ALEXANDROV TOPOLOGIES AND NON-SYMMETRIC PSEUDO-METRICS

JU-MOK OH a AND YONG CHAN KIM b, *

Abstract. In this paper, we investigate the properties of Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators on $[0, \infty]$. Moreover, we investigate the relations among Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators. We give their examples.

1. Introduction

Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Pawlak [12,13] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1-11,14,15]. Kim [6-10] investigated the properties of Alexandrov topologies, fuzzy preorders and join-preserving maps in complete residuated lattices.

In this paper, we investigate the properties of Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators on $[0, \infty]$. We give their examples. In fact, categories of Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators are isomorphic.

2. Preliminaries

Let $([0, \infty], \leq, \lor, +, \land, \to, \infty, 0)$ be a structure where

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*Corresponding author.

Definition 2.1. Let \( X \) be a set. A function \( d_X : X \times X \to [0, \infty] \) is called a non-symmetric pseudo-metric if it satisfies the following conditions:

1. \( d_X(x, x) = 0 \) for all \( x \in X \),
2. \( d_X(x, y) + d_X(y, z) \geq d_X(x, z) \) for all \( x, y, z \in X \).

The pair \((X, d_X)\) is called a non-symmetric pseudo-metric space.

Remark 2.2. (1) We define a function \( d_{[0,\infty)^X} : [0, \infty]^X \times [0, \infty]^X \to [0, \infty] \) as

\[
d_{[0,\infty)^X}(A, B) = \bigvee_{x \in X} (A(x) - B(x)) = \bigvee_{x \in X} ((B(x) - A(x)) \vee 0).
\]

Then \([0, \infty]^X, d_{[0,\infty)^X}\) is a pseudometric space.

(2) If \((X, d_X)\) is a non-symmetric pseudo-metric space and we define a function \(d_X^{-1}(x, y) = d_X(y, x)\), then \((X, d_X^{-1})\) is a non-symmetric pseudo-metric space.

(3) Let \((X, d_X)\) be a non-symmetric pseudo-metric space and define \((d_X \oplus d_X)(x, z) = \bigwedge_{y \in X} (d_X(x, y) + d_X(y, z))\) for each \( x, z \in X \). By (M2), \((d_X \oplus d_X)(x, z) \geq d_X(x, z)\) and \((d_X \oplus d_X)(x, z) \leq d_X(x, x) + d_X(x, z) = d(x, z)\). Hence \((d_X \oplus d_X) = d_X\).

(4) If \(d_X\) is a non-symmetric pseudo-metric and \(d_X(x, y) = d_X(y, x)\) for each \( x, y \in X \), then \(d_X\) is a pseudo-metric.

Example 2.3. (1) Let \( X = \{a, b, c\} \) be a set and define maps \(d_X^i : X \times X \to [0, \infty]\) for \( i = 1, 2, 3 \) as follows:

\[
d_X^1 = \begin{pmatrix} 0 & 6 & 5 \\ 6 & 0 & 1 \\ 15 & 7 & 0 \end{pmatrix}, \ d_X^2 = \begin{pmatrix} 0 & 6 & 3 \\ 7 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix}, \ d_X^3 = \begin{pmatrix} 0 & 3 & 7 \\ 6 & 0 & 9 \\ 5 & 4 & 0 \end{pmatrix}.
\]

Since \(d_X^1(c, b) + d_X^1(b, a) = 13 < d_X^1(c, a) = 15\), \(d_X^1\) is not a non-symmetric pseudo-metric. Since \(d_X^2\) and \(d_X^3\) are non-symmetric pseudo-metrics, \(d_X^k \oplus d_X^k = d_X^k\) for \( k = 2, 3 \).

3. Alexandrov Topologies and Non-Symmetric Pseudo-metrics

We define the following two definitions as a sense in [2, 5-10].

Definition 3.1. A subset \( \tau_X \subseteq [0, \infty]^X \) is called an Alexandrov topology on \( X \) iff it satisfies the following conditions:

\((AT1) \) \( \alpha_X \in \tau_X \) where \( \alpha_X(x) = \alpha \) for each \( x \in X \) and \( \alpha \in [0, \infty] \).
(AT2) If \( A_i \in \tau_X \) for all \( i \in I \), then \( \bigvee_{i \in I} A_i, \bigwedge_{i \in I} A_i \in \tau_X \).

(AT3) If \( A \in \tau_X \) and \( \alpha \in [0, \infty] \), then \( \alpha + A, \alpha \to A \in \tau_X \) where \( (\alpha \to A)(x) = (A(x) - \alpha) \vee 0 \).

The pair \((X, \tau_X)\) is called an Alexandrov topological space.

