POSITIVENESS FOR THE RIEMANNIAN GEODESIC BLOCK MATRIX

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Abstract. It has been shown that the geometric mean $A \# B$ of positive definite Hermitian matrices $A$ and $B$ is the maximal element $X$ of Hermitian matrices such that

$$
\begin{pmatrix}
  A & X \\
  X & B
\end{pmatrix}
$$

is positive semi-definite. As an extension of this result for the $2 \times 2$ block matrix, we consider in this article the block matrix $[[A \# w_{ij} B]]$ whose $(i, j)$ block is given by the Riemannian geodesics of positive definite Hermitian matrices $A$ and $B$, where $w_{ij} \in \mathbb{R}$ for all $1 \leq i, j \leq n$. Under certain assumption of the Loewner order for $A$ and $B$, we establish the equivalent condition for the parameter matrix $\omega = [w_{ij}]$ such that the block matrix $[[A \# w_{ij} B]]$ is positive semi-definite.

1. Introduction

Since Pusz and Woronowicz [7] have introduced a geometric mean of positive definite Hermitian matrices $A$ and $B$

$$
A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2},
$$

T. Ando [1, 6] has developed the theory of geometric means of positive definite matrices and operators. One of the important characterizations is the following:

$$
A \# B = \max \left\{ X : X = X^*, \begin{pmatrix} A & X \\ X & B \end{pmatrix} \succeq 0 \right\},
$$

which is very useful to prove the joint concavity of the geometric mean. Note that $X \preceq Y$ for Hermitian matrices $X$ and $Y$ means that $Y - X$ is positive semi-definite, known as the Loewner order. Furthermore, it is well known in [2, Chapter 6] that the geometric mean $A \# B$ coincides with the unique metric

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midpoint \((t = 1/2)\) of Riemannian geodesic \(\gamma\) connecting two points \(\gamma(0) = A\) and \(\gamma(1) = B\):
\[
\gamma(t) = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2} =: A\#_tB, \ t \in \mathbb{R}.
\]

From (1) we have obviously
\[
\begin{pmatrix}
A\#_0A & A\#_{1/2}B \\
A\#_{1/2}B & A\#_1B
\end{pmatrix}
= \begin{pmatrix}
A & A\#B \\
A\#B & B
\end{pmatrix} \geq 0
\]
for any positive definite matrices \(A\) and \(B\). As an extension of the above consequence, one can naturally ask whether or not the block matrix \(G(A_1, \ldots, A_m) = [[A_i \# A_j]]\) of positive definite matrices \(A_1, \ldots, A_m\) is positive semi-definite.

(3) \[G(A_1, \ldots, A_m) := \begin{bmatrix}
A_1 & A_1\# A_2 & \cdots & A_1\# A_m \\
A_2\# A_1 & A_2 & \cdots & A_2\# A_m \\
\vdots & \vdots & \ddots & \vdots \\
A_m\# A_1 & A_m\# A_2 & \cdots & A_m
\end{bmatrix}\]

Recently, the characterization of positive semi-definiteness of \(G(A_1, \ldots, A_m)\) in terms of the \(\Gamma\)-commuting family has been verified.

Theorem 1.1 ([5, Theorem 1.1]). For positive definite matrices \(A_1, \ldots, A_m\), the block matrix \(G(A_1, \ldots, A_m)\) is positive semi-definite if and only if \([A_i \# A_j]\) is \(\Gamma\)-commuting, that is, there exists an invertible matrix \(M\) such that \(MA_i M^*\) and \(MA_j M^*\) commute for all \(1 \leq i, j \leq m\).

