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CHARACTERIZATION THEOREMS OF RILEY TYPE FOR BICOMPLEX HOLOMORPHIC FUNCTIONS

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Abstract. We characterize bicomplex holomorphic functions from several estimates. Originally, Riley [5] studied such problems in local case. In our study, we treat global estimates on various unbounded domains. In many cases, we can determine the explicit form of a function.

1. Introduction

Bicomplex algebra was introduced by Segre inspired by the work of Hamilton and Clifford on quaternions, and defined by

$$\mathbb{BC} = \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}, \tag{1.1}$$

where j is another imaginary unit commuting with the imaginary unit i of $\mathbb{C}$. Since $\mathbb{BC}$ is not an integral domain, we denote the set of zero divisors by $\mathfrak{S}_0$. By the non-complex idempotent elements $\mathbf{e} = \frac{1+ij}{2}$, $\mathbf{e}^\dagger = \frac{1-ij}{2}$ satisfying $\mathbf{e}\mathbf{e}^\dagger = 0$, any bicomplex number $Z = z_1 + z_2 j$ has the idempotent representation

$$Z = (z_1 - z_2 i)\mathbf{e} + (z_1 + z_2 i)\mathbf{e}^\dagger. \tag{1.2}$$

We set $Z_{\mathbf{e}} = z_1 - z_2 i$, $Z_{\mathbf{e}^\dagger} = z_1 + z_2 i \in \mathbb{C}$.

For bicomplex functions, we define the notion of holomorphy similarly to the complex holomorphic functions. By the idempotent representation (1.2), for any bicomplex holomorphic function F there exist complex holomorphic functions $F_{\mathbf{e}}$ and $F_{\mathbf{e}^\dagger}$ in one variable such that

$$F(Z) = F_{\mathbf{e}}(Z_{\mathbf{e}})\mathbf{e} + F_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger. \tag{1.3}$$

By this formula, we can generalize immediately several properties of complex holomorphic functions to those of bicomplex holomorphic functions. For bicomplex numbers and bicomplex functions, [2] and [4] are standard textbooks. See also Section 2.

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In [5], Riley studied some properties of bicomplex holomorphic functions not obtained immediately from the idempotent representation (1.3) of them. He introduced the values

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|Z_{\mathbf{e}}|^2 + |Z_{\mathbf{e}^\dagger}|^2}{2}}, \tag{1.4}$$

$$|Z|_{\mathrm{R}} = \sqrt{|z_1^2 + z_2^2|} = \sqrt{|Z_{\mathbf{e}}||Z_{\mathbf{e}^\dagger}|} \tag{1.5}$$

for a bicomplex number $Z = z_1 + z_2 j = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ and proved local characterizations of bicomplex holomorphic functions. We introduce one of his results as follows. See Section 3 for others.

Theorem 1.1 ([5]). *Let F be a bicomplex holomorphic function on $\mathbb{BC} \setminus \mathfrak{S}_0$. If F satisfies the condition*

$$\|F(Z)\|\,|Z|_{\mathrm{R}}^n \le c \tag{1.6}$$

on $\|Z\| < r$ for a positive integer n and positive real constants c, r, then $F_{\mathbf{e}}$ (resp. $F_{\mathbf{e}^\dagger}$) has a pole of order at most $[\frac{n}{2}]$ at $Z_{\mathbf{e}} = 0$ (resp. $Z_{\mathbf{e}^\dagger} = 0$).

We study characterization theorems of Riley type for bicomplex holomorphic functions on several unbounded domains. In many cases, we can determine the explicit form of a function from an estimate. We introduce some of our main results as follows. See Section 4 for the details.

Theorem 1.2. *Let F be a bicomplex holomorphic function on $\mathbb{BC} \setminus \mathfrak{S}_0$.*

(i) *If F satisfies the condition*

$$\|F(Z)\|\,|Z|_{\mathrm{R}}^n \le c \tag{1.7}$$

on $0 < |Z|_{\mathrm{R}} < r$ for a non-negative integer n and positive real constants c, r, then F is analytically continued to an entire function, that is a constant function.

(ii) *If F satisfies the condition*

$$\|F(Z)\|\,|Z|_{\mathrm{R}}^n \le c \tag{1.8}$$

on $r < |Z|_{\mathrm{R}}$ for a positive integer n and positive real constants c, r, then F is analytically continued to an entire function, that is the constant function 0.

(iii) *If F satisfies the condition*

$$\|F(Z)\|\,|Z|_{\mathrm{R}}^n \le c \tag{1.9}$$

on $r < \|Z\|$ for a positive integer n and positive real constants c, r, then F is analytically continued to an entire function, that is the constant function 0.

Note that our results are not obtained immediately from the idempotent representation (1.3) of bicomplex holomorphic functions. For some characterization theorems of bicomplex functions by k-modulus, see [1]. For some

characterization theorems of bicomplex polynomials with a kind of symmetry, see [3].

2. Preliminaries

2.1. Bicomplex numbers

In this section, we recall the definition and fundamental properties of bicomplex numbers. See [2] and [4] for more details.

Let $\mathbb{C}$ be the field of complex numbers with the imaginary unit i. The set of bicomplex numbers is defined by

$$\mathbb{BC} = \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}, \tag{2.1}$$

where j is another imaginary unit independent of and commuting with i:

$$i \neq j, \quad ij = ji, \quad i^2 = j^2 = -1. \tag{2.2}$$

Defining the addition and multiplication naturally, $\mathbb{BC}$ has a structure of a commutative ring. The set of zero divisors of $\mathbb{BC}$ with 0 is described as

$$\mathfrak{S}_0 = \{Z = z_1 + z_2 j \in \mathbb{BC} \mid z_1^2 + z_2^2 = 0\}, \tag{2.3}$$

that is equal to the set of non-unit elements of $\mathbb{BC}$. Setting

$$\mathbf{e} = \frac{1+ij}{2}, \quad \mathbf{e}^\dagger = \frac{1-ij}{2}, \tag{2.4}$$

$\mathbf{e}$ and $\mathbf{e}^\dagger$ are the non-complex idempotent elements satisfying the property $\mathbf{e}\mathbf{e}^\dagger = 0$.

