Honam Mathematical J. **42** (2020), No. 3, pp. 425–447
https://doi.org/10.5831/HMJ.2020.42.3.425

THE BASIC KONHAUSER MATRIX POLYNOMIALS

Ayman Shehata

Abstract. The family of q-Konhauser matrix polynomials have been extended to Konhauser matrix polynomials. The purpose of the present work is to show that an extension of the explicit forms, generating matrix functions, matrix recurrence relations and Rodrigues-type formula for these matrix polynomials are given, our desired results have been established and their applications are presented.

1. Introduction

The study of special functions of matrices is a very popular topic in the literature of matrix analysis. In the last two decades, this study has been made more systematic with the consequence that many basic results of scalar orthogonality have been extended to the matrix case. Recently, q-calculus was been used as a bridge between physics and mathematics. Therefore, there is the significant increase in activity in an area of the q-calculus due to its applications in physics, mathematics, and statistics (see [1, 2, 3, 4, 6, 7, 8, 9, 10, 13, 16, 21, 37, 38]). In [5], the q-Konhauser polynomials are a q-analog of the Konhauser polynomials, introduced by Al-Salam and Verma. In [22, 23, 24, 25], Salem extended q-special functions of complex variable to q-special matrix functions. In [36], the authors have discussed a family of new q-extensions of the Humbert functions.

Motivated by their work, q-Konhauser polynomials given in [5], has been studied systematically and comprehensively in the literature. The q-extension in [5, 20, 35] and matrix extension in [12, 26, 27, 28, 29, 30, 31, 32, 33, 34, 39] of these Konhauser matrix polynomials has been given.

Received September 30, 2019. Revised June 27, 2020. Accepted June 27, 2020.
2010 Mathematics Subject Classification. 33C45, 33D15, 39A13, 33D05, 15A15, 15A60.
Key words and phrases. Matrix functional calculus, q-shifted matrix function, q-Konhauser matrix polynomials.
The author received no specific funding for this work.

In the present paper, we construct to q-Konhauser matrix polynomials and to derive different families of generating matrix functions for these matrix polynomials and give some results for these matrix polynomials. The main results of our discussion are presented in Sections 2-4.

1.1. Preliminaries

Here in the presentation, we recall some properties and notations, which will be used throughout this work below. All this paper, the symbols $\mathbb{C}^{N\times N}$ denote the complex space of all square complex matrices of common order N, for $N \in \mathbb{N}$ where $\mathbb{N}$ and $\mathcal{C}$ are the sets of natural and complex numbers. For a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A.

For the purpose of this work, we denote by $\mu(A)$ the logarithmic norm of A, which is defined as [14]

$$\mu(A) = \max\left\{z; z \text{ eigenvalue of } \frac{A + A^T}{2}\right\}, \tag{1}$$

where A^T denotes the transpose conjugate of A. We denote by the number $\tilde{\mu}(A)$

$$\tilde{\mu}(A) = \mu(-A) = \min\left\{z; z \text{ eigenvalue of } \frac{A + A^T}{2}\right\}. \tag{2}$$

From [14], it follows that $\|e^{At}\| \leq e^{t\mu(A)}$ for $t \geq 0$. Hence, $t^A = \exp(A \ln t)$, we have

$$\|t^A\| = \begin{cases} t^{\mu(A)}, & \text{if } t \geq 1, \\ t^{\tilde{\mu}(A)}, & \text{if } 0 \leq t \leq 1. \end{cases} \tag{3}$$

Recently, Salem [22, 23, 25, 24] gave some following required some definitions related to q-analysis:

Definition 1.1. *Let A be complex square matrix in $\mathbb{C}^{N\times N}$, then the q-analogue of A is defined as*

$$[A]_q = \frac{I - q^A}{1 - q}\ ;\ 0 < |q| < 1, q \in \mathcal{C} - \{1\}, q^A = e^{A \log q}. \tag{4}$$

Definition 1.2. *The q-shifted factorial matrix function $(A; q)_n$ (the q-extension of the Pochhammer symbol $(A)_n$) is defined by*

$$(A; q)_n = \prod_{k=0}^{n-1}(I - q^k A), n = 1, 2, \ldots; (A; q)_0 = I, \tag{5}$$
$$(\|q^k A\| < 1, 0 < |q| < 1, q \in \mathcal{C} - \{1\}),$$

for any complex square matrix A (see [22]). The generalization of (5) is

$$(A;q)_\infty = \lim_{n\to\infty}\prod_{k=0}^{n-1}(I-q^kA) = \prod_{k=0}^{\infty}(I-q^kA), \tag{6}$$
$$(\|q^kA\|<1, 0<|q|<1, q\in\mathcal{C}-\{1\}),$$

converges. From (5) and (6), we can easily the relation

$$(A;q)_n = (A;q)_\infty[(Aq^k;q)_\infty]^{-1}. \tag{7}$$

Lemma 1.3. *If $\mathbf{B}(k,n)$ is a matrix in $\mathbb{C}^{N\times N}$ for $k,n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the following relations are satisfied (see, Defez and Jódar [11])*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbf{B}(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathbf{B}(k,n-k) \tag{8}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathbf{B}(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbf{B}(k,n+k). \tag{9}$$

The following results and definitions will be required in our investigation. Let D_q be the q-derivative defined by means of the following

$$D_qf(x) = \frac{f(x)-f(qx)}{(1-q)x}. \tag{10}$$

By mathematical induction it is easy to verify the relation

$$(x^{k+1}D_q)^n x^A = (q^A;q^k)_n x^{A+nkI}. \tag{11}$$

Note that the q-shifted factorial is defined as

$$[aq^{-n}]_n = (-a)^n q^{-\frac{1}{2}n(n+1)}\left[\frac{q}{a}\right]_n, \tag{12}$$

and in general

$$[aq^{-kn}]_n = (-a)^n q^{\frac{1}{2}n(n-1)-kn}\frac{\left[\frac{q}{a}\right]_{kn}}{\left[\frac{q}{a}\right]_{(k-1)n}}.$$

Using (12), one gets the identity

$$[a]_{m-n} = \frac{[a]_m q^{\frac{1}{2}n(n+1)}}{\left[\frac{q^{1-m}}{a}\right]_n(-a)^n q^{mn}} = \frac{[aq^{-n}]_m}{[a]_{-n}}. \tag{13}$$

If n is a positive integer, in [15], the basic binomial theorem can be considered from another point of view as

$$(14)\qquad \begin{aligned}(1-x)^{(n)} &= (1-x)(1-qx)\dots(1-q^{n-1}x)\\ &= \sum_{r=0}^{n}\frac{[q]_n}{[q]_r[q]_{n-r}}(-1)^r x^r q^{\frac{1}{2}r(r-1)}.\end{aligned}$$

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$Re(\mu) > -1, \quad \text{for all eigenvalues } \mu\in\sigma(A)$$

and λ is a complex number with $Re(\lambda) > 0$, $k\in\mathbb{N}$, the Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k)$ and $Y_n^{(A,\lambda)}(x;k)$ are dened by Varma et al. [39]

(15)

$$Z_n^{(A,\lambda)}(x;k) = \Gamma(A+(kn+1)I)\sum_{r=0}^{n}\frac{(-1)^r(\lambda\, x)^{rk}}{r!(n-r)!}\Gamma^{-1}(A+(kr+1)I)$$

and

$$(16)\qquad Y_n^{(A,\lambda)}(x;k) = \frac{1}{n!}\sum_{r=0}^{n}\sum_{s=0}^{r}\frac{(-1)^s(\lambda x)^r}{s!(r-s)!}\left(\frac{1}{k}((s+1)I+A)\right)_n.$$

2. Definition and some new results for first kind q-Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k|q)$

This section is devoted to introduce some generating matrix functions, recurrence matrix relations, and Rodrigues type formula for the first kind q-Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k|q)$ for $0<|q|<1$ and q in $\mathcal{C}-\{1\}$.