In this article we consider another interesting block matrix
\[
G_\omega(A, B) := \begin{pmatrix}
A\#_{w_1}B & A\#_{w_2}B & \cdots & A\#_{w_m}B \\
A\#_{w_1}B & A\#_{w_2}B & \cdots & A\#_{w_m}B \\
\vdots & \vdots & \ddots & \vdots \\
A\#_{w_1}B & A\#_{w_2}B & \cdots & A\#_{w_m}B
\end{pmatrix}
\]
for given two positive definite matrices \(A\) and \(B\), where \(\omega = [w_{ij}] \in M_m\) whose entries \(w_{ij}\) are parameters in \(\mathbb{R}\) for \(1 \leq i, j \leq m\). Note that \(G_\omega(A, B)\) is another extended version of the 2-by-2 block matrix (2). For \(m \geq 3\) we give a sufficient condition for the parameter matrix \(\omega = [w_{ij}]\) that \(G_\omega(A, B)\) is positive semi-definite. Moreover, we establish the condition of \(\omega = [w_{ij}] \in M_3\) under certain assumption of the Loewner order for \(A\) and \(B\) that \(G_\omega(A, B)\) is positive semi-definite, and discuss some interesting connection with (doubly) stochastic matrix.

2. Riemannian geodesics

Let \(M_n := M_{n \times n}\) be the ring of all \(n \times n\) matrices with complex entries. Let \(H_n\) be the real vector space of all \(n \times n\) Hermitian matrices and let \(P_n \subset H_n\) be the open convex cone of all positive definite matrices. For any \(A, B \in H_n\) we write \(A \leq B\) if \(B - A\) is positive semi-definite, and \(A < B\) if \(B - A\) is positive definite. This is indeed a partial order on \(H_n\), known as the Loewner order.
Note that $A \geq 0 (> 0)$ means that $A$ is positive semi-definite (positive definite, respectively).

The cone $\mathbb{P}_n$ equipped with the trace metric $\langle X,Y \rangle_A = \text{tr}(A^{-1}XA^{-1}Y)$ for tangent vectors $X,Y \in \mathbb{H}_n$ at $A \in \mathbb{P}_n$ yields the Riemannian metric distance between $A, B \in \mathbb{P}_n$ given by $\delta(A,B) = \| \log A^{-1/2}BA^{-1/2} \|_2$, where $\|X\|_2$ denotes the Frobenius norm of $X$ $\in \mathbb{H}_n$. Note that $(\mathbb{P}_n, \delta)$ is a complete metric space satisfying a semi-parallelogram law, that is, a Hadamard space or CAT(0) space. The general linear group $GL_n$ acts isometrically on $\mathbb{P}_n$ via congruence transformations $\Gamma_X(A) = XAX^*$ for $X \in GL_n$ and $A \in \mathbb{P}_n$. Moreover, the inverse map preserves the Riemannian metric distance: $\delta(A,B) = \delta(A^{-1}, B^{-1})$ for any $A,B \in \mathbb{P}_n$.

The unique (up to parametrization) geodesic curve on $\mathbb{P}_n$ connecting from $A$ to $B$ is given by

$$R \ni t \mapsto A^\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}.$$ 

Especially, $A^\#_t B$ for $t \in [0,1]$ is called the two-variable weighted geometric mean of $A$ and $B$. Note that $A^\#_1 B = A^\#_{1/2} B$ is the unique midpoint of $A$ and $B$ with respect to the Riemannian trace metric, and is the unique solution $X \in \mathbb{P}_n$ of the Riccati equation $XA^{-1}X = B$. See [2] for more information.

We enumerate a few properties of the Riemannian geodesic of $A$ and $B$ that we will use in the following.

**Lemma 2.1.** Let $A, B, C, D \in \mathbb{P}_n$ and let $s,t,u \in \mathbb{R}$. Then the following are satisfied.

1. $A^\#_t B = A^{1-t}B^t$ if $A$ and $B$ commute.
2. $A^\#_t B = B^\#_{1-t} A$.
3. $A^\#_t B \leq C^\#_t D$ for $t \in [0,1]$ whenever $A \leq C$ and $B \leq D$.
4. $X(A^\#_t B)^* = (XAX^*)^\#_t (XBX^*)$ for any nonsingular matrix $X$.
5. $(A^\#_t B)^{-1} = A^{-1}^\#_t B^{-1}$.
6. $(A^\#_s B)^\#_t (A^\#_u B) = A^\#_{(1-t)s+tu} B$.

### 3. Schur complement

We recall some known results for the block matrix to be positive semi-definite.