We define the surjective ring homomorphisms $\Phi_{\mathbf{e}} \colon \mathbb{BC} \longrightarrow \mathbb{C}$, $\Phi_{\mathbf{e}^\dagger} \colon \mathbb{BC} \longrightarrow \mathbb{C}$ by

$$\Phi_{\mathbf{e}}(Z) = z_1 - z_2 i, \quad \Phi_{\mathbf{e}^\dagger}(Z) = z_1 + z_2 i \tag{2.5}$$

for $Z = z_1 + z_2 j \in \mathbb{BC}$, respectively. Then any bicomplex number Z has the idempotent representation

$$Z = \Phi_{\mathbf{e}}(Z)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger. \tag{2.6}$$

In this paper, we denote $\Phi_{\mathbf{e}}(Z)$ and $\Phi_{\mathbf{e}^\dagger}(Z)$ by $Z_{\mathbf{e}}$ and $Z_{\mathbf{e}^\dagger}$, respectively. By the idempotent representation, we have the equality

$$\mathfrak{S}_0 = \{Z = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathbb{BC} \mid Z_{\mathbf{e}} Z_{\mathbf{e}^\dagger} = 0\} = \mathbb{C}\mathbf{e} \cup \mathbb{C}\mathbf{e}^\dagger. \tag{2.7}$$

We define the set of hyperbolic numbers by

$$\mathbb{D} = \{X = x_1 + x_2 ij \in \mathbb{BC} \mid x_1, x_2 \in \mathbb{R}\}. \tag{2.8}$$

$\mathbb{D}$ is a subring of $\mathbb{BC}$. Since $X_{\mathbf{e}}, X_{\mathbf{e}^\dagger} \in \mathbb{R}$ for any hyperbolic number $X \in \mathbb{D}$, we define the partial order $\leq_{\mathbb{D}}$ on $\mathbb{D}$ by

$$X \leq_{\mathbb{D}} Y \quad \overset{\text{def}}{\Longleftrightarrow} \quad X_{\mathbf{e}} \leq Y_{\mathbf{e}} \text{ and } X_{\mathbf{e}^\dagger} \leq Y_{\mathbf{e}^\dagger}. \tag{2.9}$$

2.2. Bicomplex holomorphic functions

In this section, we recall the definition and fundamental properties of bicomplex holomorphic functions. See [2], [4] and [5] for more details.

For any bicomplex number $Z = z_1 + z_2 j = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathbb{BC}$, we define the norm $\|Z\|$ of Z by

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|Z_{\mathbf{e}}|^2 + |Z_{\mathbf{e}^\dagger}|^2}{2}}. \tag{2.10}$$

$\mathbb{BC}$ has a structure of a topological space induced by it. Then the topological space $\mathbb{BC}$ is isomorphic to the Euclidian space $\mathbb{C}^2$ and $\Phi_{\mathbf{e}}, \Phi_{\mathbf{e}^\dagger}$ are continuous and open.

Let $\Omega \subset \mathbb{BC}$ be an open set, $F\colon \Omega \longrightarrow \mathbb{BC}$ a bicomplex function on Ω and $Z_0 \in \Omega$. We say that F is bicomplex differentiable at Z_0 if the limit

$$\lim_{\substack{Z \to Z_0 \\ Z - Z_0 \notin \mathfrak{S}_0}} \frac{F(Z) - F(Z_0)}{Z - Z_0} \tag{2.11}$$

exists, which is denoted by $F'(Z_0)$. We also say that F is bicomplex holomorphic on Ω if F is bicomplex differentiable at any point of Ω. We denote the set of bicomplex holomorphic functions on Ω by $\mathcal{O}_{\mathbb{BC}}(\Omega)$.

The idempotent representation of bicomplex holomorphic functions, called Ringleb's theorem, plays an important role in bicomplex analysis.

Theorem 2.1. *Let F be a bicomplex function on Ω. Then F is bicomplex holomorphic on Ω if and only if there exist complex holomorphic functions $F_{\mathbf{e}}$, $F_{\mathbf{e}^\dagger}$ on $\Phi_{\mathbf{e}}(\Omega)$, $\Phi_{\mathbf{e}^\dagger}(\Omega)$ respectively such that F is of the form*

$$F(Z) = F_{\mathbf{e}}(Z_{\mathbf{e}})\mathbf{e} + F_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger. \tag{2.12}$$

By Theorem 2.1, we can immediately generalize fundamental properties of complex holomorphic functions in one variable such as Taylor's theorem, theorem of identity and so on to those of bicomplex holomorphic functions. Moreover, as a corollary of Theorem 2.1 we obtain the principle of trivial continuation.

Corollary 2.2. *Any bicomplex holomorphic function $F \in \mathcal{O}_{\mathbb{BC}}(\Omega)$ on Ω is analytically continued to a bicomplex holomorphic function on $\Phi_{\mathbf{e}}(\Omega)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(\Omega)\mathbf{e}^\dagger$.*

Note that $\Phi_{\mathbf{e}}(\Omega)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(\Omega)\mathbf{e}^\dagger$ is wider than Ω in general.

2.3. Riley's value

In [5], Riley studied some properties of bicomplex holomorphic functions not obtained immediately from ones of complex holomorphic functions in one variable and Theorem 2.1. A key tool is the following new value.

Definition ([5])**.** For any bicomplex number $Z = z_1 + z_2 j = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathbb{BC}$, we define the Riley's value $|Z|_{\mathrm{R}}$ of Z by

$$|Z|_{\mathrm{R}} = \sqrt{|z_1^2 + z_2^2|} = \sqrt{|Z_{\mathbf{e}}||Z_{\mathbf{e}^\dagger}|}. \tag{2.13}$$

The Riley's value $|Z|_{\mathrm{R}}$ of $Z \in \mathbb{BC}$ is considered to measure a kind of distance from a bicomplex number Z to the zero divisors $\mathfrak{S}_0$. Note that the norm does not satisfy the multiplicativity $\|ZW\| \neq \|Z\|\,\|W\|$ in general but the Riley's value satisfies the one $|ZW|_{\mathrm{R}} = |Z|_{\mathrm{R}}\,|W|_{\mathrm{R}}$. We have the following lemmas, which can be easily proved and play important roles in the proofs of our main results.

Lemma 2.3. *For any bicomplex number $Z \in \mathbb{BC}$, we have*

$$|Z|_{\mathrm{R}} \leq \|Z\|\,. \tag{2.14}$$

Lemma 2.4. *Let $\Omega \subset \mathbb{BC}$ be a domain. For any bicomplex holomorphic function $F \in \mathcal{O}_{\mathbb{BC}}(\Omega)$, the following two conditions are equivalent*:

(i) $|F(Z)|_{\mathrm{R}} \equiv 0$ *on* Ω,
(ii) $F_{\mathbf{e}}(Z_{\mathbf{e}}) \equiv 0$ *on* $\Phi_{\mathbf{e}}(\Omega)$ *or* $F_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger}) \equiv 0$ *on* $\Phi_{\mathbf{e}^\dagger}(\Omega)$.