**Definition 3.2.** A map \( \mathcal{H} : [0, \infty]^X \to [0, \infty]^X \) is called a lower approximation operator if it satisfies the following conditions, for all \( A, A_i \in [0, \infty]^X \), and \( \alpha \in [0, \infty] \),

- \((H1)\) \( \mathcal{H}(\alpha + A) = \alpha + \mathcal{H}(A) \) where \( (\alpha + A)(x) = \alpha + A(x) \),
- \((H2)\) \( \mathcal{H}(\bigwedge_{i \in I} A_i) = \bigwedge_{i \in I} \mathcal{H}(A_i) \),
- \((H3)\) \( \mathcal{H}(A) \leq A \),
- \((H4)\) \( \mathcal{H}(\mathcal{H}(A)) = \mathcal{H}(A) \).

**Theorem 3.3.** Let \( d_X \in [0, \infty]^{X \times X} \) be a non-symmetric pseudo-metric. Define \( \mathcal{H}_{d_X}(A) : [0, \infty]^X \to [0, \infty]^X \) as follows

\[
\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x \in X} (A(x) + d_X(x, y)).
\]

Then \( \mathcal{H}_{d_X} \) is a lower approximation operator.

**Proof.** Since \( \mathcal{H}_{d_X}(A)(y) = \bigwedge_{x \in X} (A(x) + d_X(x, y)) \),

- \((H1)\) \( \mathcal{H}_{d_X}(\alpha + A) = \alpha + \mathcal{H}_{d_X}(A) \),
- \((H2)\) \( \mathcal{H}_{d_X}(\bigwedge_{i \in I} A_i) = \bigwedge_{i \in I} \mathcal{H}_{d_X}(A_i) \),
- \((H3)\) \( \mathcal{H}_{d_X}(A)(y) = \bigwedge_{x \in X} (A(x) + d_X(x, y)) \leq A(y) + d_X(y, y) = A(y) \),
- \((H4)\) For all \( A \in [0, \infty]^X \), \( z \in X \),

\[
\mathcal{H}_{d_X}(\mathcal{H}_{d_X}(A))(z) = \bigwedge_{y \in X} (\mathcal{H}_{d_X}(A)(y) + d_X(y, z))
= \bigwedge_{y \in X} ((\bigwedge_{x \in X} (A(x) + d_X(x, y))) + d_X(y, z))
= \bigwedge_{x \in X} (A(x) + \bigwedge_{y \in X} (d_X(x, y) + d_X(y, z)))
= \bigwedge_{z \in X} (A(x) + d_X(x, z)) = \mathcal{H}_{d_X}(A)(z).
\]

Hence \( \mathcal{H}_{d_X} \) is a lower approximation operator. \( \square \)

**Theorem 3.4.** A map \( \mathcal{H} : [0, \infty]^X \to [0, \infty]^X \) is a lower approximation operator iff there exist a non-symmetric pseudo-metric \( d_\mathcal{H} \) on \( X \) such that

\[
\mathcal{H}(A)(y) = \bigwedge_{x \in X} (A(x) + d_\mathcal{H}(x, y)).
\]

**Proof.** \( (\Rightarrow) \) Put \( d_\mathcal{H} : X \times X \to [0, \infty] \) as \( d_\mathcal{H}(x, y) = \mathcal{H}(0_x)(y) \) where \( 0_x(x) = 0 \) and \( 0_x(y) = \infty \) for \( x \neq y \in X \). \((M1)\) \( d_\mathcal{H}(x, x) = \mathcal{H}(0_x)(x) \leq 0_x(x) = 0 \).
(M2) Since \( A = \bigwedge_{y \in X}(A(y) + 0_y) \) and \( \mathcal{H}(0_x) = \bigwedge_{y \in X}(\mathcal{H}(0_x)(y) + 0_y) \),

\[
\begin{align*}
\bigwedge_{y \in X}(d_H(x, y) + d_H(y, z)) &= \bigwedge_{y \in X}(\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)) \quad \text{(by H2)} \\
&= \mathcal{H}(\bigwedge_{y \in X}(\mathcal{H}(0_x)(y) + 0_y)(z)) = \mathcal{H}(\mathcal{H}(0_x))(z) \\
&= \mathcal{H}(0_x)(z) = d_H(x, z).
\end{align*}
\]

Hence \( d_H \) is a non-symmetric pseudo-metric. Moreover,

\[
\begin{align*}
\mathcal{H}(A)(y) &= \mathcal{H}(\bigwedge_{y \in X}(A(x) + 0_x))(y) \\
&= \bigwedge_{x \in X}(A(x) + \mathcal{H}(0_x)(y)) \\
&= \bigwedge_{x \in X}(A(x) + d_H(x, y)).
\end{align*}
\]

(\( \Leftarrow \)) It follows from Theorem 3.3. \( \square \)

**Theorem 3.5.** Let \( d_X \) be a non-symmetric pseudo-metric on \( X \). Define \( \tau_{d_X} = \{ A \in [0, \infty]^X \mid A(x) + d_X(x, y) \geq A(y) \} \). Then the following properties hold.

1. \( \tau_{d_X} \) is an Alexandrov topology on \( X \).
2. \( d_X(x, -) \in \tau_{d_X} \). Moreover, \( A \in \tau_{d_X} \) iff \( A = \bigwedge_{x \in X}(A(x) + d_X(x, -)) = \mathcal{H}_{d_X}(A) \).