Definition 2.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition*

$$(17)\qquad \tilde{\mu}(A) > -1, \quad \textit{for all eigenvalues } \mu\in\sigma(A)$$

and λ be a complex number with $Re(\lambda) > 0$, $0<|q|<1; q\in\mathcal{C}-\{1\}$ and $q^{-k}\notin\sigma(q^A)$ for all $0\le k\le n$, $n\in\mathbb{N}_0$. We define the first kind *q-Konhauser matrix polynomials in the form*

(18)

$$Z_n^{(A,\lambda)}(x;k|q) = \frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n} \sum_{m=0}^{n} \frac{(q^{-nk};q^k)_m}{(q^k;q^k)_m} q^{\frac{1}{2}km(km-1)I+km(A+(n+1)I))} \times \left([q^{A+I}]_{km}\right)^{-1} (\lambda\, x)^{mk}.$$

Remark 2.2. *When $q \longrightarrow 1$ in eq. (18), we obtain the Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k)$ dened in (15).*

Now, we demonstrate the current Lemma to use in the following proofs theorems

Lemma 2.3. *Let A be complex square matrix in $\mathbb{C}^{N\times N}$, $0 < |q| < 1; q \in \mathcal{C} - \{1\}$ and $q^{-n} \notin \sigma(q^A), n \in \mathbb{N}_0$. Let A is a positive stable matrix or $\mu(A) > 0$ satisfying the condition $A + nI$ is an invertible matrix for all integers $n \geq 0$. Then the identities are valid*

(19)
$$(A;q^k)_n = (-A)^n p^{-\frac{kn(n-1)}{2}} (A^{-1};p^k)_n, \quad (\|A\| < 1, \|A^{-1}\| < 1, p = \frac{1}{q}).$$

Proof. For $p = \frac{1}{q}$, we have

$$\begin{aligned}(A;q^k)_n &= \prod_{r=0}^{n-1}(I - q^{rk}A) = (I-A)(I-q^kA)(I-q^{2k}A)\dots(I-q^{kn-k}A)\\ &= (I-A)(I-p^{-k}A)(I-p^{-2k}A)\dots(I-p^{k-kn}A) = (Ap^{k-kn};p^k)_n\\ &= (-A)^n p^{-\frac{1}{2}kn(n-1)}(A^{-1};p^k)_n.\end{aligned}$$

Thus,

$$(A;q^k)_n = (-A)^n p^{-\frac{1}{2}kn(n-1)}(A^{-1};p)_n,$$

for $k = 1$, it reduces to

$$[A]_n = (-A)^n p^{-\frac{1}{2}n(n-1)}(A^{-1};p)_n.$$

Similarly, we can obtain

$$(A;p^k)_n = (-A)^n q^{-\frac{1}{2}kn(n-1)}(A^{-1};q^k)_n\,;\; p = \frac{1}{q}.$$

For $k = 1$, it reduces to

(20)
$$(A;p)_n = (-A)^n q^{-\frac{1}{2}n(n-1)}[A^{-1}]_n\,;\; p = \frac{1}{q},$$

the desired identity follows. □

Theorem 2.4. *The first kind q-Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k|q)$ has the generating matrix functions*

$$\sum_{n=0}^{\infty}\left([q^{A+I}]_{nk}\right)^{-1} Z_n^{(A,\lambda)}(x;k|q)t^n = \frac{f^{(A)}(t(\lambda\, x)^k)}{(t;q^k)_\infty}, \tag{21}$$

where

$$f^{(A)}(u) = \sum_{r=0}^{\infty}\frac{q^{\frac{1}{2}kr((kr+r)I+2A)}}{(q^k,q^k)_r}(-u)^r\left([q^{A+I}]_{kr}\right)^{-1}.$$

Proof. We consider the sum

$$\sum_{n=0}^{\infty}\left[[q^{A+I}]_{nk}\right]^{-1} Z_n^{(A,\lambda)}(x;k|q)t^n = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{1}{(q^k;q^k)_n}$$
$$\times\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\left([q^{A+I}]_{kr}\right)^{-1}(\lambda\, x)^{rk}t^n$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{(q^k;q^k)_{n+r}}\frac{(q^{-nk-kr};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+r+1)I))}$$
$$\times\left([q^{A+I}]_{kr}\right)^{-1}(\lambda\, x)^{rk}t^{n+r}.$$

By relation (20), we have

$$(q^{-nk-kr};q^k)_r = (-1)^r q^{-nkr-\frac{1}{2}r(r+k)}(q^{nk+k};q^k)_r$$

and

$$\frac{(q^{nk+k};q^k)_r}{(q^k;q^k)_{n+r}} = \frac{(q^{n+k};q^k)_r}{(q^{nk+k};q^k)_r(q^k;q^k)_n} = \frac{1}{(q^k;q^k)_n}.$$

Also, we get

$$\sum_{n=0}^{\infty}\left([q^{A+I}]_{nk}\right)^{-1} Z_n^{(A,\lambda)}(x;k|q)t^n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r}{(q^k;q^k)_n}$$
$$\times \frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r} q^{-\frac{1}{2}r(k+r)} q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$
$$\times \left([q^{A+I}]_{kr}\right)^{-1}(\lambda\, x)^{rk} t^{n+r} = \sum_{n=0}^{\infty}\frac{t^n}{(q^k;q^k)_\infty}$$
$$\times \sum_{r=0}^{\infty}\frac{q^{\frac{1}{2}kr((kr+r)I+2A)}}{(q^k,q^k)_r}\left(-t(\lambda\, x)^k\right)^r\left([q^{A+I}]_{kr}\right)^{-1}$$
$$= \frac{1}{(t;q^k)_\infty}\sum_{r=0}^{\infty}\frac{q^{\frac{1}{2}kr((kr+r)I+2A)}}{(q^k,q^k)_r}$$
$$\times \left(-t(\lambda\, x)^k\right)^r\left([q^{A+I}]_{kr}\right)^{-1} = \frac{f^{(A)}(t(\lambda\, x)^k)}{(t;q^k)_\infty},$$

where

$$f^{(A)}(u) = \sum_{r=0}^{\infty}\frac{q^{\frac{1}{2}kr((kr+r)I+2A)}}{(q^k,q^k)_r}(-u)^r\left([q^{A+I}]_{kr}\right)^{-1}.$$

It proves (21). □

Theorem 2.5. *Let $Z_n^{(A,\lambda)}(x;k|q)$. Then we get*

$$Z_n^{(A,\lambda)}(xy;k|q) = [q^{A+I}]_{kn}\sum_{r=0}^{n}\frac{(y^k;q^k)_r}{(q^k;q^k)_r}\left([q^{A+I}]_{kn-kr}\right)^{-1} \times y^{k(n-r)} Z_{n-r}^{(A,\lambda)}(x;k|q). \tag{22}$$

Proof. Replacing x by xy on the left hand member in (21), we get

$$\sum_{n=0}^{\infty}\left([q^{A+I}]_{nk}\right)^{-1}Z_n^{(A,\lambda)}(xy;k|q)t^n=\frac{f^{(A)}(t(\lambda\,xy)^k)}{(t;q^k)_\infty}$$
$$=\frac{(ty^k;q^k)_\infty}{(t;q^k)_\infty}\frac{f^{(A)}(t(\lambda\,xy)^k)}{(ty^k;q^k)_\infty}$$
$$=\frac{(ty^k;q^k)_\infty}{(t;q^k)_\infty}\sum_{n=0}^{\infty}\left([q^{A+I}]_{nk}\right)^{-1}Z_n^{(A,\lambda)}(x;k|q)(ty^k)^n$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(y^k;q^k)_r}{(q^k;q^k)_r}t^r\left([q^{A+I}]_{kn}\right)^{-1}Z_n^{(A,\lambda)}(x;k|q)(ty^k)^n$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(y^k;q^k)_r}{(q^k;q^k)_r}\left([q^{A+I}]_{k(n-r)}\right)^{-1}Z_{n-r}^{(A,\lambda)}(x;k|q)t^ny^{k(n-r)}$$
$$=\sum_{n=0}^{\infty}t^n\sum_{r=0}^{n}\frac{(y^k;q^k)_r}{(q^k;q^k)_r}\left([q^{A+I}]_{k(n-r)}\right)^{-1}Z_{n-r}^{(A,\lambda)}(x;k|q)y^{k(n-r)}.$$