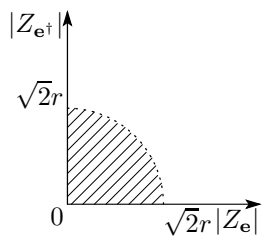
Lemma 2.5. *For any two positive real numbers $0 < s_1 \leq s_2$, there exists a bicomplex number Z such that $|Z|_{\mathrm{R}} = s_1$ and $\|Z\| = s_2$.*

3. Riley's theorem

Let $r > 0$ be a positive real number. We set

$$\Delta(r) = \{Z \in \mathbb{BC} \mid \|Z\| < r\}. \tag{3.1}$$

$\Delta(r)$ is described as a neighborhood of the origin:



In [5], Riley studied some local properties of bicomplex holomorphic functions by several estimates on $\Delta(r) \setminus \mathfrak{S}_0$. Before introducing Riley's result, let us define some terminologies in bicomplex analysis.

Definition. (i) Let $F \in \mathcal{O}_{\mathbb{BC}}(\Delta(r) \setminus \mathfrak{S}_0)$ be a bicomplex holomorphic function on $\Delta(r) \setminus \mathfrak{S}_0$ and N a hyperbolic number with $0 \leq_{\mathbb{D}} N$. We say that F has a pole of order at most N at $Z = 0$ if both $F_{\mathbf{e}}$ has a pole of order at most $N_{\mathbf{e}}$ at $Z_{\mathbf{e}} = 0$ and $F_{\mathbf{e}^\dagger}$ has a pole of at most $N_{\mathbf{e}^\dagger}$ at $Z_{\mathbf{e}^\dagger} = 0$.

(ii) Let $F \in \mathcal{O}_{\mathbb{BC}}(\Delta(r))$ be a bicomplex holomorphic function on $\Delta(r)$. We say that F has a strong zero at $Z = 0$ if both $F_{\mathbf{e}}$ has a zero at $Z_{\mathbf{e}} = 0$ and $F_{\mathbf{e}^\dagger}$ has a zero at $Z_{\mathbf{e}^\dagger} = 0$. We say that F has a weak zero at $Z = 0$ if either $F_{\mathbf{e}}$ has a zero at $Z_{\mathbf{e}} = 0$ or $F_{\mathbf{e}^\dagger}$ has a zero at $Z_{\mathbf{e}^\dagger} = 0$. Let $N_{\mathbf{e}}$ (resp. $N_{\mathbf{e}^\dagger}$) be the order of zero of $F_{\mathbf{e}}$ at $Z_{\mathbf{e}} = 0$ (resp. $F_{\mathbf{e}^\dagger}$ at $Z_{\mathbf{e}^\dagger} = 0$). We define the order of zero of F at $Z = 0$ by $N = N_{\mathbf{e}}\mathbf{e} + N_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathbb{D}$ with $0 \leq_{\mathbb{D}} N$.

Theorem 3.1 ([5]). *Let $F \in \mathcal{O}_{\mathbb{BC}}(\Delta(r) \setminus \mathfrak{S}_0)$ be a bicomplex holomorphic function on $\Delta(r) \setminus \mathfrak{S}_0$.*

(i) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}}\, |Z|_{\mathrm{R}}^n \leq c \tag{3.2}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F has a pole of order at most n at the origin $0 \in \mathbb{BC}$.

(ii) *If F satisfies the condition*

$$\|F(Z)\|\, |Z|_{\mathrm{R}}^n \leq c \tag{3.3}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F has a pole of order at most $\left[\frac{n}{2}\right]$ at the origin $0 \in \mathbb{BC}$.

(iii) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}}\, \|Z\|^n \leq c \tag{3.4}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to a bicomplex holomorphic function on $\Delta(r)$.

(iv) *If F satisfies the condition*

$$\|F(Z)\|\, \|Z\|^n \leq c \tag{3.5}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to a bicomplex holomorphic function on $\Delta(r)$.

(v) *If F satisfies the condition*

$$|F(Z)|_{\mathrm{R}} \leq c\, |Z|_{\mathrm{R}}^n \tag{3.6}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to a bicomplex holomorphic function on $\Delta(r)$ and it has a strong zero of order at least n at the origin $0 \in \mathbb{BC}$.

(vi) *If F satisfies the condition*

$$\|F(Z)\| \leq c\, |Z|_{\mathrm{R}}^n \tag{3.7}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to a bicomplex holomorphic function on $\Delta(r)$, that is the constant function 0.

(vii) *If F satisfies the condition*

$$|F(Z)|_{\mathrm{R}} \leq c\,\|Z\|^n \tag{3.8}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to a bicomplex holomorphic function on $\Delta(r)$ and it has a weak zero of order N satisfying $N_{\mathbf{e}} + N_{\mathbf{e}^\dagger} \geq 2n$ at the origin $0 \in \mathbb{BC}$.

(viii) *If F satisfies the condition*

$$\|F(Z)\| \leq c\,\|Z\|^n \tag{3.9}$$

on $\Delta(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to a bicomplex holomorphic function on $\Delta(r)$ and it has a strong zero of order at least n at the origin $0 \in \mathbb{BC}$.

We omit the proof of Theorem 3.1. See [5] for it. Note that if a function F on $\mathbb{BC}\setminus\mathfrak{S}_0$ satisfies the condition $|F(Z)|_{\mathrm{R}} \equiv 0$, then F may have singularities along $\mathfrak{S}_0$ and we can not characterize them. In fact, $F(Z) = \frac{1}{Z}\mathbf{e}$ and $G(Z) = Z\mathbf{e}$ are holomorphic functions on $\mathbb{BC} \setminus \mathfrak{S}_0$ satisfying $|F(Z)|_{\mathrm{R}} = |G(Z)|_{\mathrm{R}} \equiv 0$ on $\mathbb{BC} \setminus \mathfrak{S}_0$. Note also that each order of poles or zeros in Theorem 3.1 is best possible.

4. Main results

In this section, we generalize Theorem 3.1 and characterize bicomplex holomorphic functions on $\mathbb{BC}\setminus\mathfrak{S}_0$ from estimates of Riley type on several unbounded domains.

