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A FIXED POINT APPROACH TO THE STABILITY OF THE ADDITIVE-CUBIC FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate the stability of the additive-cubic functional equations

$$f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y)+(k^2-1)f(x) - (k^2-1)f(-x) = 0,$$

$$f(x+ky)-f(ky-x)-k^2f(x+y)+k^2f(y-x)+2(k^2-1)f(x)=0,$$

$$f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)=0$$

by using the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

Throughout this paper, let V and W be real vector spaces, Y a real Banach space, and k a fixed nonzero real number such that $|k| \neq 1$. For a given mapping $f : V \to W$, we use the following abbreviations

$$\begin{aligned}
f_o(x) :=& \frac{f(x)-f(-x)}{2},\\
Af(x,y) :=& f(x+y)-f(x)-f(y),\\
Cf(x,y) :=& f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y),\\
D_1f(x,y) :=& f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y)\\
&+(k^2-1)f(x)-(k^2-1)f(-x),\\
D_2f(x,y) :=& f(x+ky)-f(ky-x)-k^2f(x+y)+k^2f(y-x)\\
&+2(k^2-1)f(x),
\end{aligned}$$

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$$\begin{aligned} D_3 f(x,y) :=& f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y) \\ &-2f(kx)+2kf(x) \end{aligned}$$

for all $x, y \in V$.

In 1940, the problem for the stability of group homomorphism was first raised by S. M. Ulam [18]. In the next year, D. H. Hyers [8] gave a partial solution to Ulam's question for the case of additive mappings. Hyers' result has greatly influenced the study of the stability problem of the functional equation. His result was generalized by Th. M. Rassias [16] and Găvruta [7].

Each functional equation $Af(x,y) = 0$ and $Cf(x,y) = 0$ are called an additive functional equation and a cubic functional equation, respectively. Every solution of functional equations $Af(x,y) = 0$ and $Cf(x,y) = 0$ are called an additive mapping and a cubic mapping, respectively. If a mapping can be expressed by sum of a cubic mapping and an additive mapping, then we call the mapping an additive-cubic mapping. A functional equation is called an additive-cubic functional equation provided that each solution of that equation is an additive-cubic mapping and every additive-cubic mapping is a solution of that equation.

M. Arunkumar et al. [1, 2] proved the stability of the additive-cubic functional equation $D_2 f(x,y) = 0$ when $k = 2$ and S.-S. Jin et al. [12] proved the stability of the additive-cubic functional equation $D_2 f(x,y) = 0$. M. E. Gordji et al. [**?**], A. Najati et al.[14], and Z. Wang et al. [19] proved the stability of the additive-cubic functional equation $D_3 f(x,y) = 0$ when $k = 2$, and T. Z. Xu et al.[20, 22, 23, 21] proved the stability of the additive-cubic functional equation $D_3 f(x,y) = 0$ when k is an integer. Many mathematicians investigated the stability of the other types of additive-cubic functional equations [6, 9, 15, 17]. They proved the stability of the additive-cubic functional equations by handling the additive part and the cubic part of the given function f, respectively. In this paper, instead of splitting the given function $f : V \to Y$ into two parts, we will prove the stability of the functional equations $D_1 f(x,y) = 0$, $D_2 f(x,y) = 0$, $D_3 f(x,y) = 0$ by using the fixed point theory in the sense of Cădariu and Radu [3, 4] (See also [10, 11, 13]).

2. Main results

We recall the following Margolis and Diaz's fixed point theorem to prove the main theorem.

Theorem 2.1. ([5]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:

(1) *$d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;*

(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*

(4) *$d(y, y^*) \leq (1/(1 - L))d(y, Jy)$ for all $y \in Y$.*

Lemma 2.2. *Let $m \in \{1, 2, 3\}$ and $f : V \to W$ with $f(0) = 0$. Then the equality*