On equating the coefficients of t^n on both sides, we obtain (22). □

Theorem 2.6. *The first kind q-Konhauser matrix polynomials* $Z_n^{(A,\lambda)}(x;k|q)$ *satisfy the matrix relation*

$$\begin{aligned}&\left[D_p^k(\lambda\,x)^{A+I}D_p\right]Z_n^{(A,\lambda)}(x;k|q)\\&=(-\lambda)^k[q^{A+I}]_{nk}\left([q^{A+I}]_{nk-k}\right)^{-1}(\lambda\,x)^AZ_{n-1}^{(A,\lambda)}(x;k|q).\end{aligned}\tag{23}$$

Proof. Differentiating it on both sides of (18) with respect to x and replacing p by $\frac{1}{q}$, we get

$$D_pZ_n^{(A,\lambda)}(x;k|q)=\frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n}\sum_{r=0}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}$$
$$q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\left([q^{A+I}]_{kr}\right)^{-1}D_p(\lambda\,x)^{rk}$$
$$=\frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n}\sum_{r=1}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$
$$\times\left([q^{A+I}]_{kr}\right)^{-1}\lambda^{rk}\left[\frac{x^{rk}-(px)^{rk}}{x}\right]$$

$$= \frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n} \sum_{r=1}^{n} \frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r} q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$

$$\times \left([q^{A+I}]_{kr}\right)^{-1} \lambda^{rk} x^{rk-1} \Big[1-p^{rk}\Big]$$

$$= \frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n} \sum_{r=1}^{n} \frac{(q^{-nk};q^k)_r}{(q^k;q^k)_{r-1}} q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$

$$\times \left([q^{A+I}]_{kr}\right)^{-1} \lambda^{rk} x^{rk-1} (-1) q^{rk}.$$

Multiplying both sides by x^{A+I} and differentiating k times with respect to x with base p^k (replacing p by $\frac{1}{q}$), we get

$$\Big[D_p^k (\lambda\, x)^{A+I} D_p\Big] Z_n^{(A,\lambda)}(x;k|q) = \Big[D_p^k x^{A+I}\Big] D_p Z_n^{(A,\lambda)}(x;k|q)$$

$$= \frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n} \sum_{r=1}^{n} \frac{(q^{-nk};q^k)_r}{(q^k;q^k)_{r-1}} q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$

$$\times \left([q^{A+I}]_{kr}\right)^{-1} (-\lambda) q^{rk} \Big[1-p^{rk}\Big] \Big[D_p^k (\lambda\, x)^{A+I}\Big] (\lambda\, x)^{rk-1}$$

$$= \frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n} \sum_{r=1}^{n} \frac{(q^{-nk};q^k)_r}{(q^k;q^k)_{r-1}} q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$

$$\times \left([q^{A+I}]_{kr}\right)^{-1} (-\lambda) q^{rk} \big[1-p^{rk}\big] D_p^k (\lambda\, x)^{A+rkI}$$

$$= \frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n} \sum_{r=0}^{n} \frac{(q^{-nk};q^k)_r}{(q^k;q^k)_{r-1}} q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$

$$\times \left([q^{A+I}]_{kr}\right)^{-1} (-\lambda) q^{rk} \lambda^{rk} \big[1-p^{rk}\big] \big[1-p^{A+rk}\big]$$

$$\times \big[1-p^{A+rk-1}\big] \dots \big[1-p^{A+rk-k+1}\big] x^{A+(rk-k)I}$$

$$= \frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n} \sum_{r=0}^{n} \frac{(q^{-nk};q^k)_r}{(q^k;q^k)_{r-1}} q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}$$

$$\times \left([q^{A+I}]_{kr}\right)^{-1} \lambda^{rk} (-\lambda)^k q^{rk} (p^{A+(rk-k+1)I};p)_k (\lambda\, x)^{A+(rk-k)I}.$$

Using (20) and multiplying it in numerator and denominator of right hand member by $(q^{A+I};q^k)_{(n-1)k}$, we get

$$\Big[D_p^k(\lambda\, x)^{A+I}\Big]D_pZ_n^{(A,\lambda)}(x;k|q)=\frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n}(\lambda\, x)^A$$
$$\sum_{r=1}^{n}\frac{(q^{-nk+k};q^k)_r}{(q^k;q^k)_{r-1}}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\bigg([q^{A+I}]_{kr}\bigg)^{-1}$$
$$\times\,\lambda^{rk}(-1)q^{rk}(\lambda\, x)^{rk-k}(-\lambda)^kq^{-kA-krI+\frac{1}{2}k(k-1)I}[q^{A+I+k(r-1)I}]_k$$
$$=(-1)^{k+1}\frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n}(\lambda\, x)^A\sum_{r=0}^{n}\frac{(q^{-nk+k};q^k)_{r+1}}{(q^k;q^k)_r}$$
$$\times\, q^{\frac{1}{2}k(r+1)(k(r+1)-1)I+k(r+1)(A+(n+1)I))}\bigg([q^{A+I}]_{k(r+1)}\bigg)^{-1}$$
$$\times\,\lambda^{(r+1)k}(\lambda\, x)^{rk}q^{-kA-k(r+1)I+\frac{1}{2}k(k-1)I}[q^{A+I+krI}]_k$$
$$=(-1)^{k+1}\frac{[q^{A+I}]_{nk}}{(q^k;q^k)_n}(\lambda\, x)^A\sum_{r=0}^{n}\frac{(q^{-nk+k};q^k)_r}{(q^k;q^k)_{r-1}}$$
$$\times\, q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\bigg([q^{A+I}]_{kr+k}\bigg)^{-1}\lambda^k$$
$$(\lambda\, x)^{rk}q^{\frac{1}{2}kr(kr-1)I+kr(A+nI)}[q^{A+I+krI}]_k$$
$$=(-\lambda)^k\frac{[q^{A+I}]_{nk}}{(q^k;q^k)_{n-1}}(\lambda\, x)^A\sum_{r=0}^{\infty}\frac{(q^{-nk+k};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)}$$
$$\times\bigg([q^{A+I}]_{kr}\bigg)^{-1}(\lambda\, x)^{rk}q^{kr(A+I+(n-1)I)}=(-\lambda)^k[q^{A+I}]_{nk}$$
$$\times\bigg([q^{A+I}]_{nk-k}\bigg)^{-1}(\lambda\, x)^A\frac{[q^{A+I}]_{(n-1)k}}{(q^k;q^k)_{n-1}}\sum_{r=0}^{n-1}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}$$
$$\times\, q^{\frac{1}{2}kr(r-1)I+kr(A+I+(n-1)I))}\bigg([q^{A+I}]_{kr}\bigg)^{-1}(-\lambda)^k(\lambda\, x)^{rk}$$
$$=(-\lambda)^k[q^{A+I}]_{nk}\bigg([q^{A+I}]_{nk-k}\bigg)^{-1}(\lambda\, x)^AZ_{n-1}^{(A,\lambda)}(x;k|q).$$