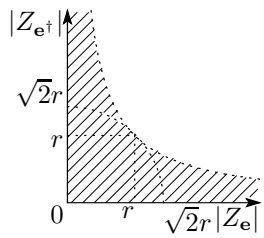
4.1. Characterizations from estimates on $|Z|_{\mathbf{R}} < r$

For a positive real number $r > 0$, we set

$$\Delta_{\mathrm{R}}(r) = \{Z \in \mathbb{BC} \mid |Z|_{\mathrm{R}} < r\}, \tag{4.1}$$

$$\Lambda_{\mathrm{R}}(r) = \{Z \in \mathbb{BC} \mid |Z|_{\mathrm{R}} = r\}. \tag{4.2}$$

$\Delta_{\mathrm{R}}(r)$ is described as a neighborhood of the set of zero divisors $\mathfrak{S}_0$:



First, we give characterizations from estimates of Riley type on $\Delta_{\mathrm{R}}(r) \setminus \mathfrak{S}_0$. Since we have $\Delta(r) \setminus \mathfrak{S}_0 \subset \Delta_{\mathrm{R}}(r) \setminus \mathfrak{S}_0$ by Lemma 2.3, we can apply Theorem 3.1. Since we consider an estimate of Riley type on the wider domain, we obtain

more explicit results than Riley's theorem (Theorem 3.1). In some cases, we can determine the explicit form of a function.

Theorem 4.1. *Let $F \in \mathcal{O}_{\mathbb{BC}}(\mathbb{BC} \setminus \mathfrak{S}_0)$ be a bicomplex holomorphic function on $\mathbb{BC} \setminus \mathfrak{S}_0$.*

(i) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}}\, |Z|_{\mathrm{R}}^n \leq c \tag{4.3}$$

on $\Delta_{\mathrm{R}}(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then there exist an integer $m \leq n$ and a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that $F(Z) = CZ^{-m}$ on $\mathbb{BC} \setminus \mathfrak{S}_0$. Moreover, if $m \leq 0$, F is analytically continued to an entire function.

(ii) *If F satisfies the condition*

$$\|F(Z)\|\, |Z|_{\mathrm{R}}^n \leq c \tag{4.4}$$

on $\Delta_{\mathrm{R}}(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is a constant function.

(iii) *If F satisfies the condition*

$$|F(Z)|_{\mathrm{R}}\, \|Z\|^n \leq c \tag{4.5}$$

on $\Delta_{\mathrm{R}}(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F satisfies the condition $|F(Z)|_{\mathrm{R}} \equiv 0$ on $\mathbb{BC} \setminus \mathfrak{S}_0$.

(iv) *If F satisfies the condition*

$$\|F(Z)\|\, \|Z\|^n \leq c \tag{4.6}$$

on $\Delta_{\mathrm{R}}(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(v) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}} \leq c\, |Z|_{\mathrm{R}}^n \tag{4.7}$$

on $\Delta(r)_{\mathrm{R}} \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function and there exist an integer $m \geq n$ and a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that $F(Z) = CZ^m$ on $\mathbb{BC}$.

(vi) *If F satisfies the condition*

$$\|F(Z)\| \leq c\, |Z|_{\mathrm{R}}^n \tag{4.8}$$

on $\Delta_{\mathrm{R}}(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(vii) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}} \leq c\, \|Z\|^n \tag{4.9}$$

on $\Delta_{\mathrm{R}}(r)\setminus\mathfrak{S}_0$ *for a positive integer* n *and a positive real constant* $c>0$*, then* F *is analytically continued to an entire function and has a weak zero of order* N *satisfying* $N_{\mathbf{e}}+N_{\mathbf{e}^\dagger}\geq 2n$*. Moreover, suppose that* $N_{\mathbf{e}}\geq N_{\mathbf{e}^\dagger}$*,* $F_{\mathbf{e}^\dagger}$ *is a polynomial of degree at most* $2n+N_{\mathbf{e}}$ *and we have the following*:

(vii-a) *In the case of* $2n+N_{\mathbf{e}^\dagger}\geq N_{\mathbf{e}}$*,* $F_{\mathbf{e}}$ *is a polynomial of degree at most* $2n+N_{\mathbf{e}^\dagger}$*,*

(vii-b) *In the case of* $2n+N_{\mathbf{e}^\dagger}<N_{\mathbf{e}}$*, there exists a constant* $c'\in\mathbb{C}\setminus\{0\}$ *such that* $F_{\mathbf{e}}(Z_{\mathbf{e}})=c'Z_{\mathbf{e}}^{N_{\mathbf{e}}}$*.*

(viii) *If* F *satisfies the condition*

$$\|F(Z)\|\leq c\,\|Z\|^n \tag{4.10}$$

on $\Delta_{\mathrm{R}}(r)\setminus\mathfrak{S}_0$ *for a non-negative integer* n *and a positive real constant* $c>0$*, then* F *is analytically continued to an entire function and there exists a constant* $C\in\mathbb{BC}$ *such that* $F(Z)=CZ^n$ *on* $\mathbb{BC}$*.*

Proof. (i) In the case of $n=0$, we obtain (i) by Theorem 3.1(v) and Corollary 2.2. Let us consider the case of $n\geq 1$. By Theorem 3.1(i), F has a pole of order at most n at the origin. By Corollary 2.2, there exists an integer $m\leq n$ such that $Z^mF(Z)$ is analytically continued to an entire function G satisfying $G(0)\neq 0$. We may assume that $G_{\mathbf{e}^\dagger}(0)\neq 0$. By (4.3) and the continuity of G, we have

$$|G(Z)|_{\mathrm{R}}\,|Z|_{\mathrm{R}}^{n-m}\leq c \tag{4.11}$$

on $\Delta_{\mathrm{R}}(r)$. For a positive real number $0<s<r$, we restrict the condition to $\Lambda_{\mathrm{R}}(s)$. By the condition $G_{\mathbf{e}^\dagger}(0)\neq 0$ and the continuity of $|G_{\mathbf{e}^\dagger}|$, for a sufficiently small positive real number $\delta>0$, $|G_{\mathbf{e}^\dagger}|$ has the positive minimum value $d>0$ on the compact set $\{z\in\mathbb{C}\,|\,|z|\leq\delta\}$. By (4.11), we have

$$|G_{\mathbf{e}}(Z_{\mathbf{e}})|\leq\frac{c^2}{s^{2(n-m)}|G_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})|}\leq\frac{c^2}{s^{2(n-m)}d} \tag{4.12}$$