**Theorem 3.1 ([4, Theorem 7.7.9]).** Let $A \in \mathbb{M}_m, B \in \mathbb{M}_n, C \in \mathbb{M}_{m \times n}$ and let

$$T = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}.$$ 

Then $T \geq 0$ if and only if

1. $A \geq 0$,
2. $B \geq 0$, and
3. $C = A^{1/2}EB^{1/2}$ for some contraction $E$, that is, $\|E\| \leq 1$ for the operator norm.
In fact, this statement holds for bounded operators on Hilbert spaces with infinite dimension [3].

**Definition.** Let
\[ M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \]
be the 2 \( \times \) 2 block matrix for \( A_{11} \in M_m \), \( A_{22} \in M_n \) and \( A_{12} \in M_{m \times n} \), \( A_{21} \in M_{n \times m} \). The Schur complement of \( A_{22} \), denoted by \( M/A_{22} \), is
\[ M/A_{22} = A_{11} - A_{12} A_{22}^{-1} A_{21}, \]
provided that \( A_{22} \) is nonsingular. Similarly, the Schur complement of \( A_{11} \), denoted by \( M/A_{11} \), is
\[ M/A_{11} = A_{22} - A_{21} A_{11}^{-1} A_{12}, \]
provided that \( A_{11} \) is nonsingular.

**Theorem 3.2** (Schur complement condition for positive semidefiniteness, [2, 4]). Let \( M \) be a 2 \( \times \) 2 block matrix partitioned as
\[ M = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}. \]
If \( A \) is invertible, then the followings are true:
(a) \( M > 0 \) if and only if \( A > 0 \) and \( M/A > 0 \).
(b) If \( A > 0 \), then \( M \geq 0 \) if and only if \( M/A \geq 0 \).

If \( B \) is invertible, then the followings are true:
(c) \( M > 0 \) if and only if \( B > 0 \) and \( M/B > 0 \).
(d) If \( B > 0 \), then \( M \geq 0 \) if and only if \( M/B \geq 0 \).

We provide the following which is useful to prove our main results.

**Lemma 3.3.** Let \( s, t, u \in \mathbb{R} \). Let \( M \) be a 2 \( \times \) 2 block Hermitian matrix partitioned as
\[ M = \begin{pmatrix} X^s & X^t \\ X^t & X^u \end{pmatrix}, \]
where \( X \in \mathbb{P}_n \). Then \( M \geq 0 \) if and only if one of the followings holds:
(i) \( t \leq \frac{s + u}{2} \) if \( X \geq I \);
(ii) \( t \geq \frac{s + u}{2} \) if \( X \leq I \);
(iii) \( t = \frac{s + u}{2} \) if neither \( X \geq I \) nor \( X \leq I \).

**Proof.** Note that \( X^t > 0 \) for any real number \( t \). By Theorem 3.2 \( M \geq 0 \) if and only if
\[ M/X^s = X^u - X^t X^{-s} X^t \geq 0. \]
By a simple calculation we obtain the equivalent condition that
\[ X^{s+u-2t} \geq I. \]
This is the same as another Schur complement case, \( M/X = X^s - X^t X^{-1} X^t \geq 0 \).

(i) If \( X \geq I \), then all eigenvalues of \( X \) are greater than or equal to 1. So \( s + u - 2t \geq 0 \), that is, \( t \leq \frac{s+u}{2} \).

(ii) If \( X \leq I \), then all eigenvalues of \( X \) are positive, and less than or equal to 1. So \( s + u - 2t \leq 0 \), that is, \( t \geq \frac{s+u}{2} \).

(iii) If neither \( X \geq I \) nor \( X \leq I \), then some of eigenvalues of \( X \) are greater than or equal to 1, and the others are less than or equal to 1. So it should be \( s + u - 2t = 0 \), that is, \( t = \frac{s+u}{2} \).

\[ \square \]

4. Main results

Let \( A, B \in \mathbb{P}_n \). Let \( \omega = [w_{ij}] \in \mathbb{M}_m \), where \( w_{ij} \) are parameters in \( \mathbb{R} \) for \( 1 \leq i, j \leq m \). Consider the \( m \times m \) block matrix consisting of Riemannian geodesics

\[
G_\omega(A, B) := \begin{pmatrix}
A \# w_{11} B & A \# w_{12} B & \cdots & A \# w_{1m} B \\
A \# w_{21} B & A \# w_{22} B & \cdots & A \# w_{2m} B \\
\vdots & \vdots & \ddots & \vdots \\
A \# w_{m1} B & A \# w_{m2} B & \cdots & A \# w_{mm} B
\end{pmatrix}.
\]

We call it the Riemannian geodesic block matrix of \( A \) and \( B \).