Hence, from (23), we have the desired result. □

Theorem 2.7. *The first kind q-Konhauser matrix polynomials satisfy the matrix recurrence relation*

$$\begin{aligned} &q^{\frac{1}{2}k(2A+(k+2n+1)I)}(\lambda\, x)^k Z_n^{(A+kI,\lambda)}(x;k|q) \\ &+(1-q^{k(n+1)})Z_{n+1}^{(A,\lambda)}(x;k|q)=[q^{A+(kn+1)I}]_k Z_n^{(A,\lambda)}(x;k|q). \end{aligned} \tag{24}$$

Proof. Multiplying (18) by $q^{\frac{1}{2}k(2A+(k+2n+1)I)}(\lambda x)^k$ with replacing $A+kI$ for A and multiplying (18) by $(1-q^{k(n+1)})$ with replacing $n+1$ for n, we get

$$q^{\frac{1}{2}k(2A+(k+2n+1)I)}(\lambda x)^k Z_n^{(A+kI,\lambda)}(x;k|q)+(1-q^{k(n+1)})Z_{n+1}^{(A,\lambda)}(x;k|q).$$

Substituting for $Z_n^{(A+kI,\lambda)}(x;k|q)$ and $Z_{n+1}^{(A,\lambda)}(x;k|q)$ from (18), we get

$$\begin{aligned} &q^{\frac{1}{2}k(2A+(k+2n+1)I)}(\lambda x)^k Z_n^{(A+kI,\lambda)}(x;k|q)+(1-q^{k(n+1)})Z_{n+1}^{(A,\lambda)}(x;k|q) \\ &=q^{\frac{1}{2}k(2A+(2n+k+1)I)}(\lambda\, x)^k \frac{[q^{A+(k+1)I}]_{nk}}{(q^k;q^k)_n}\sum_{r=0}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r} \\ &\times q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+k+1)I))}\left([q^{A+(k+1)I}]_{kr}\right)^{-1}(\lambda\, x)^{rk} \\ &+(1-q^{k(n+1)})\frac{[q^{A+I}]_{(n+1)k}}{(q^k;q^k)_{n+1}}\sum_{r=0}^{n+1}\frac{(q^{-kn-k};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+2)I))} \\ &\times\left([q^{A+I}]_{kr}\right)^{-1}(\lambda\, x)^{rk} \\ &=\frac{[q^{A+I}]_{nk}[q^{A+(nk+1)I}]_k}{(q^k;q^k)_n}\left([q^{A+I}]_k\right)^{-1}\sum_{r=0}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r} \\ &\times q^{\frac{1}{2}kr(kr-1)I}q^{kr(A+(k+n+1)I))}\left([q^{A+(k+1)I}]_{kr}\right)^{-1}(\lambda x)^{(r+1)k}q^{\frac{1}{2}k(2A+(2n+k+1)I)} \\ &+\frac{[q^{A+I}]_{(n+1)k}}{(q^k;q^k)_n}\sum_{r=0}^{n+1}\frac{(q^{-nk};q^k)_r(1-q^{-kn-k})}{(q^k;q^k)_r(1-q^{-nk+kr-k})} \\ &\times q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+2)I))}\left([q^{A+I}]_{kr}\right)^{-1}(\lambda\, x)^{rk} \end{aligned}$$

$$= \frac{[q^{A+I}]_{nk}[q^{A+(nk+1)I}]_k}{(q^k;q^k)_n}\sum_{r=0}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(k+n+1)I))}$$

$$\times\left([q^{A+I}]_{k(r+1)}\right)^{-1}(\lambda\,x)^{(r+1)k}q^{\frac{1}{2}k(2A+(k+2n+1)I)}$$

$$+\frac{[q^{A+I}]_{nk}[q^{A+(nk+1)I}]_k}{(q^k;q^k)_n}\sum_{r=0}^{n+1}\frac{(q^{-nk};q^k)_r(1-q^{-kn-k})}{(q^k;q^k)_r(1-q^{-nk+kr-k})}$$

$$\times q^{kr(A+(n+2)I))}\left([q^{A+I}]_{kr}\right)^{-1}(\lambda\,x)^{rk}.$$

Now the coefficient of $(\lambda\,x)^{kr}$ in the above expression is

$$\frac{[q^{A+(nk+1)I}]_k[q^{A+I}]_{nk}}{(q^k;q^k)_n}\Bigg[\frac{(q^{-nk};q^k)_{r-1}}{(q^k;q^k)_{r-1}}q^{\frac{1}{2}k(r-1)(kr-k-1)}$$

$$\times\left([q^{A+I}]_{kr}\right)^{-1}q^{k(r-1)(A+(n+k+1)I)+\frac{1}{2}k(2A+(2n+k+1)I)}$$

$$+\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+2)I))}\left([q^{A+I}]_{kr}\right)^{-1}$$

$$\times\frac{1-q^{-kn-k}}{1-q^{-kn+kr-k}}\Bigg]=\frac{[q^{A+(nk+1)I}]_k[q^{A+I}]_{nk}}{(q^k;q^k)_n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}$$

$$\times q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\left([q^{A+I}]_{kr}\right)^{-1}$$

$$\times\left[\frac{(1-q^{kr})q^{\frac{1}{2}k(2A+(2n+k+1)I)}}{1-q^{-kn+kr-k}}q^{-k(A+nI+\frac{1}{2}(k+1)I)}+\frac{1-q^{-kn-k}}{1-q^{-kn+kr-k}}q^{kr}\right]$$

$$=\frac{[q^{A+(nk+1)I}]_k[q^{A+I}]_{nk}}{(q^k;q^k)_n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\left([q^{A+I}]_{kr}\right)^{-1}$$

$$\times\left[\frac{1-q^{kr}}{1-q^{-kn+kr-k}}+\frac{q^{kr}-q^{-kn+kr-k}}{1-q^{-kn+kr-k}}\right]=\frac{[q^{A+(nk+1)I}]_k[q^{A+I}]_{nk}}{(q^k;q^k)_n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}$$

$$\times q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\left([q^{A+I}]_{kr}\right)^{-1}.$$

Hence,

$$\frac{[q^{A+(nk+1)I}]_k[q^{A+I}]_{nk}}{(q^k;q^k)_n}\sum_{r=0}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{\frac{1}{2}kr(kr-1)I+kr(A+(n+1)I))}\left([q^{A+I}]_{kr}\right)^{-1}$$

$$=[q^{A+(nk+1)I}]_kZ_r^{(A,\lambda)}(x;k|q).$$

Thus, we obtain (24). Therefore the first kind q-Konhauser matrix polynomials satisfy the matrix recurrence relation (24). $\square$

Remark 2.8. *The formula (18) for $k = 1$ is now*

$$\begin{aligned} &Z_n^{(A,\lambda)}(x;1|q) \\ &= \frac{[q^{A+I}]_n}{(q;q)_n}\sum_{m=0}^{n}\frac{(q^{-n};q)_m}{(q;q)_m}q^{\frac{1}{2}m(m-1)I+m(A+(n+1)I))}\left([q^{A+I}]_m\right)^{-1}(\lambda\, x)^m \\ &= \frac{[q^{A+I}]_n}{[q]_n}\sum_{m=0}^{n}\frac{[q^{-n}]_m}{[q]_m}q^{\frac{1}{2}m(m+1)I+m(A+nI))}\left([q^{A+I}]_m\right)^{-1}(\lambda\, x)^m \\ &= L_n^{(A,\lambda)}(x|q). \end{aligned}$$

Thus, for $k = 1$, $Z_n^{(A,\lambda)}(x;1|q)$ reduces to q-Laguerre matrix polynomials $L_n^{(A,\lambda)}(x|q)$ [25].

Remark 2.9. *The properties (20), (21), (22) and (23) reduce for $k =$ 1, to corresponding properties for the q-Laguerre matrix polynomials* [25].

