \text{ for all } x, y \in R.
\end{equation}

Substituting $yr$ for $y$, we get

\begin{equation}
\begin{align*}
d_2(y)[d_1(x), r] + [d_1(x), d_2(y)]r + y[d_1(x), d_2(r)] + [d_1(x), y][d_2(r) \\
- y[x, r] - [x, y]r & \in P.
\end{align*}
\end{equation}

Using (2.8) and (2.9), one can verify that

\begin{equation}
\begin{align*}
d_2(y)[d_1(x), r] + [d_1(x), y][d_2(r) & \in P \text{ for all } x, y \in R.
\end{align*}
\end{equation}

Writing $d_1(x)$ instead of $y$, one can see that

\begin{equation}
\begin{align*}
d_2(d_1(x))[d_1(x), r] & \in P \text{ for all } r, x \in R.
\end{align*}
\end{equation}

Replacing $r$ by $d_2(r)$ and using (2.8), we obtain

\begin{equation}
\begin{align*}
d_2(d_1(x))[x, r] & \in P \text{ for all } r, x \in R
\end{align*}
\end{equation}

and thus

\begin{equation}
\begin{align*}
d_2(d_1(x))R[x, r] & \subseteq P \text{ for all } r, x \in R.
\end{align*}
\end{equation}

Since $P$ is a prime ideal then either $[x, r] \in P$ or $d_2d_1(x) \in P$ for all $r, x \in R$, which implies that $R = R_1 \cup R_2$ with $R_1 = \{ x \in R \mid [x,y] \in P \text{ for all } y \in R \}$ and $R_2 = \{ x \in R \mid d_2d_1(x) \in P \}$. Since a group cannot be union of its subgroups then $R = R_1$ or $R = R_2$. If $R = R_1$, then obviously $R/P$ is commutative. If $R = R_2$, then $d_2d_1(R) \subseteq P$; using the hypothesis with
of a ring which admits derivations $d_1$ and $d_2$ instead of $x$ we get $[x, y] \in P$ for all $x, y \in R$. Accordingly, $R/P$ is commutative. Hence in all cases $R/P$ is a commutative integral domain.

(2) We are given that

$$d_1(x)d_2(y) - [x, y] \in P \text{ for all } x, y \in R.$$  

Substituting $yr$ for $y$, we get

$$d_1(x)d_2(y)r + d_1(x)yrd_2(r) - y[x, r] - [x, y]r \in P \text{ for all } r, x, y \in R.$$  

Multiplying (2.14) by $r$ on right and using (2.15) one can see that

$$d_1(x)yrd_2(r) - y[x, r] \in P \text{ for all } r, x, y \in R.$$  

By view of (2.14), the equation (2.16) yields

$$[d_1(x), y]d_2(r) \in P \text{ for all } r, x, y \in R.$$  

Therefore

$$[d_1(x), y]Rd_2(r) \subseteq P \text{ for all } r, x, y \in R.$$  

Since $P$ is prime then $[d_1(x), y] \in P$ or $d_2(r) \in P$ for all $r, x, y \in R$ and thus by Lemma 1, $d_1(R) \subseteq P$ or $R/P$ is commutative or $d_2(R) \subseteq P$. But if $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$, then the hypothesis implies that $[x, y] \in P$, hence in all cases $R/P$ is a commutative integral domain. \hfill \Box

Remark 1. Using the same technics as in the preceding proof, it is obvious to see that $[d_1(x), d_2(y)] + [x, y] \in P$ (resp. $d_1(x)d_2(y) + [x, y] \in P$) for all $x, y \in R$ implies that $R/P$ is a commutative integral domain.

As a consequence of Theorem 1, if $R$ is a prime ring, then $[d_1(x), d_2(y)] - [x, y] = 0$ (resp. $d_1(x)d_2(y) - [x, y] = 0$) for all $x, y \in R$ assures that $R$ is a commutative integral domain. The following corollary proves that the similar conclusion remains valid on semiprime rings.

**Corollary 1.** Let $R$ be a semiprime ring, $d_1$ and $d_2$ derivations of $R$. Then

1. $[d_1(x), d_2(y)] - [x, y] = 0$ for all $x, y \in R$ if and only if $R$ is commutative.
2. $d_1(x)d_2(y) - [x, y] = 0$ for all $x, y \in R$ implies that $R$ is commutative.