**Remark 4.1.** The Riemannian geodesic block matrix \( G_\omega(A, B) \) is Hermitian if and only if \( w_{ji} = w_{ij} \), that is, \( \omega = [w_{ij}] \) is a symmetric matrix. Thus, we assume that \( \omega = [w_{ij}] \) is a symmetric matrix in the following.

**Lemma 4.2.** Let \( \omega = [w_{ij}] \in \mathbb{M}_m \), where \( w_{ij} \) are parameters in \( \mathbb{R} \) for \( 1 \leq i, j \leq m \). Then the followings are equivalent:

1. \( G_\omega(A, B) \geq 0 \) (\( > 0 \))
2. \( G_\omega(I, X) \geq 0 \) (\( > 0 \))
3. \( G_\omega(I, X) \geq 0 \) (\( > 0 \), respectively),

where \( X = A^{-1/2} B A^{-1/2} \) and \( \hat{\omega} = [\hat{w}_{ij}] \) is the symmetric matrix whose diagonal entries are zero and off-diagonal \((i, j)\) entries are \( \hat{w}_{ij} = \frac{w_{ij} + w_{ji}}{2} \).

**Proof.** Note that

\[
G_\omega(I, X) = (I_m \otimes A^{-1/2})G_\omega(A, B)(I_m \otimes A^{-1/2}),
\]

where \( \otimes \) denotes the tensor product and \( I_m \) is the \( m \times m \) identity matrix. The above equality follows from the definition of the Riemannian geodesic. Moreover,

\[
G_\omega(I, X) = \begin{pmatrix}
X^{-\frac{w_{11}}{2}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X^{-\frac{w_{mm}}{2}}
\end{pmatrix} \begin{pmatrix}
X^{-\frac{w_{11}}{2}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X^{-\frac{w_{mm}}{2}}
\end{pmatrix}.
\]
Since the congruence transformation by the invertible matrix preserves the positive semi-definiteness (the positive definiteness, respectively), the equivalent relations hold.

**Theorem 4.3.** Let $w_{ij} \in \mathbb{R}$ be parameters for $i, j \in \{1, 2\}$. Then the $2 \times 2$ block matrix partitioned as

$$
\begin{pmatrix}
A \# w_{11} B & A \# w_{12} B \\
A \# w_{21} B & A \# w_{22} B
\end{pmatrix}
$$

if and only if $w_{12} = w_{21}$ and one of the followings holds:

(i) $w_{12} \leq \frac{w_{11} + w_{22}}{2}$ if $B \geq A$;
(ii) $w_{12} \geq \frac{w_{11} + w_{22}}{2}$ if $B \leq A$;
(iii) $w_{12} = \frac{w_{11} + w_{22}}{2}$ otherwise.

**Proof.** By Lemma 4.2 we obtain that

$$
\begin{pmatrix}
A \# w_{11} B & A \# w_{12} B \\
A \# w_{21} B & A \# w_{22} B
\end{pmatrix} \geq 0 \iff \begin{pmatrix} X w_{11} & X w_{12} \\
X w_{21} & X w_{22} \end{pmatrix} \geq 0,
$$

where $X = A^{-1/2}BA^{-1/2}$. By Lemma 3.3 it is proved.

The positive semi-definiteness of the Riemannian geodesic block matrix for $m \geq 3$ is totally different from the consequence in Theorem 4.3. We see the sufficient condition of the positive semi-definiteness of the Riemannian geodesic block matrix for $m \geq 3$.

**Theorem 4.4.** Let $w_{ij} \in \mathbb{R}$ be parameters for $1 \leq i, j \leq m$. Then the $m \times m$ block matrix $G_\omega(A, B)$ is positive semi-definite for $m \geq 3$ if $w_{ij} = \frac{w_{ii} + w_{jj}}{2}$ for all $i, j$.

