UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING PASCAL DISTRIBUTION SERIES

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Abstract. The aim of this article is to make a connection between the Pascal distribution series and some subclasses of normalized analytic functions whose coefficients are probabilities of the Pascal distribution. To be more precise, we investigate such connections with the classes of analytic univalent functions with positive coefficients in the open unit disk U.

1. Introduction

Let \( H \) denote the class of analytic functions in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) on the complex plane \( \mathbb{C} \). Also, let \( A \) denote the subclass of \( H \) comprising of functions \( f \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \), and let \( S \subset A \) denote the class of functions which are univalent in \( U \). For the functions \( f \in A \) given by

\[
(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U,
\]

and \( g \in A \) given by \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \), we define the Hadamard product (or convolution product) of \( f \) and \( g \) by

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.
\]

Denote by \( V \) the subclass of \( A \) consisting of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \quad \text{with} \quad a_n \geq 0, \quad n \in \mathbb{N}, \quad n \geq 2.
\]

The class \( M(\alpha) \) of starlike functions of order \( \alpha \), with \( 1 < \alpha \leq \frac{4}{3} \), defined by

\[
M(\alpha) := \left\{ f \in A : \text{Re} \frac{zf'(z)}{f(z)} < \alpha, \quad z \in U \right\},
\]

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and the class $\mathcal{N}(\alpha)$ of convex functions of order $\alpha$, with $1 < \alpha \leq 4/3$, defined by

$$\mathcal{N}(\alpha) := \left\{ f \in \mathcal{A} : \text{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha, \ z \in \mathbb{U} \right\} = \left\{ f \in \mathcal{A} : zf'(z) \in \mathcal{M}(\alpha) \right\},$$

were introduced by Uralegaddi et al. [17] (see [4,6]). Also, let $\mathcal{M}^*(\alpha) \equiv \mathcal{M}(\alpha) \cap \mathcal{V}$ and $\mathcal{N}^*(\alpha) \equiv \mathcal{N}(\alpha) \cap \mathcal{V}$.

In this paper we consider two subclasses of $\mathcal{S}$, namely $\mathcal{M}(\lambda, \alpha)$ and $\mathcal{N}(\lambda, \alpha)$, to discuss some inclusion properties based on Pascal distribution.

For some $\alpha \ (1 < \alpha \leq 4/3)$ and $\lambda \ (0 \leq \lambda < 1)$, we introduce the classes $\mathcal{M}(\lambda, \alpha)$ and $\mathcal{N}(\lambda, \alpha)$ like two new subclass of $\mathcal{S}$ consisting of functions of the form and satisfying the analytic criteria

$$\mathcal{M}(\lambda, \alpha) := \left\{ f \in \mathcal{A} : \text{Re} \left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)}\right) < \alpha, \ z \in \mathbb{U} \right\},$$

$$\mathcal{N}(\lambda, \alpha) := \left\{ f \in \mathcal{A} : \text{Re} \left(f'(z) + zf''(z)f'(z) + \lambda zf''(z)\right) < \alpha, \ z \in \mathbb{U} \right\}.$$

We also let $\mathcal{M}^*(\lambda, \alpha) \equiv \mathcal{M}(\lambda, \alpha) \cap \mathcal{V}$, and $\mathcal{N}^*(\lambda, \alpha) \equiv \mathcal{N}(\lambda, \alpha) \cap \mathcal{V}$. Note that $\mathcal{M}(0, \alpha) := \mathcal{M}(\alpha)$, $\mathcal{N}(0, \alpha) := \mathcal{N}(\alpha)$, $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ are the subclasses of studied by Uralegaddi et al. [17].

It is well known that the special functions play an important role in Geometric Function Theory, especially in the proof given by de Branges [3] for the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized Gaussian hypergeometric functions [2,8,9,15,16].

A variable $x$ is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities $\left(1 - q\right)^m, \frac{qm(1-q)^m}{m}, \frac{q^2m(m+1)(1-q)^m}{2}, \frac{q^3m(m+1)(m+2)(1-q)^m}{3}, \ldots$ respectively, where $q$ and $m$ are the parameters, and thus

$$P(x = k) = \binom{k + m - 1}{m - 1} q^k (1 - q)^m, k = 0, 1, 2, 3, \ldots.$$  

Very recently, El-Deeb [7] introduced a power series whose coefficients are probabilities of Pascal distribution

$$\Phi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}(1 - q)^m z^n, \ z \in \mathbb{U},$$

where $m \geq 1, 0 \leq q \leq 1$, and we note that, by ratio test, the radius of convergence of above series is infinity.