**Proof.** For the non-trivial implication assume that $[d_1(x), d_2(y)] - [x, y] = 0$ for all $x, y \in R$. By view of the semiprimeness of the ring $R$, there exists a family $\Gamma$ of prime ideals such that $\bigcap_{\Gamma \in \Gamma} P = \{0\}$, thereby obtaining $[d_1(x), d_2(y)] - [x, y] \in P$ for all $P \in \Gamma$. Invoking Theorem 1, we conclude that $R/P$ is a commutative integral domain for all $P \in \Gamma$ which, because of $\bigcap_{\Gamma \in \Gamma} P = \{0\}$, assures that $R$ is commutative. Similarly, if $d_1(x)d_2(y) - [x, y] = 0$ for all $x, y \in R$, then the same reasoning proves that $R$ is commutative. \hfill \Box

The fundamental aim of the next theorem is to investigate commutativity of a ring which admits derivations $d_1$ and $d_2$ satisfying

$$(d_1(x) \circ d_2(y) - x \circ y \in P \text{ for all } x, y \in P) \quad \text{or}$$
Our result is of a special kind because we will describe not only the structure of the ring $R/P$ but we will also determine its characteristic. More precisely we will prove the following result.

**Theorem 2.** Let $R$ be a ring, $P$ a prime ideal, $d_1$ and $d_2$ derivations of $R$. The following assertions are equivalent:

1. $d_1(x) \circ d_2(y) - x \circ y \in P$ for all $x, y \in R$;
2. $[d_1(x), d_2(y)] - x \circ y \in P$ for all $x, y \in R$;
3. $R/P$ is a commutative integral domain and $\text{char}(R/P) = 2$.

**Proof.** It is clear that (3) implies both of (1) and (2). So we need to prove that (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3).

(1) $\Rightarrow$ (3) We are given that

$$d_1(x) \circ d_2(y) - x \circ y \in P \quad \text{for all } x, y \in R. \tag{2.19}$$

Substituting $yr$ for $y$ we arrive at

$$d_1(x) \circ d_2(y)r - d_2(y)[d_1(x), r] + y(d_1(x) \circ d_2(r)) + [d_1(x), y]d_2(r) - (x \circ y)r + y[x, r] \in P. \tag{2.20}$$

Right multiplying the equation (2.19) by $r$ and using (2.20), we obtain

$$-d_2(y)[d_1(x), r] + y(d_1(x) \circ d_2(r)) + [d_1(x), y]d_2(r) + y[x, r] \in P \quad \text{for all } r, x, y \in R. \tag{2.21}$$

Using the equations (2.19) and (2.21) one can see that

$$-d_2(y)[d_1(x), r] + y(x \circ r) + [d_1(x), y]d_2(r) + y[x, r] \in P; \tag{2.22}$$

thereby obtaining

$$-d_2(y)[d_1(x), r] + 2yxr + [d_1(x), y]d_2(r) \in P \quad \text{for all } r, x, y \in R. \tag{2.23}$$

Replacing $r$ by $d_1(x)$ we get

$$2ydx_1 + [d_1(x), y]d_2(d_1(x)) \in P \quad \text{for all } x, y \in R. \tag{2.24}$$

Substituting $ry$ for $y$ and using the last equation one can verify that

$$[d_1(x), y]Rd_2(d_1(x)) \subseteq P \quad \text{for all } x, y \in R. \tag{2.25}$$

By view of the primeness of $P$ we conclude that either $[d_1(x), y] \subseteq P$ for all $x, y \in R$ or $d_2(d_1(R)) \subseteq P$. In the first case; using Lemma 1 one can see that $d_1(R) \subseteq P$ or $R/P$ is a commutative integral domain.

Now assume $d_2(d_1(R)) \subseteq P$; substituting $d_1(y)$ for $y$ in (2.19), we obviously obtain

$$x \circ d_1(y) \in P \quad \text{for all } x, y \in R. \tag{2.26}$$

Writing $zrx$ instead of $x$ in the last relation, one can easily verify that

$$zr[x, d_1(y)] + ((zr) \circ d_1(y))x \in P.$$
which, in light of (2.26), yields $zR[x, d_1(y)] \subseteq P$. In particular, we may write
\begin{equation}
[x, d_1(y)]R[x, d_1(y)] \subseteq P \quad \text{for all } x, y \in R.
\end{equation}

Since $P$ is prime, it follows that $[x, d_1(y)] \in P$ for all $x, y \in R$. Arguing as above, we have either $d_1(R) \subseteq P$ or $R/P$ is a commutative integral domain.

Finally, we claim that $R/P$ is of characteristic $2$. Indeed, assume that $R/P$ is commutative then the hypothesis assures that $d_1(R) \subseteq P$, and thus the equation (2.19) reduces to $x \circ y \in P$ for all $x, y \in R$. Replacing $y$ by $yr$ we find that $[x, y]r \in P$ and using the fact that a prime ideal is proper, we conclude that $[R, R] \subseteq P$ so that $R/P$ is commutative. Consequently, in both cases $R/P$ is a commutative integral domain.