**Proof.** Let the parameters $w_{ij} \in \mathbb{R}$ satisfy $w_{ij} = \frac{w_{ii} + w_{jj}}{2}$ for all $i, j$. Then the $m \times m$ block matrix $G_\omega(A, B)$ coincides with the block matrix given in (3) with $A_i = A \# w_{ii}, B$, since

$$
A_i \# A_j = (A \# w_{ij}, B) \# (A \# w_{ij}, B) = A \# \frac{w_{ii} + w_{jj}}{2} B = A \# w_{ij} B.
$$

The second equality follows from the affine property of parameters in Lemma 2.1 (5). Clearly, the family $\{A \# w_{ii}, B, A \# w_{22} B, \ldots, A \# w_{mm} B\}$ is $\Gamma$-commuting. Therefore, it is proved by Theorem 1.1.

**Remark 4.5.** The converse of Theorem 4.4 is not true in general. For instance, let $x > 3$ and $\omega = [w_{ij}] \in \mathbb{M}_3$, where $w_{ii} = 1$ and $w_{ij} = 1/2$ for all $i, j = 1, 2, 3$. Then the matrix $G_\omega(1, x)$

$$
\begin{pmatrix}
x & x^{1/2} & x^{1/2} \\
x^{1/2} & x & x^{1/2} \\
x^{1/2} & x^{1/2} & x
\end{pmatrix}
$$
is positive semi-definite since all principal minors are non-negative. However, \( \omega = [w_{ij}] \) does not satisfy the condition \( w_{ij} = \frac{w_{ii} + w_{jj}}{2} \) for all \( i, j \).

**Theorem 4.6.** Let \( \omega = [w_{ij}] \in M_3 \), where \( w_{ij} \in \mathbb{R} \) for \( i, j = 1, 2, 3 \) satisfying \( 2w_{ij} < w_{ii} + w_{jj} \) for all \( i \neq j \). Assume that \( A < B \). Then the \( 3 \times 3 \) block matrix \( G_\omega(A, B) \) is positive semi-definite if and only if

\[
\sum_{i<j} \lambda^2 w_{ij} - w_{ii} - w_{jj} - 2 \lambda w_{12} + w_{13} + w_{23} - w_{11} - w_{22} - w_{33} \leq 1,
\]

where \( \lambda \) denotes any eigenvalue of \( X = A^{-1/2}BA^{-1/2} \).

**Proof.** For convenience of the proof, set \( X_{ij} = X w_{ij} - w_{ii} + w_{jj} \) for all \( i \neq j \), where \( X = A^{-1/2}BA^{-1/2} > I \). It is equivalent to

\[
(I - \lambda^2 w_{ij}) > 0
\]

by the Schur complement condition for positive definiteness in Theorem 3.2. Therefore, Lemma 4.2 yields that \( G_\omega(A, B) \geq 0 \) if and only if

\[
G_\omega(I, X) = \begin{pmatrix} I & X_{12} & X_{13} \\ X_{12} & I & X_{23} \\ X_{13} & X_{23} & I \end{pmatrix} \geq 0.
\]

By Theorem 3.2, (5) is equivalent to

\[
I - \begin{pmatrix} X_{13} & X_{23} \end{pmatrix} \left( \begin{pmatrix} I & X_{12} \\ X_{12} & I \end{pmatrix} \right)^{-1} \begin{pmatrix} X_{13} \\ X_{23} \end{pmatrix} \geq 0.
\]

Note that

\[
\begin{pmatrix} I & X_{12} \\ X_{12} & I \end{pmatrix}^{-1} = \begin{pmatrix} (I - X_{12}^2)^{-1} & -X_{12}(I - X_{12}^2)^{-1} \\ -X_{12}(I - X_{12}^2)^{-1} & (I - X_{12}^2)^{-1} \end{pmatrix},
\]

and \( X_{12}, X_{13}, X_{23} \) commute, since \( X_{ij} \)'s are powers of \( X > 0 \). So we have

\[
I - (I - X_{12}^2)^{-1}(X_{13}^2 + X_{23}^2 - 2X_{12}X_{13}X_{23}) \geq 0.
\]

Thus, by a simple calculation

\[
X_{12}^2 + X_{13}^2 + X_{23}^2 - 2X_{12}X_{13}X_{23} \leq I,
\]

which is equivalent to the inequality (4) for any eigenvalue of \( X \). \( \square \)