3. Definition and some new relations for second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$

This section is devoted to introduce some recurrence relations and Rodrigues type formula for the second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$ for $0 < |q| < 1$ and q in $\mathcal{C} - \{1\}$.

Definition 3.1. *We define the second kind q-Konhauser matrix polynomials in the form*

(25)
$$Y_n^{(A,\lambda)}(x;k|q) = \frac{1}{[q]_n}\sum_{r=0}^{n}\sum_{s=0}^{r}\frac{q^{\frac{1}{2}r(r-1)}[q^{-r}]_s q^s}{[q]_r[q]_s}(q^{A+(s+1)I};q^k)_n(\lambda x)^r,$$

where λ be a complex number with $Re(\lambda) > 0$ and A a complex square matrix satisfying the spectral conditions (17), $0 < |q| < 1; q \in \mathcal{C} - \{1\}$ and $q^{-k} \notin \sigma(q^A)$ for all $k = 0, 1, 2, \ldots, n$ or $0 \le k \le n$ and $n \in \mathbb{N}_0$.

Remark 3.2. *When $q \longrightarrow 1$ in eq. (25), we get the Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k)$ dened in (16).*

Now, we prove the current Lemma to use in the following proofs theorems.

Lemma 3.3. *Let A be complex square matrix in $\mathbb{C}^{N\times N}$, $0<|q|<1; q\in\mathcal{C}-\{1\}$ and $q^{-n}\notin\sigma(q^A), n\in\mathbb{N}_0$. Let A is a positive stable matrix or $\mu(A)>0$. Then we have the following identities*

$$D_q^m(xA;q^k)_n=\sum_{r=m}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{knr}A^r(-1)^m \times q^{mr-\frac{1}{2}m(m-1)}[q^{-r}]_m x^{r-m};\ \|xA\|<1. \tag{26}$$

Proof. Let $\|xA\|<1$ and $0<|q|<1; q\in\mathcal{C}-\{1\}$, then we have

$$(xA;q^k)_n=\sum_{r=0}^{n}[q^k]_n[q^k]_r[q^k]_{n-r}(-1)^r q^{\frac{1}{2}kr(r-1)}x^rA^r$$

and

$$\begin{aligned}
D_q^m(xA;q^k)_n&=D_q^m\sum_{r=0}^{n}[q^k]_n[q^k]_r[q^k]_{n-r}A^r(-1)^r q^{\frac{1}{2}kr(r-1)}x^r\\
&=\sum_{r=0}^{n}[q^k]_n[q^k]_r[q^k]_{n-r}A^r(-1)^r q^{\frac{1}{2}kr(r-1)}D_q^m x^r\\
&=\sum_{r=m}^{n}[q^k]_n[q^k]_r[q^k]_{n-r}A^r(-1)^{r+m} q^{\frac{1}{2}kr(r-1)}[q^{r-m+1}]_m x^{r-m}\\
&=\sum_{r=m}^{n}[q^k]_n[q^k]_r[q^k]_{n-r}A^r(-1)^{m+r} q^{\frac{1}{2}kr(r-1)}\\
&\times q^{mr-\frac{1}{2}m(m-1)}[q^{-r}]_m x^{r-m}=\sum_{r=m}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}\frac{(-q^k)^r}{q^{\frac{1}{2}kr(r+1)}}\\
&\times q^{knr}A^r(-1)^{m+r}q^{\frac{1}{2}kr(r-1)}q^{mr-\frac{1}{2}m(m-1)}[q^{-r}]_m x^{r-m},
\end{aligned}$$

or

$$D_q^m(xA;q^k)_n=\sum_{r=m}^{n}\frac{(q^{-nk};q^k)_r}{(q^k;q^k)_r}q^{knr}A^r(-1)^m q^{mr-\frac{1}{2}m(m-1)}[q^{-r}]_m x^{r-m}.$$

This proves the Lemma. □

Theorem 3.4. *The second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$ has the generating matrix function*

$$\sum_{n=0}^{\infty}\frac{[q]_n}{(q^k;q^k)_n}Y_n^{(A,\lambda)}(x;k|q)t^n=\frac{\Phi(\lambda\, x,tq^{A+I})}{(t;q^k)_\infty}, \tag{27}$$

where

$$\Phi(\lambda\, x, tq^{A+I}) = \sum_{s=0}^{\infty} \frac{q^{\frac{1}{2}ks(s-1)}(-1)^s q^{s(A+I)} t^s}{(q^k;q^k)_s}(1-\lambda x)^{(s)}.$$

Proof. By use of relation (10), we can write

$$Y_n^{(A,\lambda)}(x;k|q) = \frac{1}{[q]_n}\sum_{r=0}^{n}\frac{q^{\frac{1}{2}r(r-1)}}{[q]_r}(\lambda x)^r D_q^r(\lambda x q^{A+I};q^k)_n\bigg|_{x=1}.$$

By the relation (14), we have

$$\begin{aligned}(\lambda\, xq^{A+I};q^k)_n &= (I-\lambda\, xq^{A+I})^{(n)} \\ &= \sum_{s=0}^{n}\frac{(q^k;q^k)_n(-1)^s q^{\frac{1}{2}ks(ks-1)}}{(q^k;q^k)_s(q^k;q^k)_{n-s}}(\lambda\, x)^s q^{s(A+I)}.\end{aligned}$$

Therefore, we give

$$D_q^r(\lambda\, xq^{A+I};q^k)_n = \sum_{s=r}^{n}\frac{(q^k;q^k)_n(-1)^s q^{\frac{1}{2}ks(ks-1)}}{(q^k;q^k)_s(q^k;q^k)_{n-s}} q^{s(A+I)} D_q^r(\lambda\, x)^s$$

$$= \sum_{s=r}^{n}\frac{(q^k;q^k)_n(-1)^s q^{\frac{1}{2}ks(ks-1)}}{(q^k;q^k)_s(q^k;q^k)_{n-s}} q^{s(A+I)}[q^{s-r+1}]_r(\lambda\, x)^{s-r}.$$

By relation (12), we have

$$[q^{s-r+1}]_r = (-1)^r q^{sr} q^{-\frac{1}{2}r(r-1)}[q^{-s}]_r,$$

and

$$(q^k;q^k)_{n-s} = \frac{(q^{-nk};q^k)_s(-q^k)^s q^{kns}}{(q^k;q^k)_n q^{\frac{1}{2}ks(s+1)}}.$$

Thus, we get

$$\begin{aligned}&D_q^r(\lambda\, xq^{A+I};q^k)_n\bigg|_{x=1} \\ &\quad = \sum_{s=r}^{n}\frac{(q^{-nk};q^k)_s}{(q^k;q^k)_s} q^{kns} q^{s(A+I)}(-1)^r q^{sr} q^{-\frac{1}{2}r(r-1)}[q^{-s}]_r \lambda^{s-r}.\end{aligned}$$

Hence

$$\begin{aligned}Y_n^{(A,\lambda)}(x;k|q) =& \frac{1}{[q]_n}\sum_{r=0}^{n}\frac{q^{\frac{1}{2}r(r-1)}}{[q]_r}(\lambda x)^r \sum_{s=r}^{n}\frac{(q^{-nk};q^k)_s}{(q^k;q^k)_s} q^{kns} q^{s(A+I)} \\ &\times (-1)^r q^{sr} q^{-\frac{1}{2}r(r-1)}[q^{-s}]_r \lambda^{s-r}.\end{aligned}$$

Since

$$\sum_{r=0}^{n} a_r \sum_{s=r}^{n} b_s = \sum_{s=0}^{n} b_s \sum_{r=0}^{s} a_r, \tag{28}$$

we have

(29)

$$Y_n^{(A,\lambda)}(x;k|q) = \frac{1}{[q]_n} \sum_{s=0}^{n} \frac{(q^{-nk};q^k)_s q^{s(A+(kn+1)I)}}{(q^k;q^k)_s} \sum_{r=0}^{s} \frac{[q^{-s}]_r}{[q]_r}(-\lambda x q^s)^r.$$