**Proof.**

1. (AT1) Since \( \alpha_X(x) + d_X(x, y) \geq \alpha_X(y) \), we have \( \alpha_X \in \tau_{d_X} \).

2. (AT2) If \( A_i \in \tau_{d_X} \) for all \( i \in I \), then

\[
\begin{align*}
(\bigwedge_{i \in I} A_i)(x) + d_X(x, y) &= \bigwedge_{i \in I}(A_i(x) + d_X(x, y)) \\
&\geq \bigwedge_{i \in I} A_i(y), \\
(\bigvee_{i \in I} A_i)(x) + d_X(x, y) &= \bigvee_{i \in I}(A_i(x) + d_X(x, y)) \\
&\geq \bigvee_{i \in I} A_i(y).
\end{align*}
\]

Hence \( \bigwedge_{i \in I} A_i, \bigvee_{i \in I} A_i \in \tau_{d_X} \).

3. (AT3) If \( A \in \tau_{d_X} \) and \( \alpha \in [0, \infty) \), then

\[
\begin{align*}
(\alpha + A)(x) + d_X(x, y) &\geq (\alpha + A)(y), \\
(\alpha \to A)(x) + d_X(x, y) &= ((A(x) - \alpha) \vee 0) + d(x, y) \\
&= ((A(x) - \alpha) + d(x, y)) \lor d(x, y) \\
&\geq (A(y) - \alpha) \lor 0 = (\alpha \to A)(y).
\end{align*}
\]

So, \( \alpha + A, \alpha \to A \in \tau_{d_X} \). Hence \( \tau_{d_X} \) is an Alexandrov topology on \( X \).

(2) Since \( d_X(x, y) + d_X(y, z) \geq d_X(x, z), d_X(x, -) \in \tau_{d_X} \). Let \( A \in \tau_{d_X} \). Then

\[
\bigwedge_{x \in X}(A(x) + d_X(x, y)) \geq A(y) \text{ and } \bigwedge_{x \in X}(A(x) + d_X(x, y)) \leq A(y) + d_X(y, y) = A(y).
\]

Hence \( A = \bigwedge_{x \in X}(A(x) + d_X(x, -)) = \mathcal{H}_{d_X}(A) \).

Conversely, since \( \mathcal{H}_{d_X}(A)(y) + d_X(y, z) = \bigwedge_{x \in X}(A(x) + d_X(x, y)) + d_X(y, z) \geq \bigwedge_{x \in X}(A(x) + d_X(x, z)) = \mathcal{H}_{d_X}(A)(z) \). So, \( A = \mathcal{H}_{d_X}(A) \in \tau_{d_X} \). \( \square \)
Theorem 3.6. Let \( \mathcal{H} : [0, \infty]^X \to [0, \infty]^X \) be a lower approximation operator. Then the following properties hold.

1. \( \tau_\mathcal{H} = \{ A \in [0, \infty]^X \mid \mathcal{H}(A) = A \} \) is an Alexandrov topology on \( X \) such that \( \tau_\mathcal{H} = \{ \mathcal{H}(A) \mid A \in [0, \infty]^X \} \).

2. Define \( d_\mathcal{H} : X \times X \to [0, \infty] \) as \( d_\mathcal{H}(x, y) = \mathcal{H}(0_x)(y) \) where \( 0_x(x) = 0 \) and \( 0_x(y) = \infty \) for \( x \neq y \in X \). Then \( d_\mathcal{H} \) is a non-symmetric pseudo-metric on \( X \) such that \( \mathcal{H}_{d_\mathcal{H}} = \mathcal{H} \) and \( \tau_{d_\mathcal{H}} = \tau_\mathcal{H} \).

Proof. (1) (AT1) Since \( \mathcal{H}(\alpha_X) = \mathcal{H}(\alpha + 0_X) = \alpha + 0_X = \alpha_X \), then \( \alpha_X \in \tau_\mathcal{H} \).

(2) (AT2) For \( A_i \in \tau_\mathcal{H} \) for each \( i \in \Gamma \), by (H2), \( \bigwedge_{i \in \Gamma} A_i \in \tau_\mathcal{H} \). Since \( \bigvee_{i \in \Gamma} \mathcal{H}(A_i) \leq \mathcal{H}(\bigvee_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} A_i \), thus, \( \bigvee_{i \in \Gamma} A_i \in \tau_\mathcal{H} \).

(3) (AT3) For \( A \in \tau_\mathcal{H} \), by (H1), \( \alpha + A \in \tau_\mathcal{H} \).