on the set $\{z\in\mathbb{C}\,|\,|z|\geq\frac{s^2}{\delta}\}$. By Liouville's theorem, $G_{\mathbf{e}}$ is a constant function. By the condition $|G(Z)|_{\mathrm{R}}\not\equiv 0$, there exists a constant $c'\in\mathbb{C}\setminus\{0\}$ such that $G_{\mathbf{e}}(Z_{\mathbf{e}})\equiv c'$. Since $G_{\mathbf{e}}(0)\neq 0$, we can prove that $G_{\mathbf{e}^\dagger}$ is also a non-zero constant function, similarly. Thus G is a constant function, that is, there exists a constant $C\in\mathbb{BC}\setminus\mathfrak{S}_0$ such that $G(Z)\equiv C$. Therefore we have $F(Z)=CZ^{-m}$. Moreover, if $m\leq 0$, $F(Z)=CZ^{-m}$ is analytically continued to an entire function.

(ii) For a positive real number $0<s<r$, we restrict the condition to $\Lambda_{\mathrm{R}}(s)$. By (4.4), we have

$$|F_{\mathbf{e}}(Z_{\mathbf{e}})|\leq\frac{\sqrt{2}c}{s^n} \tag{4.13}$$

on $\mathbb{C}\setminus\{0\}$. By Riemann's theorem on removable singularities and Liouville's theorem, $F_{\mathbf{e}}$ is analytically continued to an entire function, that is a constant

function. Similarly, so is $F_{\mathbf{e}^\dagger}$. Therefore F is analytically continued to an entire function, that is a constant function.

(iii) Assume that $|F(Z)|_{\mathrm{R}} \not\equiv 0$. Since by Lemma 2.3 and (4.5) we have

$$|F(Z)|_{\mathrm{R}}\,|Z|_{\mathrm{R}}^{n} \le |F(Z)|_{\mathrm{R}}\,\|Z\|^{n} \le c \tag{4.14}$$

on $\Delta_{\mathrm{R}}(r)\setminus\mathfrak{S}_0$, by Theorem 4.1(i) there exist an integer $m \le n$ and a constant $C \in \mathbb{BC}\setminus\mathfrak{S}_0$ such that $F(Z) = CZ^{-m}$. Restricting the condition to $\Lambda_{\mathrm{R}}(s)$ for a positive real number $0 < s < r$, we have

$$|F(Z)|_{\mathrm{R}}\,\|Z\|^{n} = |C|_{\mathrm{R}}\,|Z|_{\mathrm{R}}^{-m}\,\|Z\|^{n} = |C|_{\mathrm{R}}\,s^{-m}\,\|Z\|^{n} \le c \tag{4.15}$$

on $\Lambda_{\mathrm{R}}(s)$. This contradicts Lemma 2.5.

(iv) By Lemma 2.3 and Theorem 4.1(ii), F is analytically continued to an entire function, that is a constant function $C \in \mathbb{BC}$. If C is not equal to 0, then by (4.6) $\|Z\|$ is bounded on $\Delta_{\mathrm{R}}(r)\setminus\mathfrak{S}_0$. This contradicts Lemma 2.5. Thus we have $C = 0$.

(v) By Theorem 3.1(v) and Corollary 2.2, F is analytically continued to an entire function and it has a strong zero of order at least n at $Z = 0$. By the factorization theorem of bicomplex holomorphic functions, there exist an integer $m \ge n$ and $G \in \mathcal{O}_{\mathbb{BC}}(\mathbb{BC})$ such that $F(Z) = Z^m G(Z)$ and $G(0) \ne 0$. By (4.7) and the continuity of G, we have

$$|G(Z)|_{\mathrm{R}}\,|Z|_{\mathrm{R}}^{m-n} \le c \tag{4.16}$$

on $\Delta_{\mathrm{R}}(r)$. By the proof of Theorem 4.1(i) and $G(0) \ne 0$, there exists a constant $C \in \mathbb{BC}\setminus\mathfrak{S}_0$ such that $G(Z) \equiv C$. Therefore we have $F(Z) = CZ^m$.

(vi) By Theorem 3.1(vi) and Corollary 2.2, F is analytically continued to an entire function, that is the constant function 0.

(vii) By Theorem 3.1(vii) and Corollary 2.2, F is analytically continued to an entire function and it has a weak zero of degree N at $Z = 0$, satisfying $N_{\mathbf{e}}+N_{\mathbf{e}^\dagger} \ge 2n$. By the factorization theorem of complex holomorphic functions, there exist an entire function G such that F is of the form

$$F(Z) = Z_{\mathbf{e}}^{N_{\mathbf{e}}} G_{\mathbf{e}}(Z_{\mathbf{e}})\mathbf{e} + Z_{\mathbf{e}^\dagger}^{N_{\mathbf{e}^\dagger}} G_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger \tag{4.17}$$

and $G(0) \notin \mathfrak{S}_0$. By (4.9) and the continuity of G, we have

$$|Z_{\mathbf{e}}|^{N_{\mathbf{e}}}|Z_{\mathbf{e}^\dagger}|^{N_{\mathbf{e}^\dagger}}|G_{\mathbf{e}}(Z_{\mathbf{e}})||G_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})| \le \frac{c^2}{2^n}(|Z_{\mathbf{e}}|^2 + |Z_{\mathbf{e}^\dagger}|^2)^n \tag{4.18}$$

on $\Delta_{\mathrm{R}}(r)$.

Suppose that $N_{\mathbf{e}} \ge N_{\mathbf{e}^\dagger}$. For a positive real number $0 < s < r$, we restrict (4.18) to the set $\{Z \in \mathbb{BC} \mid |Z|_{\mathrm{R}} = s,\ |Z_{\mathbf{e}}| > |Z_{\mathbf{e}^\dagger}|\}$. Then we have

$$s^{2N_{\mathbf{e}^\dagger}}|Z_{\mathbf{e}}|^{N_{\mathbf{e}}-N_{\mathbf{e}^\dagger}}|G_{\mathbf{e}}(Z_{\mathbf{e}})||G_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})| \le c^2|Z_{\mathbf{e}}|^{2n} \tag{4.19}$$

on the set $\{Z \in \mathbb{BC} \mid |Z|_{\mathrm{R}} = s,\ |Z_{\mathbf{e}}| > |Z_{\mathbf{e}^\dagger}|\}$. By the condition $G_{\mathbf{e}^\dagger}(0) \ne 0$ and the continuity of $|G_{\mathbf{e}^\dagger}|$, for a sufficiently small positive real number $\delta > 0$, $|G_{\mathbf{e}^\dagger}|$