Multiplying (29) by $\frac{(q;q)_n}{(q^k;q^k)_n} t^n$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[q]_n}{(q^k;q^k)_n} Y_n^{(A,\lambda)}(x;k|q) t^n = \sum_{n=0}^{\infty} \frac{t^n}{(q^k;q^k)_n} \\
&\sum_{s=0}^{n} \frac{(q^{-nk};q^k)_s q^{s(A+(kn+1)I)}}{(q^k;q^k)_s} \sum_{r=0}^{s} \frac{[q^{-s}]_r}{[q]_r}(-\lambda x q^s)^r \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{q^{\frac{1}{2}ks(s+1)} q^{s(A+(kn+1)I)} t^n}{(q^k;q^k)_{n-s}(q^k;q^k)_s(-q^k)^s q^{kns}} \sum_{r=0}^{s} \frac{[q^{-s}]_r}{[q]_r}(-\lambda x q^s)^r \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{q^{\frac{1}{2}ks(s-1)}(-1)^s}{(q^k;q^k)_n (q^k;q^k)_s} q^{s(A+I)} t^{n+s} \sum_{r=0}^{s} \frac{[q]_s q^{\frac{1}{2}r(r+1)}}{[q]_r [q]_{s-r}(-q)^r q^{rs}}(-\lambda x q^s)^r.
\end{aligned}$$

By using relation (13), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[q]_n}{(q^k;q^k)_n} Y_n^{(A,\lambda)}(x;k|q) t^n \\
&\qquad = \sum_{n=0}^{\infty} \frac{t^n}{(q^k;q^k)_n} \sum_{s=0}^{\infty} \frac{q^{\frac{1}{2}ks(s-1)}(-1)^s q^{s(A+I)} t^s}{(q^k;q^k)_s}(1-\lambda\, x)^{(s)}.
\end{aligned}$$

By use of the relation (14), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[q]_n}{(q^k;q^k)_n} Y_n^{(A,\lambda)}(x;k|q) t^n \\
&= \frac{1}{(t;q^k)_\infty} \sum_{s=0}^{\infty} \frac{q^{\frac{1}{2}ks(s-1)}(-1)^s q^{s(A+I)} t^s}{(q^k;q^k)_s}(1-\lambda\, x)^{(s)} \\
&= \frac{\Phi(\lambda\, x, t q^{A+I})}{(t;q^k)_\infty},
\end{aligned}$$

where

$$\Phi(\lambda\, x, tq^{A+I}) = \sum_{s=0}^{\infty} \frac{q^{\frac{1}{2}ks(s-1)}(-1)^s q^{s(A+I)} t^s}{(q^k;q^k)_s} (1-\lambda\, x)^{(s)}.$$

□

Theorem 3.5. *Let A, B and $A-B$ be complex square matrices satisfying the conditions (17), $0<|q|<1; q\in\mathcal{C}-\{1\}$ and $q^{-k}\notin\sigma(q^A)$ for all $k\in\mathbb{N}_0$, $q^{-s}\notin\sigma(q^B)$ for all $s\in\mathbb{N}_0$ and $q^{-r}\notin\sigma(q^{A-B})$ for all $r\in\mathbb{N}_0$. Then, we get*

(30)

$$Y_n^{(A,\lambda)}(xy;k|q) = \sum_{r=0}^{n} \frac{(q^k;q^k)_n [q]_r (q^{A-B};q^k)_{n-r}}{(q^k;q^k)_r [q]_n (q^k;q^k)_{n-r}} q^{r(A-B)} Y_r^{(B,\lambda)}(x;k|q).$$

Proof. Now, we consider

$$\sum_{n=0}^{\infty} \frac{[q]_n}{(q^k;q^k)_n} Y_n^{(A,\lambda)}(x;k|q) t^n = \frac{(tq^{A-B};q^k)_\infty}{(t;q^k)_\infty} \frac{\Phi(\lambda\, x, tq^{A-B}q^{B+I})}{(tq^{A-B};q^k)_\infty}$$
$$= \sum_{n=0}^{\infty} \frac{(q^{A-B};q^k)_n}{(q^k;q^k)_n} t^n \sum_{m=0}^{\infty} \frac{[q]_m}{(q^k;q^k)_m} Y_m^{(B,\lambda)}(x;k|q)(q^{A-B}t)^m.$$

By use of q-Binomial theorem, interchanging the order of summation, we get

$$\sum_{n=0}^{\infty} \frac{[q]_n}{(q^k;q^k)_n} Y_n^{(A,\lambda)}(x;k|q) t^n$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{[q]_m}{(q^k;q^k)_m} \frac{(q^{A-B};q^k)_{n-m}}{(q^k;q^k)_{n-m}} Y_m^{(B,\lambda)}(x;k|q) q^{m(A-B)} t^n.$$

Equating the coefficient of t^n, on both the sides, we obtain the series expansion for $Y_n^{(A,\lambda)}(x;k|q)$, which proves (30). □

In order to obtain the Rodrigues-type formula for the second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$, we derive the following theorem.

Theorem 3.6. *The second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$, we get Rodrigues formula*

(31)

$$Y_n^{(A,\lambda)}(x;k|q) = \frac{[-\lambda\, x]_\infty}{[q]_n} (\lambda\, x)^{(k-1)I-A} D_{q^k}^n \left[\frac{(\lambda\, x)^{nI+\frac{1}{k}(A+(1-k)I)}}{[-(\lambda\, x)^{\frac{1}{k}}]_\infty} \right]\Bigg|_{x^k}.$$

Proof. Using the q-binomial theorem (Slater [37])

$$\sum_{n=0}^{\infty}\frac{[a]_n}{[q]_n}x^n=\frac{[ax]_\infty}{[x]_\infty},|x|<1. \tag{32}$$

From (28) and (32), we have

$$Y_n^{(A,\lambda)}(x;k|q)=\frac{1}{[q]_n}\sum_{s=0}^{n}\frac{(q^{-nk};q^k)_s}{(q^k;q^k)_s}q^s q^{s(A+(kn+1)I)}\sum_{r=0}^{s}\frac{[q^{-s}]_r}{[q]_r}(-\lambda x q^s)^r$$
$$=\frac{1}{[q]_n}\sum_{s=0}^{n}\frac{(q^{-nk};q^k)_s}{(q^k;q^k)_s[q]_s}q^{s(A+(kn+1)I)}\frac{[q^{-s}(-\lambda\, xq^s)]_\infty}{[-\lambda\, xq^s]_\infty}.$$

Using (26), we get

$$Y_n^{(A,\lambda)}(x;k|q)=\frac{[-\lambda\, x]_\infty}{[q]_n}\sum_{s=0}^{n}\frac{(q^{-nk};q^k)_s}{(q^k;q^k)_s[q]_s[-\lambda\, xq^s]_\infty}q^{s(A+(kn+1)I)}$$
$$=\frac{[-\lambda\, x]_\infty}{[q]_n}(\lambda\, x)^{(k-kn-1)I-A}\sum_{s=0}^{n}\frac{(q^{-nk};q^k)_s}{(q^k;q^k)_s}q^{sk}(\lambda\, x)^{A+(kn-k+1)I)}$$
$$\times\, q^{s(A+(kn+1-k)I)}\frac{1}{[-\lambda\, xq^s]_\infty}=\frac{[-\lambda\, x]_\infty}{[q]_n}(\lambda\, x)^{(k-1)I-A}$$
$$\times\left[(\lambda\, x)^{-n}\sum_{s=0}^{n}\frac{(q^{-nk};q^k)_s}{q^k;q^k)_s}q^{sk}\frac{(\lambda\, xq^{ks})^{\frac{1}{k}(A+(1-k)I)+nI}}{[-(\lambda\, xq^{sk})^{\frac{1}{k}}]_\infty}\right]\Bigg|_{x^k}$$
$$=\frac{[-\lambda\, x]_\infty}{[q]_n}(\lambda\, x)^{(k-1)I-A}\left[D_{q^k}^n\frac{(\lambda\, x)^{\frac{1}{k}(A+(1-k)I)+nI}}{[-(\lambda\, x)^{\frac{1}{k}}]_\infty}\right]\Bigg|_{x^k}.$$