**Remark 4.7.** Upon proving Theorem 4.6, we can apply the Schur complement condition for positive semi-definiteness to the other \( 2 \times 2 \) block matrices of (5). In this case, we also obtain the same result as Theorem 4.6.
Corollary 4.8. Let $\omega = [w_{ij}] \in M_3$, where $w_{ij} \in \mathbb{R}$ for $i, j = 1, 2, 3$ satisfying $2w_{ij} > w_{ii} + w_{jj}$ for all $i \neq j$. Assume that $A > B$. Then the $3 \times 3$ block matrix $G_\omega(A, B)$ is positive semi-definite if and only if

\[
\sum_{i<j} \lambda^{2w_{ij} - w_{ii} - w_{jj}} - 2\lambda^{w_{12} + w_{13} + w_{23} - w_{11} - w_{22} - w_{33}} \leq 1,
\]

where $\lambda$ denotes any eigenvalue of $X = A^{-1/2}BA^{-1/2}$.

Proof. By Lemma 2.1(2),

\[G_\omega(A, B) = G_{1-\omega}(B, A),\]

where $1$ is the $3 \times 3$ matrix whose entries are all $1$. So $1 - \omega = [1 - w_{ij}] \in M_3$, and $2(1 - w_{ij}) < (1 - w_{ii}) + (1 - w_{jj})$ for all $i \neq j$. Thus, we can easily get the conclusion. \(\square\)

5. Conclusion and open problem with stochastic matrices

Using the Schur complement, we have mainly verified a characterization of $3 \times 3$ block matrix $G_\omega(A, B)$ consisting of Riemannian geodesics of positive definite matrices $A$ and $B$ to be positive semi-definite. In general, it is natural to ask under which equivalent condition the $n \times n$ block matrix $G_\omega(A, B)$ is positive semi-definite. On the other hand, it might be difficult to clarify because we have to consider many different kinds of Schur complement and the explicit forms for the inverse of block matrices. So we set aside an open problem to find the characterization of $n \times n$ block matrix $G_\omega(A, B)$ to be positive semi-definite.

We finally see some connection of our result with stochastic matrix and give an open problem. A non-negative matrix $A \in M_m$ is called a row (column) stochastic matrix if all of its row (column, respectively) sums are $1$. A non-negative matrix $A \in M_m$ is called a doubly stochastic matrix if all of its row and column sums are $1$. Stochastic matrices also arise in the study of Markov chains and in a variety of modeling problems in economics and operation research. See more information about stochastic matrices in [4, Section 8.7].

One can naturally ask for which kinds of stochastic matrices the Riemannian geodesic block matrix $G_\omega(A, B)$ is positive semi-definite. We provide a simple consequence from Theorem 4.4 to answer this question.

Remark 5.1. Let $w_{ij} = 1/m$ for all $1 \leq i, j \leq m$. Then $\omega = [w_{ij}] \in M_m$ is a doubly stochastic matrix, and $w_{ij} = \frac{w_{ij} + w_{ji}}{2}$ hold for all $i, j$. Thus, by Theorem 4.4 $G_\omega(A, B)$ is positive semi-definite.

Remark 5.1 tells that the special doubly stochastic matrix $\omega = [w_{ij}] \in M_m$ whose all rows and columns are uniformly distributed probability vectors makes the Riemannian geodesic block matrix $G_\omega(A, B)$ positive semi-definite. On the other hand, other stochastic matrices may make the Riemannian geodesic block
matrix $G_\omega(A, B)$ positive semi-definite. For instance, let $x \geq 1$ and

$$\omega = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Then it is a doubly stochastic matrix, and the Riemannian geodesic matrix

$$G_\omega(1, x) = \begin{pmatrix} x^{1/2} & x^{1/2} & 1 \\ x^{1/2} & x^{1/2} & 1 \\ 1 & 1 & x \end{pmatrix}$$

is positive semi-definite since all principal minors are non-negative. Thus, it is an interesting and open question to find the characterization of stochastic matrices to make the Riemannian geodesic block matrix $G_\omega(A, B)$ positive semi-definite.

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