Since \( \alpha + \mathcal{H}(\alpha \to A) = \mathcal{H}(\alpha + (\alpha \to A)) \geq \mathcal{H}(A), \mathcal{H}(\alpha \to A) \geq (\mathcal{H}(A) - \alpha) \vee 0 = \alpha \to A \). Then \( \alpha \to A \in \tau_\mathcal{H} \). Hence \( \tau_\mathcal{H} \) is an Alexandrov topology on \( X \). Let \( A \in \tau_\mathcal{H} \). Then \( A = \mathcal{H}(A) \in \{ \mathcal{H}(A) \mid A \in [0, \infty]^X \} \). Let \( \mathcal{H}(A) \in \{ \mathcal{H}(A) \mid A \in [0, \infty]^X \} \). Since \( \mathcal{H}(\mathcal{H}(A)) = \mathcal{H}(A), \mathcal{H}(A) \in \tau_\mathcal{H} \).

(2) By Theorem 3.4, \( d_\mathcal{H} \) is a non-symmetric pseudo-metric on \( X \). Moreover, \( \mathcal{H}_{d_\mathcal{H}}(A)(y) = \bigwedge_{x \in X} (A(x) + d_\mathcal{H}(x, y)) \) and \( \tau_{d_\mathcal{H}} = \tau_\mathcal{H} \) from:

\[ A \in \tau_{d_\mathcal{H}} \text{ iff } \mathcal{H}_{d_\mathcal{H}}(A) = A \text{ iff } \mathcal{H}(A) = A \text{ iff } A \in \tau_\mathcal{H}. \]

\[ \square \]

Example 3.7. (1) Define maps \( d^i : [0, \infty] \times [0, \infty] \to [0, \infty] \) for \( i = 0, 1, 2, 3 \) as follows:

\[ d^0(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \infty, & \text{if } x \neq y. \end{cases} \]

\[ d^1(x, y) = \begin{cases} 0, & \text{if } x \geq y, \\ \infty, & \text{if } x < y. \end{cases} \]

\[ d^2(x, y) = \begin{cases} 0, & \text{if } x \leq y, \\ \infty, & \text{if } x > y. \end{cases} \]

\[ d^3(x, y) = 0. \]

Since \( \mathcal{H}_{d_X}(A)(y) = \bigwedge_{x \in X} (A(x) + d_X(x, y)) \), we can obtain

\[ \mathcal{H}_{d^0}(A)(y) = \bigwedge_{x \in X} (A(x) + d^0_X(x, y)) = A(y), \]

\[ \mathcal{H}_{d^1}(A) = \bigwedge_{x \geq y} A(x), \]

\[ \mathcal{H}_{d^2}(A) = \bigwedge_{x \leq y} A(x), \]

\[ \mathcal{H}_{d^3}(A) = \bigwedge_{x \in X} A(x). \]
\[ \tau_0 = [0, \infty)^0, \]
\[ \tau_1 = \{ A \in [0, \infty] \mid A(x) \leq A(y) \text{ if } x \leq y \}, \]
\[ \tau_2 = \{ A \in [0, \infty] \mid A(x) \geq A(y) \text{ if } x \leq y \}, \]
\[ \tau_3 = \{ \alpha \in [0, \infty] \mid \alpha \in [0, \infty] \}. \]

**Theorem 3.8.** Let \( \tau \) be Alexandrov topology on \( X \). Then the following properties hold.

1. Define \( \mathcal{H}_\tau : [0, \infty]^X \to [0, \infty]^X \) as follows:
   \[ \mathcal{H}_\tau(A) = \bigvee \{ B \mid B \leq A, \ B \in \tau \}. \]
   Then \( \mathcal{H}_\tau \) is a lower approximation operator such that \( \tau_{\mathcal{H}_\tau} = \tau, \ \mathcal{H}_{\tau_{\mathcal{H}_\tau}} = \mathcal{H}. \)
2. Define \( d_\tau : X \times X \to [0, \infty] \) as
   \[ d_\tau(x, y) = \bigvee_{A \in \tau} (A(x) \to A(y)) = \bigvee_{A \in \tau} ((A(y) - A(x)) \lor 0) \]
   Then \( d_\tau \) is a non-symmetric pseudo-metric such that \( \tau = \tau_{d_\tau} = \tau_{d_\tau}. \) Moreover, \( \mathcal{H}_\tau = \mathcal{H}_{d_\tau} \) and \( d_\tau = d_{\mathcal{H}_\tau}. \)
3. If \( \mathcal{H} : [0, \infty]^X \to [0, \infty]^X \) is a lower approximation operator, then
   \[ \bigwedge_{y \in X} (\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)) = \mathcal{H}(0_x)(z) \]
   for all \( x, y, z \in X \) and \( d_{\tau_{\mathcal{H}}} = d_{\mathcal{H}}. \)

**Proof.**
(1) We show \( \mathcal{H}_\tau(A) = \bigvee \{ B \mid B \leq A, \ B \in \tau \} \) is a lower approximation operator.

(H1) For \( \alpha \in [0, \infty], A \in [0, \infty]^X, \)
   \[ \alpha + \mathcal{H}_\tau(A) = \alpha + \bigvee \{ B \mid B \leq A, \ B \in \tau \} = \bigvee \{ \alpha + B \mid \alpha + B \leq \alpha + A, \ \alpha + B \in \tau \} = \mathcal{H}_\tau(\alpha + A). \]

(H2) Since \( \mathcal{H}_\tau(A) \leq \mathcal{H}_\tau(B) \) for \( A \leq B, \) we have \( \bigwedge_{i \in \mathcal{I}} \mathcal{H}_\tau(A_i) \geq \mathcal{H}_\tau(\bigwedge_{i \in \mathcal{I}} A_i) \).