has the positive minimum value $d > 0$ on the compact set $\{z \in \mathbb{C} \mid |z| \leq \delta\}$. By (4.19), we have

$$|G_{\mathbf{e}}(Z_{\mathbf{e}})| \leq \frac{c^2}{s^{2N_{\mathbf{e}^\dagger}}|G_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})|}|Z_{\mathbf{e}}|^{2n-N_{\mathbf{e}}+N_{\mathbf{e}^\dagger}} \leq \frac{c^2}{s^{2N_{\mathbf{e}^\dagger}}d}|Z_{\mathbf{e}}|^{2n-N_{\mathbf{e}}+N_{\mathbf{e}^\dagger}} \tag{4.20}$$

on the set $\{z \in \mathbb{C} \mid |z| \geq \frac{s^2}{\delta}\}$.
(vii-a) In the case of $2n + N_{\mathbf{e}^\dagger} \geq N_{\mathbf{e}}$, by (4.20) $G_{\mathbf{e}}$ is a polynomial of degree at most $2n - N_{\mathbf{e}} + N_{\mathbf{e}^\dagger}$. Thus $F_{\mathbf{e}}$ is a polynomial of degree at most $2n + N_{\mathbf{e}^\dagger}$.
(vii-b) In the case of $2n + N_{\mathbf{e}^\dagger} < N_{\mathbf{e}}$, by (4.20) we have

$$|G_{\mathbf{e}}(Z_{\mathbf{e}})| \leq \frac{c^2}{s^{2N_{\mathbf{e}^\dagger}}d}|Z_{\mathbf{e}}|^{2n-N_{\mathbf{e}}+N_{\mathbf{e}^\dagger}} \leq \frac{c^2 s^{4n-2N_{\mathbf{e}}}}{d\delta^{2n-N_{\mathbf{e}}+N_{\mathbf{e}^\dagger}}} \tag{4.21}$$

on the set $\{z \in \mathbb{C} \mid |z| \geq \frac{s^2}{\delta}\}$. By Liouville's theorem, $G_{\mathbf{e}}$ is a constant function. By $G_{\mathbf{e}}(0) \neq 0$, there exists a constant $c' \in \mathbb{C} \setminus \{0\}$ such that $G_{\mathbf{e}}(Z_{\mathbf{e}}) \equiv c'$. Thus we have $F_{\mathbf{e}}(Z_{\mathbf{e}}) = c' Z_{\mathbf{e}}^{N_{\mathbf{e}}}$.

Similarly, we obtain that $G_{\mathbf{e}^\dagger}$ is a polynomial of degree at most $2n+N_{\mathbf{e}}-N_{\mathbf{e}^\dagger}$. Thus $F_{\mathbf{e}^\dagger}$ is a polynomial of degree at most $2n + N_{\mathbf{e}}$ with a zero of order at least $N_{\mathbf{e}^\dagger}$ at $Z_{\mathbf{e}^\dagger} = 0$.

(viii) In the case of $n = 0$, we obtain (viii) by Theorem 4.1(ii). Let us consider the case of $n \geq 1$. By Theorem 3.1(viii) and Corollary 2.2, F is analytically continued to an entire function and it has a strong zero of order at least n at $Z = 0$. Then $G(Z) = \frac{F(Z)}{Z^n}$ is also an entire function. By (4.10) and limiting $|Z_{\mathbf{e}^\dagger}|$ to 0, we have

$$|G_{\mathbf{e}}(Z_{\mathbf{e}})| \leq \frac{c}{2^{\frac{n-1}{2}}} \tag{4.22}$$

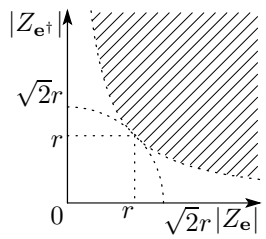
on $\mathbb{C}$. By Liouville's theorem, $G_{\mathbf{e}}$ is a constant function. Similarly, so is $G_{\mathbf{e}^\dagger}$. Thus G is a constant function, that is, there exists a constant $C \in \mathbb{BC}$ such that $G(Z) \equiv C$. Therefore we have $F(Z) = CZ^n$. □

4.2. Characterizations from estimates on $|Z|_{\mathbf{R}} > r$

For a positive real number $r > 0$, we set

$$\nabla_{\mathrm{R}}(r) = \{Z \in \mathbb{BC} \mid |Z|_{\mathrm{R}} > r\}. \tag{4.23}$$

$\nabla_{\mathrm{R}}(r)$ is described as an exterior domain of the set of zero divisors $\mathfrak{S}_0$:



In this subsection, we give characterizations from estimates of Riley type on $\nabla_{\mathrm{R}}(r)$. In spite of the shape of domains, we obtain a similar result as Theorem 4.1.

Theorem 4.2. *Let $F \in \mathcal{O}_{\mathbb{BC}}(\mathbb{BC} \setminus \mathfrak{S}_0)$ be a bicomplex holomorphic function on $\mathbb{BC} \setminus \mathfrak{S}_0$.*

(i) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}}\, |Z|_{\mathrm{R}}^n \leq c \tag{4.24}$$

on $\nabla_{\mathrm{R}}(r)$ for a positive integer n and a positive real constant $c > 0$, then there exist an integer $m \geq n$ and a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that $F(Z) = CZ^{-m}$ on $\mathbb{BC} \setminus \mathfrak{S}_0$.

(ii) *If F satisfies the condition*

$$\|F(Z)\|\, |Z|_{\mathrm{R}}^n \leq c \tag{4.25}$$

on $\nabla_{\mathrm{R}}(r)$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(iii) *If F satisfies the condition*

$$|F(Z)|_{\mathrm{R}}\, \|Z\|^n \leq c \tag{4.26}$$

on $\nabla_{\mathrm{R}}(r)$ for a positive integer n and a positive real constant $c > 0$, then F satisfies the condition $|F(Z)|_{\mathrm{R}} \equiv 0$ on $\mathbb{BC} \setminus \mathfrak{S}_0$.