We considered the linear operator

$$T_q^m : \mathcal{A} \to \mathcal{A}$$
defined by the convolution (or Hadamard) product

\[ T^m_q f(z) = \Phi^m_q(z) \ast f(z) = z + \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) q^{n-1} (1 - q)^m a_n z^n, \quad z \in \mathbb{U}, \]

where \( m \geq 1 \) and \( 0 \leq q \leq 1 \).

Motivated by the results in connections between various subclasses of analytic univalent functions, by using hypergeometric functions (see \([2,8,9,15,16]\)) and Poisson distributions \([10, 12, 14]\), we obtain necessary and sufficient condition for the function \( \Phi^m_q \) to be in the classes \( M^*\lambda,\alpha \) and \( N^*\lambda,\alpha \), and information regarding the images of functions belonging in \( R^*(A, B) \) by applying convolution operator. Finally, we obtain necessary and sufficient condition for the image of the function \( \Phi^m_q \) via the Alexander integral operator to be in the classes \( M^*\lambda,\alpha \) and \( N^*\lambda,\alpha \).

2. Inclusion results

We start with we recall the following results that we will use in our proofs.

**Lemma 2.1** ([11, Theorem 2.1.]). For some \( \alpha \left( 1 < \alpha \leq \frac{4}{3} \right) \) and \( \lambda \left( 0 \leq \lambda < 1 \right) \), and if \( f \in V \), then \( f \in M^*\lambda,\alpha \) if and only if

\[ \sum_{n=2}^{\infty} [n - (1 + n\lambda - \lambda)\alpha] a_n \leq \alpha - 1. \]

**Lemma 2.2** ([11, Theorem 2.2.]). For some \( \alpha \left( 1 < \alpha \leq \frac{4}{3} \right) \) and \( \lambda \left( 0 \leq \lambda < 1 \right) \), and if \( f \in V \), then \( f \in N^*\lambda,\alpha \) if and only if

\[ \sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda)\alpha] a_n \leq \alpha - 1. \]

In the sequel we use the following formulas:

\[
\begin{align*}
\sum_{n=0}^{\infty} \left( \frac{n + m - 1}{m - 1} \right) q^n &= \frac{1}{(1 - q)^m}, & \sum_{n=0}^{\infty} \left( \frac{n + m - 2}{m - 2} \right) q^n &= \frac{1}{(1 - q)^{m-1}}, \\
\sum_{n=0}^{\infty} \left( \frac{n + m}{m} \right) q^n &= \frac{1}{(1 - q)^{m+1}}, & \sum_{n=0}^{\infty} \left( \frac{n + m + 1}{m + 1} \right) q^n &= \frac{1}{(1 - q)^{m+2}}, & |q| < 1.
\end{align*}
\]

Further, throughout this paper unless otherwise stated we let \( m \geq 1 \) and \( 0 \leq q \leq 1 \).

**Theorem 2.3.** If \( m \geq 1 \), then \( \Phi^m_q \in M^*\lambda,\alpha \) if and only if

\[ \frac{(1 - \lambda \alpha)qm}{(1 - q) [2 - (1 - q)^m]} \leq \alpha - 1. \]
Proof. Since $\Phi_m^q$ is given by (2), according to Lemma 2.1 it is sufficient to prove that

$$L_1(m, \lambda, \alpha) := \sum_{n=2}^{\infty} [n - (1 + n\lambda - \lambda)\alpha] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} (1 - q)^m \leq \alpha - 1.$$ 

Now, it is easy to show that

$$L_1(m, \lambda, \alpha) = \sum_{n=2}^{\infty} n \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} (1 - q)^m - \lambda \alpha \sum_{n=2}^{\infty} (n - 1) \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} (1 - q)^m - \alpha \sum_{n=2}^{\infty} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} (1 - q)^m$$

$$= (1 - q)^m \sum_{n=2}^{\infty} (n - 1) \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} - \lambda \alpha (1 - q)^m \sum_{n=2}^{\infty} (n - 1) \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} - (1 - \alpha) (1 - q)^m \sum_{n=2}^{\infty} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}$$

$$= (1 - q)^m \left\{ (1 - \lambda \alpha) \sum_{n=2}^{\infty} qm \left(\frac{n + m - 2}{m}\right) q^{n-2} + (1 - \alpha) \sum_{n=2}^{\infty} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} \right\}$$