Since \( \bigwedge_{i \in \mathcal{I}} A_i \geq \bigwedge_{i \in \mathcal{I}} \mathcal{H}_\tau(A_i) \in \tau, \) then \( \mathcal{H}_\tau(\bigwedge_{i \in \mathcal{I}} A_i) \geq \bigwedge_{i \in \mathcal{I}} \mathcal{H}_\tau(A_i). \)

(H3) It follows from the definition.

(H4) Since \( \mathcal{H}_\tau(A) \in \tau, \) we have \( \mathcal{H}_\tau(\mathcal{H}_\tau(A)) = \mathcal{H}_\tau(A). \)

Let \( A \in \tau_{\mathcal{H}_\tau}. \) Then \( A = \mathcal{H}_\tau(A) \in \tau. \) Hence \( \tau_{\mathcal{H}_\tau} \subset \tau. \)

Let \( A \in \tau. \) Then \( \mathcal{H}_\tau(A) = A. \) So, \( A \in \tau_{\mathcal{H}_\tau}. \) Hence \( \tau \subset \tau_{\mathcal{H}_\tau}. \)

Since \( \mathcal{H}_{\tau_{\mathcal{H}_\tau}}(A) = \bigvee \{ B \mid B \leq A, \ B \in \tau_{\mathcal{H}_\tau} \} \) and \( A \geq \mathcal{H}(\mathcal{H}(A)) = \mathcal{H}(A) \), we have
\[ \mathcal{H}(A) \leq \mathcal{H}_{\tau_{\mathcal{H}_\tau}}(A). \]

For \( B_i \in \tau_{\mathcal{H}_\tau}, \) since \( \mathcal{H}(\bigvee_{i \in \mathcal{I}} B_i) \geq \bigvee_{i \in \mathcal{I}} \mathcal{H}(B_i) = \bigvee_{i \in \mathcal{I}} B_i, \) then
\[ \mathcal{H}(\mathcal{H}(\mathcal{H}_\tau(A)) = \mathcal{H}_{\tau_{\mathcal{H}_\tau}}(A). \]

So, \( \mathcal{H}(A) \geq \mathcal{H}_{\tau_{\mathcal{H}_\tau}}(A). \)
(2) We easily show that $d_x$ is a non-symmetric pseudo-metric.

Let $A \in \tau$. Then $\mathcal{H}_{d_x}(A) = A$ because

\[
\mathcal{H}_{d_x}(A)(y) = \bigwedge_{x \in X} (A(x) + d_x(x, y)) = \bigwedge_{x \in X} (A(x) + \bigvee_{B \in \tau} (B(x) \rightarrow B(y)) \\
\geq \bigwedge_{x \in X} (A(x) + (A(y) - A(x)) = \bigwedge_{x \in X} (A(x) + ((A(y) - A(x)) \lor 0)) \geq A(y).
\]

So, $\tau \subset \tau_{\mathcal{H}_{d_x}}$.

Let $A = \mathcal{H}_{d_x}(A)$. Then

\[
A = \mathcal{H}_{d_x}(A) = \bigwedge_{x \in X} (A(x) + \bigvee_{B \in \tau} (B(x) \rightarrow B)) \in \tau.
\]

So, $\tau_{\mathcal{H}_{d_x}} \subset \tau$.

Let $A \in \tau$. Then $A \in \tau_{d_x}$ because

\[
A(x) + d_x(x, y) = A(x) + \bigvee_{B \in \tau} (B(x) \rightarrow B(y)) \\
\geq A(x) + (A(x) \rightarrow A(y)) = A(x) + ((A(y) - A(x)) \lor 0)) \geq A(y).
\]

So, $\tau \subset \tau_{d_x}$.

Let $A \in \tau_{d_x}$. Then $A \in \tau$ because

\[
A = \mathcal{H}_{d_x}(A) = \bigwedge_{x \in X} (A(x) + \bigvee_{B \in \tau} (B(x) \rightarrow B)) \in \tau.
\]

Since $A \geq \mathcal{H}_{d_x}(A) \in \tau$, then $\mathcal{H}(A) \geq \mathcal{H}_{d_x}(A)$. Since

\[
\mathcal{H}_{d_x}(A)(y) = \bigwedge_{x \in X} (A(x) + \bigvee_{B \in \tau} (B(x) \rightarrow B(y)) \\
\geq \bigwedge_{x \in X} (\mathcal{H}(A)(x) + (\mathcal{H}(A)(x) \rightarrow \mathcal{H}(A)(y)) \geq \mathcal{H}(A)(y),
\]

Hence $\mathcal{H}_{d_x}(A) = \mathcal{H}(A)$.