(iv) *If F satisfies the condition*

$$\|F(Z)\|\, \|Z\|^n \leq c \tag{4.27}$$

on $\nabla_{\mathrm{R}}(r)$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(v) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}} \leq c\, |Z|_{\mathrm{R}}^n \tag{4.28}$$

on $\nabla_{\mathrm{R}}(r)$ for a non-negative integer n and a positive real constant $c > 0$, then there exist an integer $m \leq n$ and a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that $F(Z) = CZ^m$ on $\mathbb{BC} \setminus \mathfrak{S}_0$. Moreover, if $m \geq 0$, F is analytically continued to an entire function.

(vi) *If F satisfies the condition*

$$\|F(Z)\| \leq c\, |Z|_{\mathrm{R}}^n \tag{4.29}$$

on $\nabla_{\mathrm{R}}(r)$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(vii) *If F is an entire function and satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}} \leq c\, \|Z\|^n \tag{4.30}$$

on $\nabla_{\mathrm{R}}(r)$ *for a non-negative integer* n *and a positive real constant* $c > 0$*, then* F *is a polynomial of degree at most* $2n$*.*

(viii) *If* F *is an entire function and satisfies the condition*

$$\|F(Z)\| \leq c\,\|Z\|^n \tag{4.31}$$

on $\nabla_{\mathrm{R}}(r)$ *for a non-negative integer* n *and a positive real constant* $c > 0$*, then* F *is a polynomial of degree at most* n*.*

Proof. Note that by the transformation $W = \frac{1}{Z}$ of variables and applying Theorem 4.1 we obtain (i), (ii), (v), immediately.

(iii) Similarly to Theorem 4.1(iii), we obtain (iii).

(iv) Similarly to Theorem 4.1(iv), we obtain (iv).

(vi) Set $W = \frac{1}{Z}$ and $G(Z) = F(\frac{1}{Z})$. Then we have $G \in \mathcal{O}_{\mathbb{BC}}(\Delta_{\mathrm{R}}(\frac{1}{r}) \setminus \mathfrak{S}_0)$ and by (4.29) we have

$$\|G(W)\|\,|W|_{\mathrm{R}}^n \leq c \tag{4.32}$$

on $\Delta_{\mathrm{R}}(\frac{1}{r}) \setminus \mathfrak{S}_0$. By Theorem 4.1(ii), G is analytically continued to an entire function, that is a constant function. Thus F is a constant function on $\mathbb{BC} \setminus \mathfrak{S}_0$. By (4.29) and limiting Z to 0, F is also analytically continued to an entire function, that is the constant function 0.

(vii) By the condition $|F(Z)|_{\mathrm{R}} \not\equiv 0$, there exists a point $Z_0 = Z_{0\mathbf{e}}\mathbf{e} + Z_{0\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \nabla_{\mathrm{R}}(r)$ such that $F_{\mathbf{e}^\dagger}(Z_{0\mathbf{e}^\dagger}) \neq 0$ and $|Z_{0\mathbf{e}}| > |Z_{0\mathbf{e}^\dagger}|$. By (4.30), we have

$$|F_{\mathbf{e}}(Z_{\mathbf{e}})| \leq \frac{c^2}{2^n|F_{\mathbf{e}^\dagger}(Z_{0\mathbf{e}^\dagger})|}(|Z_{\mathbf{e}}|^2 + |Z_{0\mathbf{e}^\dagger}|^2)^n < \frac{c^2}{|F_{\mathbf{e}^\dagger}(Z_{0\mathbf{e}^\dagger})|}|Z_{\mathbf{e}}|^{2n} \tag{4.33}$$

on the set $\{z \in \mathbb{C} \mid |z| > |Z_{0\mathbf{e}}|\}$. Then $F_{\mathbf{e}}$ is a polynomial of degree at most $2n$. Similarly, so is $F_{\mathbf{e}^\dagger}$. Therefore F is a polynomial of degree at most $2n$.

(viii) Taking a point $Z_0 \in \nabla_{\mathrm{R}}(r)$ satisfying $|Z_{0\mathbf{e}}| > |Z_{0\mathbf{e}^\dagger}|$, by (4.31) we have

$$|F_{\mathbf{e}}(Z_{\mathbf{e}})| \leq \sqrt{2}\left\|F(Z\mathbf{e} + Z_0\mathbf{e}^\dagger)\right\| \leq \sqrt{2}c\left\|Z\mathbf{e} + Z_0\mathbf{e}^\dagger\right\|^n \leq \sqrt{2}c|Z_{\mathbf{e}}|^n \tag{4.34}$$

on the set $\{z \in \mathbb{C} \mid |z| > |Z_{0\mathbf{e}}|\}$. Then $F_{\mathbf{e}}$ is a polynomial of degree at most n. Similarly, so is $F_{\mathbf{e}^\dagger}$. Therefore F is a polynomial of degree at most n. □

Remark 4.3. (i) In Theorem 4.2(vii) and (viii), if $n \geq 1$, we could not remove the assumption that F is an entire function. In fact, $F(Z) = \frac{cr^{2n}}{2^{\frac{n-1}{2}}Z^n}$ is a counter-example. The situation is different from Theorem 4.4 below.

(ii) In Theorem 4.2(vii), the degree of the polynomial F is best possible. In fact, $F(Z) = \frac{c}{2^n}\mathbf{e}Z^{2n} + c\mathbf{e}^\dagger$ is a polynomial of degree $2n$ satisfying (4.30).

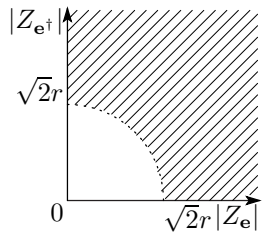
(iii) In Theorem 4.2(viii), the degree of the polynomial F is best possible. In fact, $F(Z) = \frac{c}{2^{\frac{n}{2}}}Z^n$ is a polynomial of degree n satisfying (4.31).

4.3. Characterizations from estimates on $\|Z\| > r$

For a positive real number $r > 0$, we set

$$\nabla(r) = \{Z \in \mathbb{BC} \mid \|Z\| > r\}. \tag{4.35}$$

$\nabla(r)$ is described as an exterior domain of the origin:



Finally, we give characterizations from estimates of Riley type on $\nabla(r) \setminus \mathfrak{S}_0$. In this case, the transformation $W = \frac{1}{Z}$ of variables does not work well directly, since the norm does not satisfy the multiplicativity in general. However, since we have $\nabla_{\mathrm{R}}(r) \subset \nabla(r) \setminus \mathfrak{S}_0$ by Lemma 2.3, we can apply Theorem 4.4.