$$= (1 - q)^m \left\{ (1 - \lambda \alpha) \sum_{n=0}^{\infty} qm \left(\frac{n + m}{m}\right) q^n + (1 - \alpha) \left[ \sum_{n=0}^{\infty} \left(\frac{n + m - 1}{m - 1}\right) q^{n-1} \right] \right\}$$

$$= (1 - q)^m \left\{ (1 - \lambda \alpha) qm \frac{1}{(1 - q)^{m+1}} + (1 - \alpha) \left[ \frac{1}{(1 - q)^m - 1} \right] \right\}$$

$$= \frac{(1 - \lambda \alpha) qm}{1 - q} - (\alpha - 1) [1 - (1 - q)^m].$$

But this last expression is bounded above by $\alpha - 1$ if and only if (3) holds, which completes our proof. □
Theorem 2.4. If $m \geq 1$, then $\Phi_m^q \in N^\ast(\lambda, \alpha)$ if and only if

\begin{equation}
\frac{(1 - \lambda \alpha)m(m + 1)q^2}{[2 - (1 - q)m](1 - q)^2} + \frac{(3 - 2\lambda \alpha - \alpha)qm}{[2 - (1 - q)m](1 - q)} \leq \alpha - 1.
\end{equation}

Proof. Let $f$ be of the form (1) belong to the class $V$. By virtue of Lemma 2.2, it is sufficient to show that

$$L_2(m, \lambda, \alpha) = \sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda)n] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m \leq \alpha - 1.$$ 

Since

$$L_2(m, \lambda, \alpha) = \sum_{n=2}^{\infty} [(1 - \lambda \alpha)n^2 - \alpha(1 - \lambda)n] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m,$$

writing $n = (n - 1) + 1$, and $n^2 = (n - 1)(n - 2) + 3(n - 1) + 1$, we can rewrite the above term as

$$L_2(m, \lambda, \alpha) = (1 - \lambda \alpha) \sum_{n=2}^{\infty} (n - 1)(n - 2) \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m$$

$$+ (3 - 2\lambda \alpha - \alpha) \sum_{n=2}^{\infty} (n - 1) \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m$$

$$+ (1 - \alpha) \sum_{n=2}^{\infty} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m,$$

hence

$$L_2(m, \lambda, \alpha) = (1 - \lambda \alpha)q^2(1 - q)^m \sum_{n=3}^{\infty} (n - 1)(n - 2) \left(\frac{n + m - 2}{m - 1}\right) q^{n-3}$$

$$+ (3 - 2\lambda \alpha - \alpha)(1 - q)^m \sum_{n=2}^{\infty} q^n \left(\frac{n + m - 2}{m}\right) q^n$$

$$+ (1 - \alpha)(1 - q)^m \sum_{n=2}^{\infty} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}$$

$$= (1 - \lambda \alpha)m(m + 1)q^2(1 - q)^m \sum_{n=0}^{\infty} \left(\frac{n + m + 1}{m + 1}\right) q^n$$

$$+ (3 - 2\lambda \alpha - \alpha)qm(1 - q)^m \sum_{n=0}^{\infty} \left(\frac{n + m}{m}\right) q^n$$

$$+ (1 - \alpha)(1 - q)^m \left[\frac{1}{(1 - q)^m} - 1\right]$$

$$= \frac{(1 - \lambda \alpha)m(m + 1)q^2}{(1 - q)^2} + \frac{(3 - 2\lambda \alpha - \alpha)qm}{(1 - q)} - (\alpha - 1) [1 - (1 - q)^m].$$
But this expression is bounded above by $\alpha - 1$ if and only if (4) holds, and the proof is complete. □

By taking $\lambda = 0$ in the above two theorems we state the following corollary:

**Corollary 2.5.** If $m \geq 1$, then:

1. $\Phi_q^m \in \mathcal{M}^*(\alpha)$ if and only if
   \[
   \frac{qm}{2 - (1 - q)^m(1 - q)} \leq \alpha - 1;
   \]

2. $\Phi_q^m \in \mathcal{N}^*(\alpha)$ if and only if
   \[
   \frac{m(m + 1)q^2}{2 - (1 - q)^m(1 - q)^2} + \frac{(3 - \alpha)qm}{2 - (1 - q)^m(1 - q)} \leq \alpha - 1.
   \]

3. **Image properties of $I_q^m$ and $L_m$ operators**

   A function $f \in A$ is said to be in the class $\mathcal{R}^\tau(A, B)$, with $\tau \in \mathbb{C} \setminus \{0\}$ and $-1 \leq B < A \leq 1$, if it satisfies the inequality
   \[
   \left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U}.
   \]

   The class $\mathcal{R}^\tau(A, B)$ was introduced earlier by Dixit and Pal [5]. It is of interest to note that if we consider
   \[
   \tau = 1, \quad A = \beta, \quad B = -\beta \quad (0 < \beta \leq 1),
   \]
then we obtain the class of functions $f \in A$ satisfying the inequality
   \[
   \left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta, \quad z \in \mathbb{U},
   \]
which was studied by, among others, Padmanabhan [13] and Caplinger and Causey [1].