(3) Since $\mathcal{H}(0_x) = \bigwedge_{y \in X} (\mathcal{H}(0_x)(y) + 0_y(-)), \bigwedge_{y \in X} (\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)) = \mathcal{H}(0_x)(z)$ because

\[
\mathcal{H}(0_x)(z) = \mathcal{H}(\mathcal{H}(0_x)(z) = \mathcal{H}((\bigwedge_{y \in X} (\mathcal{H}(0_x)(y) + 0_y(-))) = \bigwedge_{y \in X} (\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)).
\]

\[
d_{\tau_{\mathcal{H}}}(x, y) = \bigvee_{A \in \tau_{\mathcal{H}}} ((A(x) - A(x)) \lor 0) \\
\geq \bigvee_{x \in X} ((\mathcal{H}(0_x)(y) - \mathcal{H}(0_x)(x)) \lor 0) \geq (\mathcal{H}(0_x)(y) - \mathcal{H}(0_x)(x)) \lor 0 \\
\geq (\mathcal{H}(0_x)(y) - \mathcal{H}(0_x)(x)) \lor 0 = \mathcal{H}(0_x)(y),
\]

\[
d_{\tau_{\mathcal{H}}}(x, y) = \bigvee_{A \in \tau_{\mathcal{H}}} ((A(x) - A(x)) \lor 0) = \bigwedge_{A \in [0, \infty]} ((\mathcal{H}(A)(y) - \mathcal{H}(A)(x)) \lor 0) \\
= \bigwedge_{A \in [0, \infty]} ((\bigwedge_{x \in X} (A(x) + \mathcal{H}(0_x)(y)) - \bigwedge_{x \in X} (A(x) + \mathcal{H}(0_x)(x))) \lor 0) \\
\leq \bigwedge_{x \in X} ((\mathcal{H}(0_x)(y) - \mathcal{H}(0_x)(x)) \lor 0) \leq \mathcal{H}(0_x)(y).
\]

Hence $d_{\tau_{\mathcal{H}}}(x, y) = \mathcal{H}(0_x)(y) = d_{\mathcal{H}}(x, y)$.

\textbf{Theorem 3.9.} Let $d_X \in [0, \infty]^{X \times X}$ be a non-symmetric pseudo-metric. Then

\[d_{\mathcal{H}_{d_x}}(x, y) = \mathcal{H}_{d_x}(0_x)(y) = d_X(x, y) = d_{\tau_{\mathcal{H}}}(x, y) \text{ for each } x, y \in X.\]
Proof. Since $\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x \in X} (A(x) + d_X(x, y))$, $\mathcal{H}_{d_X}(0_x)(y) = \bigwedge_{x \in X} (0_x(x) + d_X(x, y)) = d_X(x, y)$. Since $d_X(x, -) \in \tau_{d_X}$ and $A = \mathcal{H}_{d_X}(A)$ for $A \in \tau_{d_X}$,

$$d_{\tau_{d_X}}(x, y) = \bigvee_{A \in \tau_{d_X}} ((A(y) - A(x)) \vee 0)$$

$$\geq (d_X(x, y) - d_X(x, x)) \vee 0 = d_X(x, y),$$

$$d_{\tau_{d_X}}(x, y) = \bigvee_{A \in \tau_{d_X}} ((A(y) - A(x)) \vee 0)$$

$$= \bigvee_{A \in \tau_{d_X}} ((\mathcal{H}_{d_X}(A)(y) - \mathcal{H}_{d_X}(A)(x)) \vee 0)$$

$$= \bigvee_{A \in \tau_{d_X}} ((\bigwedge_{z \in X} (A(z) + d_X(z, y)) - (\bigwedge_{z \in X} (A(z) + d_X(z, x))) \vee 0)$$

$$\leq \bigvee_{z \in X} ((d_X(z, y) - d_X(z, x)) \vee 0) \leq d_X(x, y).$$

Example 3.10. Let $X = \{a, b, c\}$ be a set and $A \in [0, \infty]^X$ as

$$A(a) = 7, A(b) = 5, A(c) = 10$$

(1) Define $d_A(x, y) = (A(y) - A(x)) \vee 0$ as

$$d_A = \begin{pmatrix} 0 & 0 & 3 \\ 2 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{H}_d_A(B) = \begin{pmatrix} B(a) \land (2 + B(b)) \land B(c) \\ B(a) \land B(b) \land B(c) \\ (3 + B(a)) \land (5 + B(b)) \land B(c) \end{pmatrix}.$$ 

Then $d_A(a, \cdot) = (0, 0, 3), d_A(b, \cdot) = (2, 0, 5), d_A(c, \cdot) = (0, 0, 0), A \in \tau_{d_A}$. Moreover, $\tau_{d_A} = \{\mathcal{H}_d_A(B) \mid B \in [0, \infty]^X\}.$

(2) Define $d_A^1(x, y) = |A(x) - A(y)|$ as

$$d_A^1 = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 0 \end{pmatrix}, \mathcal{H}_{d_A^1}(B) = \begin{pmatrix} B(a) \land (2 + B(b)) \land (3 + B(c)) \\ (2 + B(a)) \land B(b) \land (5 + B(c)) \\ (3 + B(a)) \land (5 + B(b)) \land B(c) \end{pmatrix}.$$ 