Thus, we obtain (31), which is the Rodrigue's formula. □

In order to obtain the operational representation for the polynomials $Y_n^{(A,\lambda)}(x;k|q)$, from (25), we can write

$$Y_n^{(A,\lambda)}(x;k|q)=\frac{1}{[q]_n}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{[q^{-r}]_s}{[q]_r[q]_s}(q^{A+(s+1)I};q^k)_n q^{\frac{1}{2}r(r-1)+s}(\lambda x)^r$$
$$=\frac{1}{[q]_n}\sum_{r=0}^{\infty}\frac{(\lambda x)^r}{[q]_r}q^{\frac{1}{2}r(r-1)}\sum_{s=0}^{\infty}\frac{(-\lambda\, x)^s}{[q]_s}(q^{A+(s+1)I};q^k)_n$$
$$=\frac{[-\lambda\, x]_\infty}{[q]_n}\sum_{s=0}^{\infty}\frac{1}{[q]_s}(-\lambda x)^s(q^{A+(s+1)I};q^k)_n.$$

This we give

$$Y_n^{(A,\lambda)}(x;k|q) = \frac{[-\lambda\, x]_\infty}{[q]_n} \sum_{s=0}^{\infty} \frac{(-\lambda x)^s}{[q]_s} (\lambda\, x)^{-(1+s+nk)I-A} \times (x^{k+1} D_q)^n (\lambda x)^{A+(s+1)I},$$

where property (11) of the q-difference operator D_q is used. Finally, we have

$$(33) \quad Y_n^{(A,\lambda)}(x;k|q) = \frac{[-\lambda\, x]_\infty}{[q]_n} (\lambda\, x)^{-(1+nk)I-A} (x^{k+1} D_q)^n \left[\frac{(\lambda x)^{A+I}}{[-\lambda x]_\infty} \right].$$

More generally, one can obtain

$$(34) \quad (x^{k+1} D_q)^m \left[\frac{(\lambda\, x)^{A+(nk+1)I}}{[-\lambda\, x]_\infty} Y_n^{(A,\lambda)}(x;k|q) \right] = [q^{n+1}]_m \frac{(\lambda\, x)^{A+(nk+mk+1)I}}{[-\lambda\, x]_\infty} Y_{n+m}^{(A,\lambda)}(x;k|q).$$

For $m = 1$, this reduces to a matrix recurrence relation

$$(35) \quad (1 - q^{n+1}) Y_{n+1}^{(A,\lambda)}(x;k|q) = Y_n^{(A,\lambda)}(x;k|q) - q^{A+(nk+1)I} (\lambda x + 1) Y_n^{(A,\lambda)}(x;k|q).$$

Summary of these results is given in the following theorem.

Theorem 3.7. *Let $n \in \mathbb{N}_0$ and let λ be a complex number with $Re(\lambda) > 0$ and A a complex square matrix satisfying the conditions (17), $0 < |q| < 1; q \in \mathcal{C} - \{1\}$ and $q^{-k} \notin \sigma(q^A), k = 0, 1, 2, \ldots$ for all $0 \leq k \leq n$. Then the second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$ satisfy the following properties:*

1. The second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$ is given by the Rodrigues formula (33).

2. The second kind q-Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k|q)$ is a solution of second order matrix differential equation (34).

3. For $n \geq 0$:

$$(1 - q^{n+1}) Y_{n+1}^{(A,\lambda)}(x;k|q) = Y_n^{(A,\lambda)}(x;k|q) - q^{A+(nk+1)I} (\lambda x + 1) Y_n^{(A,\lambda)}(x;k|q).$$

Remark 3.8. *Now consider for $k=1$, we get*

$$Y_n^{(A,\lambda)}(x;1|q) = \frac{1}{[q]_n}\sum_{r=0}^{n}\sum_{s=0}^{r}\frac{q^{\frac{1}{2}r(r-1)}[q^{-r}]_s q^s}{[q]_r[q]_s}(q^{A+(s+1)I};q)_n(\lambda x)^r$$
$$= \frac{1}{[q]_n}\sum_{r=0}^{n}\frac{q^{\frac{1}{2}r(r-1)}}{[q]_r}(\lambda x)^r\sum_{s=0}^{r}\frac{[q^{-r}]_s q^s}{[q]_s}[q^{A+(s+1)I}]_n,$$

and using the relation

$$[q^A]_{n+s} = [q^A]_s[q^{A+sI}]_n.$$

The sum

$$\sum_{s=0}^{r}\frac{[q^{A+(s+1)I}]_n[q^{-rI}]_s q^s}{[q]_s}$$
$$= \sum_{s=0}^{r}\frac{[q^{A+I}]_s[q^{A+(s+1)I}]_n[q^{-rI}]_s q^s}{[q]_s}\left([q^{A+I}]_s\right)^{-1}$$
$$= \sum_{s=0}^{r}\frac{[q^{A+I}]_{n+s}[q^{-rI}]_s q^s}{[q]_s}\left[[q^{A+I}]_s\right]^{-1}$$
$$= [q^{A+I}]_n\sum_{s=0}^{r}\frac{[q^{A+(n+1)I}]_s[q^{-rI}]_s q^s}{[q]_s}\left([q^{A+I}]_s\right)^{-1}.$$

Hence, we have

$$Y_n^{(A,\lambda)}(x;1|q) = \frac{[q^{A+I}]_n}{[q]_n}\sum_{r=0}^{n}\frac{q^{\frac{1}{2}r(r-1)}}{[q]_r}(\lambda\, x)^r$$
$$\times\sum_{s=0}^{r}\frac{[q^{A+(n+1)I}]_s[q^{-r}]_s q^s}{[q]_s}\left([q^{A+I}]_s\right)^{-1}$$
$$= \frac{[q^{A+I}]_n}{[q]_n}\sum_{r=0}^{n}\frac{q^{\frac{1}{2}r(r-1)}}{[q]_r}(\lambda x)^r\ {}_2\phi_1\left(q^{-rI},q^{A+(n+1)I};q^{A+I};q;q\right),$$

where (see [23]*)*

$${}_2\phi_1\left(q^{-rI},q^{A+(n+1)I};q^{A+I};q;q\right)$$
$$= \sum_{s=0}^{r}\frac{[q^{A+(n+1)I}]_s[q^{-r}]_s q^s}{[q]_s}\left([q^{A+I}]_s\right)^{-1}.$$

Now using q-vandermonde's theorem, we get

$$
\begin{aligned}
&Y_n^{(A,\lambda)}(x;1|q)\\
&=\frac{[q^{A+I}]_n}{[q]_n}\sum_{r=0}^{n}\frac{q^{\frac{1}{2}r(r-1)}}{[q]_r}(\lambda x)^r[q^{-n}]_r q^{r(A+(n+1)I)}\left([q^{A+I}]_r\right)^{-1}\\
&=\frac{[q^{A+I}]_n}{[q]_n}\sum_{r=0}^{n}\frac{[q^{-n}]_r}{[q]_r}q^{\frac{1}{2}r(r+1)I+r(A+nI))}\left([q^{A+I}]_r\right)^{-1}(\lambda\, x)^r\\
&=L_n^{(A,\lambda)}(x|q).
\end{aligned}
$$

Thus, for $k=1$, $Y_n^{(A,\lambda)}(x;1|q)$ also reduces to q-Laguerre matrix polynomials.