Theorem 4.4. *Let $F \in \mathcal{O}_{\mathbb{BC}}(\mathbb{BC} \setminus \mathfrak{S}_0)$ be a bicomplex holomorphic function on $\mathbb{BC} \setminus \mathfrak{S}_0$.*

(i) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}}\, |Z|_{\mathrm{R}}^n \leq c \tag{4.36}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then there exists a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that $F(Z) = CZ^{-n}$ on $\mathbb{BC} \setminus \mathfrak{S}_0$.

(ii) *If F satisfies the condition*

$$\|F(Z)\|\, |Z|_{\mathrm{R}}^n \leq c \tag{4.37}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(iii) *If F satisfies the condition*

$$|F(Z)|_{\mathrm{R}}\, \|Z\|^n \leq c \tag{4.38}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F satisfies the condition $|F(Z)|_{\mathrm{R}} \equiv 0$ on $\mathbb{BC} \setminus \mathfrak{S}_0$.

(iv) *If F satisfies the condition*

$$\|F(Z)\|\, \|Z\|^n \leq c \tag{4.39}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(v) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}} \leq c\,|Z|_{\mathrm{R}}^{n} \tag{4.40}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function and there exists a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that $F(Z) = CZ^n$ on $\mathbb{BC}$.

(vi) *If F satisfies the condition*

$$\|F(Z)\| \leq c\,|Z|_{\mathrm{R}}^{n} \tag{4.41}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a positive integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is the constant function 0.

(vii) *If F satisfies the conditions $|F(Z)|_{\mathrm{R}} \not\equiv 0$ and*

$$|F(Z)|_{\mathrm{R}} \leq c\,\|Z\|^{n} \tag{4.42}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is a polynomial of degree at most $2n$.

(viii) *If F satisfies the condition*

$$\|F(Z)\| \leq c\,\|Z\|^{n} \tag{4.43}$$

on $\nabla(r) \setminus \mathfrak{S}_0$ for a non-negative integer n and a positive real constant $c > 0$, then F is analytically continued to an entire function, that is a polynomial of degree at most n.

Proof. Note that since we can apply Theorem 4.2 by Lemma 2.3 we obtain (ii), (iii), (iv), (vi), immediately.

(i) By Theorem 4.2(i), there exist a non-negative integer $m \geq n$ and a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that F is of the form $F(Z) = CZ^{-m}$. By (4.36), we have

$$|C|_{\mathrm{R}}\,|Z|_{\mathrm{R}}^{n-m} \leq c \tag{4.44}$$

on $\nabla(r) \setminus \mathfrak{S}_0$. By Lemma 2.5, this holds in the only case of $m = n$.

(v) By Theorem 4.2(v), there exist an integer $m \leq n$ and a constant $C \in \mathbb{BC} \setminus \mathfrak{S}_0$ such that F is of the form $F(Z) = CZ^m$. By (4.40), we have

$$|C|_{\mathrm{R}}\,|Z|_{\mathrm{R}}^{m} \leq c\,|Z|_{\mathrm{R}}^{n} \tag{4.45}$$

on $\nabla(r) \setminus \mathfrak{S}_0$. By Lemma 2.5, this holds in the only case of $m = n$.

(vii) By the condition $|F(Z)|_{\mathrm{R}} \not\equiv 0$, there exists a point $Z_0 = Z_{0\mathbf{e}}\mathbf{e} + Z_{0\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \nabla(r) \setminus \mathfrak{S}_0$ such that $F_{\mathbf{e}^\dagger}(Z_{0\mathbf{e}^\dagger}) \neq 0$ and $|Z_{0\mathbf{e}^\dagger}| > r$. By (4.42), we have

$$|F_{\mathbf{e}}(Z_{\mathbf{e}})| \leq \frac{c^2}{2^n |F_{\mathbf{e}^\dagger}(Z_{0\mathbf{e}^\dagger})|}(|Z_{\mathbf{e}}|^2 + |Z_{0\mathbf{e}^\dagger}|^2)^n \tag{4.46}$$

on a complex domain $\{z \in \mathbb{C} \,|\, 0 < |z| \ll 1\}$. By Riemann's theorem on removable singularities, $F_{\mathbf{e}}$ is analytically continued to an entire function. Similarly, so is $F_{\mathbf{e}^\dagger}$. Then by Theorem 4.2(vii) F is a polynomial of degree at most $2n$.

(viii) Taking a point $Z_0 \in \nabla(r) \setminus \mathfrak{S}_0$ satisfying $|Z_{0\mathbf{e}^\dagger}| > r$, by (4.43) we have

$$\begin{aligned} |F_{\mathbf{e}}(Z_{\mathbf{e}})| &\leq \sqrt{2}\left\|F(Z\mathbf{e} + Z_0\mathbf{e}^\dagger)\right\| \\ &\leq \sqrt{2}c\left\|Z\mathbf{e} + Z_0\mathbf{e}^\dagger\right\|^n = \frac{c}{2^{\frac{n-1}{2}}}(|Z_{\mathbf{e}}|^2 + |Z_{0\mathbf{e}^\dagger}|^2)^n \end{aligned} \tag{4.47}$$

on a complex domain $\{z \in \mathbb{C} \,|\, 0 < |z| \ll 1\}$. By Riemann's theorem on removable singularities, $F_{\mathbf{e}}$ is analytically continued to an entire function. Similarly, so is $F_{\mathbf{e}^\dagger}$. Then by Theorem 4.2(viii) F is a polynomial of degree at most n. $\square$

Remark 4.5. (i) In Theorem 4.4(vii), the degree of the polynomial F is best possible. In fact, $F(Z) = \frac{c}{2^n}\mathbf{e}Z^{2n} + c\mathbf{e}^\dagger$ is a polynomial of degree $2n$ satisfying (4.42).

(ii) In Theorem 4.4(viii), the degree of the polynomial F is best possible. In fact, $F(Z) = \frac{c}{2^{\frac{n}{2}}}Z^n$ is a polynomial of degree n satisfying (4.43).

Conclusion

We summarize this paper as follows.

- Riley studied some local properties of bicomplex holomorphic functions by several estimates, called estimates of Riley type, on a bounded domain. See Theorem 3.1 for the details.
- We studied some global characterizations of bicomplex holomorphic functions from estimates of Riley type on several unbounded domains. See Section 4 for the details.
- By assuming estimates of Riley type on these unbounded domains, we obtain more detailed and explicit results than Riley's ones. Especially, we can determine the explicit form of a function in many cases. See Theorems 4.1, 4.2 and 4.4 for the details.

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