Then $d_A^1(a, \cdot) = (0, 2, 3), d_A^1(b, \cdot) = (2, 0, 5), d_A^1(c, \cdot) = (3, 5, 0), A \in \tau_{d_A^1}$. Moreover, $\tau_{d_A^1} = \{\mathcal{H}_d_{d_A^1}(B) \mid B \in [0, \infty]^X\}.$

(3) Define $d_X^2$ and $d_X^2 \oplus d_X^2(x, z) = \bigwedge_{y \in X} (d_X^2(x, y) + d_X^2(y, z))$ as

$$d_X^2 = \begin{pmatrix} 0 & 4 & 1 \\ 7 & 0 & 3 \\ 2 & 9 & 0 \end{pmatrix}, d_X^2 \oplus d_X^2 = \begin{pmatrix} 0 & 4 & 1 \\ 5 & 0 & 3 \\ 2 & 6 & 0 \end{pmatrix}.$$ 

Since $d_X^2(b, c) + d_X^2(c, a) = 5 < d_X^1(b, a) = 7$ and $d_X^2(c, a) + d_X^2(a, b) = 6 < d_X^1(b, a) = 9$, $d_X^2$ is not a non-symmetric pseudo-metric.

(4) Define $d_X^3 = d_X^2 \oplus d_X^2$.

$$d_X^3 = \begin{pmatrix} 0 & 4 & 1 \\ 5 & 0 & 3 \\ 2 & 6 & 0 \end{pmatrix}, \mathcal{H}_{d_X^3}(B) = \begin{pmatrix} B(a) \land (5 + B(b)) \land (2 + B(c)) \\ (4 + B(a)) \land B(b) \land (6 + B(c)) \\ (1 + B(a)) \land (3 + B(b)) \land B(c) \end{pmatrix}.$$
Since $d_X^3$ is a non-symmetric pseudo-metric, $d_X^2 \oplus d_X^3 = d_X^3$. Then $d_X^3(a, c) = (0, 4, 1), d_X^3(b, c) = (5, 0, 3), d_X^3(c, c) = (2, 6, 0) \in \tau_{d_X^3}$. Moreover, $\tau_{d_X^3} = \{\mathcal{H}_d^3(B) \mid B \in [0, \infty]^X\}.$

4. Categories of Non-symmetric Pseudo-metrics, Lower Approximation Operators and Alexandrov Topologies

Let $\textbf{LA}$ be a category with object $(X, \mathcal{H}_X)$ where $\mathcal{H}_X$ is a lower approximation operator with a morphism $f : (X, \mathcal{H}_X) \to (Y, \mathcal{H}_Y)$ such that $f^-(\mathcal{H}_Y(B)) \leq \mathcal{H}_Y(f^-(B))$ for all $B \in [0, \infty]^Y$.

Let $\textbf{NPM}$ be a category with object $(X, d_X)$ where $d_X$ is a non-symmetric pseudo-metric with a morphism $f : (X, d_X) \to (Y, d_Y)$ such that $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$.

**Theorem 4.1.** Two categories $\textbf{LA}$ and $\textbf{NPM}$ are isomorphic.

**Proof.** Define $H : \textbf{NPM} \to \textbf{LA}$ as $H(X, d_X) = (X, \mathcal{H}_d^X)$ where $\mathcal{H}_d^X(A)(y) = \bigwedge_{x \in X}(A(x) + d_X(x, y))$ from Theorem 3.3. Let $d_Y(f(x), f(y)) \leq d_X(x, y)$. Then

$$f^-(\mathcal{H}_Y(B))(x) = \bigwedge_{w \in Y}(B(w) + d_Y(w, f(x)))$$
$$\leq \bigwedge_{z \in X}(B(f(z)) + d_Y(f(z), f(x)))$$
$$\leq \bigwedge_{z \in X}(f^-(B)(z) + d_X(z, x) = \mathcal{H}_Y(f^-(B))(x).$$

Hence $H$ is a functor.

Define a functor $G : \textbf{LA} \to \textbf{NPM}$ as $G(X, \mathcal{H}_X) = (X, d_{\mathcal{H}_X})$ where $d_{\mathcal{H}_X}(x, y) = \mathcal{H}_X(0_x)(y)$ from Theorem 3.6(2). Let $f^-(\mathcal{H}_Y(B)) \leq \mathcal{H}_Y(f^-(B))$. Since

$$d_{\mathcal{H}_Y}(f(x), f(y)) = \mathcal{H}_Y(0_{f(x)})(f(y)) = f^-(\mathcal{H}_Y(0_{f(x)}))(y)$$
$$\leq \mathcal{H}_X(f^-(0_{f(x)}))(y) \leq \mathcal{H}_Y(0_x)(y) = d_X(x, y).$$