**Lemma 3.1** ([5, Theorem 1]). If $f \in \mathcal{R}^\tau(A, B)$ is of form (1), then
   \[
   |a_n| \leq (A - B)\frac{\tau}{n}, \quad n \in \mathbb{N} \setminus \{1\}.
   \]

The result is sharp.

Making use of Lemma 3.1 we will study the action of the Pascal distribution series on the class $\mathcal{R}^\tau(A, B) \cap \mathcal{V}$.

**Theorem 3.2.** Let $m \geq 1$, and $f \in \mathcal{R}^\tau(A, B) \cap \mathcal{V}$. Then, $I_q^m f \in \mathcal{N}^*(\lambda, \alpha)$ if
   \[
   \frac{(A - B)\tau(1 - \lambda \alpha)qm}{(1 - q)(1 + (A - B)\tau)(1 - (1 - q)^m)} \leq \alpha - 1.
   \]
Proof. Let $f$ be of the form (1) belong to the class $R^\tau(A, B) \cap V$. According to Lemma 2.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda\alpha)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m |a_n| \leq \alpha - 1.$$ 

Since $f \in R^\tau(A, B) \cap V$, then by Lemma 3.1 we have

$$|a_n| \leq \frac{|\tau|}{\tau}, \quad n \in N \setminus \{1\}.$$

Letting

$$L_3(m, \lambda, \alpha) := \sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda\alpha)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m |a_n|$$

we get

$$L_3(m, \lambda, \alpha) \leq (A - B)|\tau| \sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda\alpha)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m.$$ 

Proceeding like in the proof of Theorem 2.3 we get

$$L_3(m, \lambda, \alpha) \leq (A - B)|\tau| \left[\frac{1 - \lambda\alpha}{1 - q} - (\alpha - 1) \left(1 - (1 - q)^m\right)\right],$$

and this last expression is bounded above by $\alpha - 1$ if and only if (6) holds. □

For the special case $\lambda = 0$, the above theorem reduces to the following result:

**Corollary 3.3.** Let $m \geq 1$, and $f \in R^\tau(A, B) \cap V$. Then, $I^m_q f \in N^*(\alpha)$ if

$$(7) \quad \frac{(A - B)|\tau| \ qm}{(1 - q)[1 + (A - B)|\tau|][1 - (1 - q)^m]} \leq \alpha - 1.$$ 

**Theorem 3.4.** Let $f \in R^\tau(A, B) \cap V$. Then, $I^m_q f \in M^*(\lambda, \alpha)$ if

$$(8) \quad \frac{(A - B)|\tau|}{1 - (A - B)|\tau|}[1 - (1 - q)^m] \left\{1 - \alpha\lambda - \frac{(1 - \lambda)(1 - q)}{q(m - 1)} \left[1 - (1 - q)^{m-1}\right]\right\} \leq \alpha - 1.$$ 

**Proof.** Let $f$ be of the form (1) belong to the class $R^\tau(A, B) \cap V$. According to Lemma 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda\alpha)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m |a_n| \leq \alpha - 1.$$ 

Since $f \in R^\tau(A, B) \cap V$, then by Lemma 3.1 the inequality (5) holds. Letting

$$L_4(m, \lambda, \alpha) := \sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda\alpha)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m |a_n|$$
we get
\[ L_4(m, \lambda, \alpha) \leq (A - B) \left\| \tau \right\| \sum_{n=2}^{\infty} \left[ n - (1 + n\lambda - \lambda)\alpha \right] \frac{1}{n} \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \]
\[ = (A - B) \left\| \tau \right\| \sum_{n=2}^{\infty} \left[ (1 - \alpha\lambda) - \frac{\alpha}{n}(1 - \lambda) \right] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \]
\[ = (A - B) \left\| \tau \right\| (1 - q)^m \left\{ (1 - \alpha\lambda) \left[ \sum_{n=0}^{\infty} \left( \frac{n + m - 1}{m - 1} \right) q^n - 1 \right] q^{n-1} \right\} \]
\[ = (A - B) \left\| \tau \right\| (1 - q)^m \left\{ (1 - \alpha\lambda) \left[ \sum_{n=0}^{\infty} \left( \frac{n + m - 2}{m - 2} \right) q^n - 1 \right] - \frac{\alpha(1 - \lambda)}{q(m - 1)} \left[ \sum_{n=0}^{\infty} \left( \frac{n + m - 2}{m - 2} \right) q^n - 1 \right] \right\} \]
\[ = (A - B) \left\| \tau \right\| (1 - q)^m \left\{ (1 - \alpha\lambda) \left[ 1 - (1 - q)^m \right] - \frac{\alpha(1 - \lambda)}{q(m - 1)} \left[ 1 - (1 - q)(1 - q)^m \right] \right\} \]
and this last expression is bounded above by \( \alpha - 1 \) if and only if (8) holds. \( \Box \)