Using Brauer’s trick one can see that
\begin{equation}
\text{Right multiplying the equation (2.28) by } \begin{aligned}
&d_2(y)[d_1(x), y] + [d_1(x), d_2(y)]r + y[d_1(x), d_2(r)] + [d_1(x), y]d_2(r) \\
&= (x \circ y)r + y[x, r] \in P.
\end{aligned}
\end{equation}

Using the equations (2.28) and (2.30) one can see that
\begin{equation}
d_2(y)[d_1(x), r] + 2yxr + [d_1(x), y]d_2(r) \in P \quad \text{for all } r, x, y \in R.
\end{equation}

Replacing $r$ by $d_1(x)$ we get
\begin{equation}
2yxd_1(x) + [d_1(x), y]d_2(d_1(x)) \in P \quad \text{for all } x, y \in R.
\end{equation}

Substituting $ry$ for $y$ and using the last equation one can verify that
\begin{equation}
[d_1(x), y]rd_2(d_1(x)) \in P \quad \text{for all } r, x, y \in R.
\end{equation}

Since $P$ is a prime ideal then $[d_1(x), y] \in P$ or $d_2(d_1(x)) \in P$ for all $x, y \in R$. Using Brauer’s trick one can see that $[d_1(x), y] \in P$ for all $x, y \in R$ or $d_2(R) \subseteq P$. In the first case we are forced to $R/P$ is commutative or $d_1(R) \subseteq P$ in which case the hypothesis implies that $x \circ y \in P$ and thus $R/P$ is commutative. If $d_2d_1(R) \subseteq P$, using the hypothesis one can verify that $x \circ d_1(y)$. That is just the equation (2.26) then we may argue as before that $R/P$ is commutative. Now suppose that $\text{char}(R/P) \neq 2$ then $x \circ y \in P$ implies that $R = P$, a contradiction. Hence $\text{char}(R/P) = 2$. $\square$
Corollary 2. Let $R$ be a 2-torsion free ring which is either semiprime or unitary. There are no derivations $d_1$ and $d_2$ satisfying one of the following conditions:

1. $d_1(x) \circ d_2(y) - x \circ y = 0$ for all $x, y \in R$;
2. $[d_1(x), d_2(y)] - x \circ y = 0$ for all $x, y \in R$.

Proof. Assume that $R$ is a 2-torsion free semiprime ring. Then there exists a family $\Gamma$ of prime ideals such that $\bigcap_{P \in \Gamma} P = \{0\}$. Since the proof is similar and does not depend on the chosen identity we may assume existence of derivations $d_1$ and $d_2$ satisfying $d_1(x) \circ d_2(y) - x \circ y = 0$ for all $x, y \in R$. Accordingly, $d_1(x) \circ d_2(y) - x \circ y \in P$ for all $P \in \Gamma$ and Theorem 2 yields $R/P$ is commutative. Therefore, for all $x, y \in R$ we have $[x, y] \in P$ for all $P \in \Gamma$ so that $[x, y] = 0$ proving that $R$ is commutative. Hence our hypothesis reduces to $d_1(x) d_2(y) - xy = 0$ for all $x, y \in R$. Replacing $x$ by $xy$ in the above relation, we find that $d_1(x) yd_2(y) = 0$ for all $x, y \in R$ and the semiprimeness forces $d_1 = 0$ or $d_2 = 0$. Consequently, $xy = 0$ for all $x, y \in R$ so that $R = \{0\}$, a contradiction. In the case $R$ is an arbitrary 2-torsion free unitary ring, setting $y = 1$ our hypothesis reduces to $x = 0$ for all $x \in R$, a contradiction. □

Proposition 1. Let $R$ be a ring, $P$ a prime ideal, $d_1$ and $d_2$ derivations of $R$. If $d_1(x) d_2(y) - x \circ y \in P$ for all $x, y \in R$, then $R/P$ is a commutative integral domain with characteristic 2 and $(d_1(R) \subseteq P$ or $d_2(R) \subseteq P)$.

Proof. We are given that

\[d_1(x) d_2(y) - x \circ y \in P\] for all $x, y \in R$.

Substituting $y$ by $yr$, we get

\[d_1(x) d_2(y) r + d_1(x) yd_2(r) - (x \circ y) r + y[x, r] \in P\] for all $r, x, y \in R$.

Multiplying (2.34) by $r$ on right and using (2.35) one can see that

\[d_1(x) yd_2(r) + y[x, r] \in P\] for all $r, x, y \in R$.