Remark 3.9. *The properties (27), (30) and (31) reduces for $k=1$, to corresponding properties for the q-Laguerre matrix polynomials.*

4. Conclusion

In our study, we introduce the first and second kind q-Konhauser matrix polynomials (18) and (25) hold for $\tilde{\mu}(A) > -1$ for all eigenvalues $\mu \in \sigma(A)$. The generating matrix functions of first and second kind q-Konhauser matrix polynomials are investigated. The different explicit forms of the $Z_n^{(A,\lambda)}(x;k|q)$ and $Y_n^{(A,\lambda)}(x;k|q)$ are introduced. The matrix recurrence relations and Rodrigues-type formula of this $Z_n^{(A,\lambda)}(x;k|q)$ and $Y_n^{(A,\lambda)}(x;k|q)$ are given.

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments and suggestions which improved the quality and clarity of the paper.

References

[1] P. Agarwal, F. Qi, M. Chand and S. Jain, *Certain integrals involving the generalized hypergeometric function and the Laguerre polynomials*, J. Comput. Appl. Math. **313** (2017), 307-317.

[2] P. Agarwal, S. Jain and J. Choi, *Certain q-series identities*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. **111** (2017), 139-146.

[3] P. Agarwal, S.S. Dragomir, M. Jleli and B. Samet, *A Study of New Trends and Analysis of Special Function*, LAP LAMBERT Academic Publishing 1, 164, 2013

[4] P. Agarwal, S.S. Dragomir, M. Jleli and B. Samet, *Advances in Mathematical Inequalities and Applications*, Birkhäuser, 2018

[5] W.A. Al-Salam and A. Verma, *q-Konhauser polynomials*, Pacic J. Math. **108 (1)** (1983), 1-7.

[6] R. Askey, *The q-Gamma and q-Beta functions*, Applicable Anal. **8** (1978), 125-141.

[7] R. Askey, *Ramanujan's extension of the Gamma and Beta functions*, Amer. Math. Mothly **87** (1980), 346-359.

[8] B. Çekim, *q-Matrix polynomials in several variables*, Filomat **29 (9)** (2015), 2059-2067.

[9] J. Choi and P. Agarwal *Certain unified integrals involving a product of Bessel functions of the first kind*, Honam Math. J. **35 (4)** (2013), 667-677.

[10] J. Choi and P. Agarwal, *Some new Saigo type fractional integral inequalities and their q-analogues*, Abstr. Appl. Anal. **2014** (2014) Article ID 579260, pp.11.

[11] E. Defez and Jódar, *Chebyshev matrix polynomails and second order matrix differential equations*, Util. Math., **61** (2002), 107-123.

[12] E. Erkuş-Duman and B. Çekim, *New generating functions for Konhauser matrix polynomials*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **63 (1)** (2014), 35-41.

[13] H. Exton, *q-Hypergcometric Functions and Applications*, Horwood, Chichcster, 1983.

[14] I. Higueras and B. Garcia-Celaeta, *Logarithmic norms for matrix pencil*, SIAM J. Matrix Anal. **20** (1999), 646-666.

[15] F.H. Jackson, *On q-definite Integrals*, Quart. J. Pure Appl. Math. **14** (1910), 193-203.

[16] V.K. Jain and H.M. Srivastava, *New results involving a certain class of q-orthogonal polynomials*, J. Math. Anal. Appl. **166** (1992), 331–344.

[17] L. Jódar and J.C. Cortés *Some properties of Gamma and Beta matrix functions*, Appl. Math. Lett. **11 (1)** (1998), 89-93.

[18] L. Jódar and J.C. Cortés, *On the hypergeometric matrix function*, J. Comput. Anal. Appl. **99** (1998), 205-217.

[19] L. Jódar and J.C. Cortés, *Closed form general solution of the hypergeometric matrix differential equation*, Math. Comput. Model. **32** (2000), 1017-1028.

[20] H.C. Madhekar and V.T. Chamle, *On the q-Konhauser biorthogonal polynomials*, Internat. J. Math. and Math. Sci., **10 (2)** (1987), 413-415.

[21] M. Ruzhansky, Y.J. Cho, P. Agarwal and I. Area, *Advances in real and complex analysis with applications*, Springer Singapore, 2017.

[22] A. Salem, *On a q-gamma and q-beta matrix functions*, Linear Multilinear Algebra, **60 (6)** (2012), 683-696.

[23] A. Salem, *The basic Gauss hypergeometric matrix function and its matrix q-difference equation*, Linear Multilinear Algebra, **62 (3)** (2014), 347-361.

[24] A. Salem, *The q-Laguerre matrix polynomials*, Springer Plus, **5** (2016) 550-570.

[25] A. Salem, *On the discrete q-Hermite matrix polynomials*, Int. J. Appl. Comput. Math., **3** (2016), 3147-3158.

[26] A. Shehata, *Some relations on Konhauser matrix polynomials*, Miskolc Math. Notes, **17 (1)** (2016), 605-633.

[27] A. Shehata, *Certain generating relations of Konhauser matrix polynomials from the view point of Lie algebra method*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **79 (4)** (2017), 123-136 .

[28] A. Shehata, *Some new results for Chebyshev matrix polynomials of first kind*, Inter. Bull. Math. Res. **4 (4)** (2017), 50-55.

[29] A. Shehata, *Some new results for Struve matrix functions*, Milan J. Math. **86 (1)** (2018), 81-96.

[30] A. Shehata, *Extended Bessel matrix functions*, Math. Sci. Appl. E-Notes **6 (1)** (2018), 1-11.

[31] A. Shehata, *Certain properties of generalized Hermite-Type matrix polynomials using Weisner's group theoretic techniques*, Bull. Braz. Math. Soc. (N.S.) **50** (2019), 419-434.

[32] A. Shehata, *Certain properties of Konhauser matrix polynomials via Lie Algebra techniques*, Bol. Soc. Mat. Mex. (3) **26 (1)** (2020), 99-120.

[33] A. Shehata, *Certain generating matrix functions of Legendre matrix polynomials using Lie algebraic method*, Kragujevac J. Math. **44 (3)** (2020), 353368.

[34] A. Shehata, *A note on Konhauser marix polynomials*, Palest. J. Math. **9 (1)** (2020), 549-556.

[35] H.M. Srivastava, Fatma Taşdelen and Burak Şekeroğlu, *Some familes of generating functions for the q-Konhauser polynomials*, Taiwanese J. Math., **12 (3)** (2008), 841-850.

[36] H.M. Srivastava and A. Shehata, *A family of new q-Extensions of the Humbert functions*, European Journal of Mathematical Sciences, **4 (1)** (2018), 13-26.

[37] L.J. Slater, *Generalized Hvpergeometric Functions*, Cambridge University Press, 1966.

[38] J. Tariboon, S.K. Ntouyas and P. Agarwal, *New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations*, Adv. Dierence Equ. **2015 (1)** (2015), 18.

[39] S. Varma, B. Çekim and F. Taşdelen, *On Konhauser matrix polynomials*, Ars Combin **100** (2011), 193-204.

[40] X. Zhang, P. Agarwal, Z. Liu and H. Peng, *The general solution for impulsive differential equations with Riemann-Liouville fractional-order* $q \in (1, 2)$, Open Math. **13 (1)** (2015), 908-931.

Ayman Shehata
Department of Mathematics, Faculty of Science,
Assiut University, Assiut 71516, Egypt.
Department of Mathematics, College of Science and Arts,
Unaizah, Qassim University, Qassim, Saudi Arabia.
E-mail: drshehata2006@yahoo.com,
aymanshehata@science.aun.edu.eg, A.Ahmed@qu.edu.sa