Hence $G$ is a functor. Moreover, by Theorem 3.9, $G(H(X, d_X)) = G(X, \mathcal{H}_{d_X}) = (X, d_{\mathcal{H}_{d_X}}) = (X, d_X)$ and, by Theorem 3.6(2), $H(G(X, \mathcal{H}_X)) = H(X, d_{\mathcal{H}_X}) = (X, d_{\mathcal{H}_{d_X}}) = (X, \mathcal{H}_X)$. Thus, $\textbf{LA}$ and $\textbf{NPM}$ are isomorphic. $\square$

Let $\textbf{ATOP}$ be a category with object $(X, \tau_X)$ where $\tau_X$ is an Alexandrov topology with a morphism $f : (X, \tau_X) \to (Y, \tau_Y)$ such that $f^-(B) \in \tau_X$ for all $B \in \tau_Y$.

**Theorem 4.2.** Two categories $\textbf{ATOP}$ and $\textbf{LA}$ are isomorphic.
Proof. Define $U: \text{ATOP} \to \text{LA}$ as $U(X, \tau_X) = (X, \mathcal{H}_{\tau_X})$ where $\mathcal{H}_{\tau_X}(A) = \bigvee \{B \in [0, \infty]^X \mid B \leq A, B \in \tau_X\}$ from Theorem 3.8(1). For $B \in \tau_{d_X}$,

$$f^-(\mathcal{H}_{\tau_X}(B)) = \bigvee \{f^-(C) \mid C \leq B, C \in \tau_Y\} \leq \bigvee \{f^-(C) \mid f^-(C) \leq f^-(B), f^-(C) \in \tau_X\} \leq \mathcal{H}_{\tau_X}(f^-(B)).$$

Hence $U$ is a functor.

Define $W: \text{LA} \to \text{ATOP}$ as $W(X, \mathcal{H}_X) = (X, \tau_{\mathcal{H}_X})$ where $\tau_{\mathcal{H}_X} = \{A \in [0, \infty]^X \mid A = \mathcal{H}_X(A)\}$ from Theorem 3.6(1). For $B = \mathcal{H}_Y(B)$, $f^-(B) \in \tau_{\mathcal{H}_X}$ because

$$f^-(B) = f^-(\mathcal{H}_{\tau_Y}(B)) \leq \mathcal{H}_{\tau_X}(f^-(B)) \leq f^-(B).$$

Hence $W$ is a functor. Moreover, by Theorem 3.8(1), $U(W(X, \mathcal{H}_X)) = U(X, \tau_{\mathcal{H}_X}) = (X, \mathcal{H}_{\tau_X}) = (X, \mathcal{H}_X)$ and, by Theorem 3.8(1), $W(U(X, \tau_X)) = W(X, \mathcal{H}_{\tau_X}) = (X, \tau_{\mathcal{H}_X}) = (X, \tau_X)$. Thus, LA and NPM are isomorphic. \hfill $\square$

Theorem 4.3. Two categories ATOP and NPM are isomorphic.

Proof. Define $T: \text{NPM} \to \text{ATOP}$ as $T(X, d_X) = (X, \tau_{d_X})$ where $\tau_{d_X} = \{A \in [0, \infty]^X \mid A(x) + d_X(x, y) \geq A(y)\}$ from Theorem 3.5. Let $d_Y(f(x), f(z)) \leq d_X(x, z)$.

For $B \in \tau_{d_Y}$, we have

$$f^-(B)(x) + d_X(x, z) \geq B(f(x)) + d_Y(f(x), f(z)) \geq B(f(z)) = f^-(B)(z).$$

Hence $T$ is a functor.

Define $P: \text{ATOP} \to \text{NPM}$ as $P(X, \tau_X) = (X, d_{\tau_X})$ where $d_{\tau_X}(x, y) = \bigvee_{A \in \tau_X}(A(x) \to A(y))$ from Theorem 3.8(2). Let $f^-(B) \in \tau_X$ for $B \in \tau_Y$. We have

$$d_{\tau_Y}(f(x), f(y)) = \bigvee_{B \in \tau_Y}(B(f(x)) \to B(f(x))) = \bigvee_{B \in \tau_Y}(f^-(B)(x) \to f^-(B)(x)) \leq \bigvee_{A \in \tau_X}(A(x) \to A(y)) = d_{\tau_X}(x, y)$$

Hence $P$ is a functor. Moreover, by Theorem 3.8(2), $T(P(X, \tau_X)) = T(X, d_{\tau_X}) = (X, \tau_{d_X}) = (X, \tau_X)$ and, by Theorem 3.9, $P(T(X, d_X)) = P(X, \tau_{d_X}) = (X, d_{\tau_X}) = (X, d_X)$ \hfill $\square$

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\textit{Mathematics Department, Gangneung-Wonju National University, Gangneung, 25457, Korea}
\textit{Email address: jumokoh@gwnu.ac.kr}

\textit{Mathematics Department, Gangneung-Wonju National University, Gangneung, 25457, Korea}
\textit{Email address: yck@gwnu.ac.kr}