By view of (2.34), (2.36) yields

\[[d_1(x), y] d_2(r) + 2yxr \in P\] for all $r, x, y \in R$.

Therefore

\[[d_1(x), y] R d_2(r) \subseteq P\] for all $r, x, y \in R$.

That is just the equation (2.18), so we may argue as before that $R/P$ is commutative. Using the equation (2.36) one can see that $d_1(x) yd_2(r)$ for all $r, x, y \in P$ and thus $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$. Now suppose that $\text{char}(R/P) \neq 2$ then $x \circ y \in P$ implies that $R = P$, a contradiction. Hence $\text{char}(R/P) = 2$. □

The fundamental aim of the next theorem is to treat the special identity

\[d_1(x) \circ d_2(y) - [x, y] \in P\] for all $x, y \in R$
involving commutator and anti-commutator. In fact, we will characterize the structure of the ring \( R/P \) and prove that either \( d_1 \) or \( d_2 \) has its range in the prime ideal \( P \). More precisely we will prove the following result.

**Theorem 3.** Let \( R \) be a ring, \( P \) a prime ideal, \( d_1 \) and \( d_2 \) derivations of \( R \). If \( d_1(x) \circ d_2(y) - [x, y] \in P \) for all \( x, y \in R \), then \( R/P \) is a commutative integral domain. Moreover, if \( R/P \) is 2-torsion free, then \( d_1(R) \subseteq P \) or \( d_2(R) \subseteq P \).

**Proof.** We are given that

\[
(2.39) \quad d_1(x) \circ d_2(y) - [x, y] \in P \quad \text{for all } x, y \in R.
\]

Substituting \( yr \) for \( y \) we arrive at

\[
(2.40) \quad (d_1(x) \circ d_2(y)r - d_2(y)[d_1(x), r] + y(d_1(x) \circ d_2(r)) + [d_1(x), y]d_2(r)
\]

Using the equations (2.39) and (2.40) one can see that

\[
(2.41) \quad - d_2(y)[d_1(x), r] + [d_1(x), y]d_2(r) \in P \quad \text{for all } r, x, y \in R.
\]

Replacing \( y \) by \( d_1(x) \) we get

\[
(2.42) \quad d_2d_1(x)[d_1(x), r] \in P \quad \text{for all } r, x \in R
\]

and therefore

\[
(2.43) \quad d_2d_1(x)R[d_1(x), y] \subseteq P \quad \text{for all } x, y \in R.
\]

Invoking arguments used in the proof of Theorem 1 it follows that either \( R/P \)

is commutative or \( d_2d_1(R) \subseteq P \). In the latter case, writing \( d_1(y) \) instead of \( y \) in (2.32), we get \([d_1(y), x] \in P \) for all \( x, y \in R \). Therefore \( R/P \)

is commutative or \( d_1(R) \subseteq P \) in which case our hypothesis reduces to \([x, y] \in P \) for all \( x, y \in R \) proving again that \( R/P \)

is a commutative integral domain. Now assume that \( R/P \)

is 2-torsion free, then \( d_1(x)d_2(y) \in P \) for all \( x, y \in R \); replacing \( x \) by \( x^r \)

we get \( d_1(x)Rd_2(y) \subseteq P \) so that \( d_1(R) \subseteq P \) or \( d_2(R) \subseteq P \).

**Remark 2.** Using similar technics as in the preceding proof with a slight modification, it is obvious to prove that \( d_1(x) \circ d_2(y) + [x, y] \in P \) for all \( x, y \in R \) assures that \( R/P \)

is a commutative integral domain.

The following result is an immediate consequence of the above theorem.

**Corollary 3.** Let \( R \) be a semiprime ring, \( d_1 \) and \( d_2 \) derivations of \( R \). If \( d_1 \) and \( d_2 \)

satisfies \( d_1(x) \circ d_2(y) - [x, y] = 0 \) for all \( x, y \in R \), then \( R \)

is a commutative ring. Moreover, if \( R \) is 2-torsion free, then \( d_1 = 0 \) or \( d_2 = 0 \).

**References**


Hajar El Mir
Department of Mathematics
Faculty of Science and Technology of Fez
University S. M. Ben Abdellah Fez
Box 2202, Morocco
Email address: hajar.elmir@usmba.ac.ma

Abdellah Mamouni
Department of Mathematics
Faculty of Science and Technology Box 509-Boutalamin
University Moulay Ismail
Errachidia, Morocco
Email address: a.mamouni.fste@gmail.com

Lahcen Oukhtite
Department of Mathematics
Faculty of Science and Technology of Fez
University S. M. Ben Abdellah Fez
Box 2202, Morocco
Email address: oukhtitela@hotmail.com