Putting \( \lambda = 0 \) in the above theorem we obtain the next special case:

**Corollary 3.5.** Let \( m \geq 1 \), and \( f \in \mathcal{R}^T(A, B) \cap \mathcal{V} \). Then, \( I_q^m f \in \mathcal{M}^\ast(\alpha) \) if
\[ (A - B) \left\| \tau \right\| (1 - q)^m \left\{ 1 - \frac{\alpha(1 - q)}{q(m - 1)} \left[ 1 - (1 - q)^m \right] \right\} \leq \alpha - 1. \]
The next results deal with the images of the function $\Phi_m^q$ by the well-known Alexander integral operator.

**Theorem 3.7.** If $m \geq 1$, then the function $L_m(z) := \int_0^z \Phi_m^q(t) \, dt \in N^*(\lambda, \alpha)$ if and only if

\[
\frac{(1 - \lambda \alpha)qm}{[2 - (1 - q)^m] (1 - q)} \leq \alpha - 1.
\]

**Proof.** Since

\[
L_m(z) = z + \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) q^{n-1} (1 - q)^m \frac{n}{n} z^n,
\]

we note that $\Phi_m^q(z) = z (L_m(z))'$, $z \in \mathbb{U}$, and by Theorem 2.3 we get

$L_m \in N^*(\lambda, \alpha) \iff \Phi_m^q \in M^*(\lambda, \alpha) \iff (10)$ holds. $\Box$

**Theorem 3.8.** Let $m \geq 1$. Then the function $L_m(z) = \int_0^z \Phi_m^q(t) \, dt \in M^*(\lambda, \alpha)$ if and only if

\[
\frac{1}{1 - (1 - q)^m} \left( 1 - \alpha \lambda \right) - \frac{\alpha(1 - \lambda)(1 - q)}{q(m - 1)} \left[ 1 - (1 - q)^{m-1} \right] \leq \alpha - 1.
\]

**Proof.** Since $L_m$ is given by (11), according to Lemma 2.1 it suffices to prove that

\[
\sum_{n=2}^{\infty} \left( n - (1 + n\lambda - \lambda) \alpha \right) \left( \frac{n + m - 2}{m - 1} \right) q^{n-1} (1 - q)^m \frac{1}{n} \leq \alpha - 1.
\]

Letting

\[
L_5(m, \lambda, \alpha) = \sum_{n=2}^{\infty} \left[ n - (1 + n\lambda - \lambda) \alpha \right] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1} (1 - q)^m
\]

and proceeding as in the proof of Theorem 3.4, we get

\[
L_5(m, \lambda, \alpha) = (1 - \alpha \lambda) + (\alpha - 1)(1 - q)^m - \frac{\alpha(1 - \lambda)(1 - q)}{q(m - 1)} \left[ 1 - (1 - q)^{m-1} \right]
\]

which is bounded above by $\alpha - 1$ if and only if (12) holds. $\Box$

Taking $\lambda = 0$ in the above two theorems we get the next special cases:
Corollary 3.9. If \( m \geq 1 \), and \( \mathcal{L}_m(z) = \int_0^z \frac{\Phi_0^m(t)}{t} dt \), then:

(i) \( \mathcal{L}_m \in \mathcal{N}^*(\alpha) \) if and only if

\[
\frac{q^m}{2 - (1-q)^m}(1-q) \leq \alpha - 1;
\]

(ii) \( \mathcal{L}_m \in \mathcal{M}^*(\alpha) \) if and only if

\[
\frac{1}{1 - (1-q)^m} \left\{ 1 - \frac{\alpha(1-q)}{q(m-1)} \left[ 1 - (1-q)^{m-1} \right] \right\} \leq \alpha - 1.
\]

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