$$f(4x) - 10f(2x) + 16f(x) = E_m f(x) \tag{1}$$

holds for all $x \in V$, where $E_m f : V \to W$ is given by

$$\begin{aligned}
E_1 f(x) :=\ & \frac{1}{k^4 - k^2}\big((4k^2-3)D_1 f_o(x,x) - 2k^2 D_1 f_o(2x,x) + 2k^2 D_1 f_o(x,2x) \\
& - 2D_1 f_o((k+1)x,x) + 2D_1 f_o((k-1)x,x) - k^2 D_1 f_o(2x,2x) \\
& + D_1 f_o(x,3x) - D_1 f_o((2k+1)x,x) + D_1 f_o((2k-1)x,x)\big) \\
& - \frac{1}{2(k^2-1)}\big(D_1 f(4x,0) - 10 D_1 f(2x,0) + 16 D_1 f(x,0)\big), \\
E_2 f(x) :=\ & \frac{1}{k^4 - k^2}\big((4k^2-3)D_2 f_o(x,x) - 2k^2 D_2 f_o(2x,x) + 2k^2 D_2 f_o(x,2x) \\
& - 2D_2 f_o((k+1)x,x) + 2D_2 f_o((k-1)x,x) - k^2 D_2 f_o(2x,2x) \\
& + D_2 f_o(x,3x) - D_2 f_o((2k+1)x,x) + D_2 f_o((2k-1)x,x)\big) \\
& + \frac{1}{2(k^2-1)}\big(D_2 f(4x,0) - 10 D_2 f(2x,0) + 16 D_2 f(x,0)\big), \\
E_3 f(x) :=\ & \frac{1}{k - k^3}\big(8D_3 f_o(x/2, kx/2) - 8k D_3 f_o(x/2,(2k+1)x/2) \\
& + 8k D_3 f_o(x/2,(2k-1)x/2) - 8 D_3 f_o(x/2, 3kx/2) \\
& + (1-8k^2) D_3 f_o(x,x) - D_3 f_o(x,kx) + 2D_3 f_o(x,(k+1)x) \\
& + 2D_3 f_o(x,(k-1)x) + (k+1) D_3 f_o(x,(2k+1)x) \\
& - (k-1) D_3 f_o(x,(2k-1)x) + D_3 f_o(x,3kx) - 2D_3 f_o(2x,x) \\
& + k^2 D_3 f_o(2x,2x) - 2D_3 f_o(2x,kx) - D_3 f_o(2x,2kx) - D_3 f_o(3x,x)\big) \\
& + \frac{1}{2-2k}\big(D_3 f(0,4x) - 10 D_3 f(0,2x) + 16 D_3 f(0,x)\big)
\end{aligned}$$

for all $x \in V$.

Now we can prove some stability results of the functional equation $D_m F(x,y) = 0$ $(m = 1,2,3)$ by using the fixed point theory.

Theorem 2.3. *Let m be a fixed integer such that $m \in \{1,2,3\}$ and let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality*

$$\|D_m f(x,y)\| \le \varphi(x,y) \tag{2}$$

holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property

$$\varphi(2x,2y) \le (\sqrt{41}-5)L\varphi(x,y) \tag{3}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying the functional equation $D_m F(x,y) = 0$ and the inequality

$$\|f(x) - F(x)\| \le \frac{\Phi_m(x)}{16(1-L)} \tag{4}$$

for all $x \in V$, *where* $\varphi_e : V^2 \to [0,\infty)$ *and* Φ_m *are defined by*

$$\begin{aligned}
\varphi_e(x,y) &:= \frac{\varphi(x,y)+\varphi(-x,-y)}{2},\\
\Phi_1(x) &:= \frac{1}{|k^4-k^2|}\big(|4k^2-3|\varphi_e(x,x)+2k^2\varphi_e(2x,x)+2k^2\varphi_e(x,2x)\\
&\quad+2\varphi_e((k+1)x,x)+2\varphi_e((k-1)x,x)+k^2\varphi_e(2x,2x)\\
&\quad+\varphi_e(x,3x)+\varphi_e((2k+1)x,x)+\varphi_e((2k-1)x,x)\big)\\
&\quad+\frac{1}{|k^2-1|}\big(\varphi_e(4x,0)+5k^2\varphi_e(2x,0)+8k^2\varphi_e(x,0)\big),
\end{aligned}$$

$$\begin{aligned}
\Phi_2(x) &:= \frac{1}{|k^4-k^2|}\big(|4k^2-3|\varphi_e(x,x)+2k^2\varphi_e(2x,x)+2k^2\varphi_e(x,2x)\\
&\quad+2\varphi_e((k+1)x,x)+2\varphi_e((k-1)x,x)+k^2\varphi_e(2x,2x)\\
&\quad+\varphi_e(x,3x)+\varphi_e((2k+1)x,x)+\varphi_e((2k-1)x,x)\big)\\
&\quad+\frac{1}{|k^2-1|}\big(\varphi_e(4x,0)+5k^2\varphi_e(2x,0)+8k^2\varphi_e(x,0)\big),\\
\Phi_3(x) &:= \frac{1}{|k^3-k|}\big(8\varphi_e(x/2,kx/2)+8k\varphi_e(x/2,(2k+1)x/2)\\
&\quad+8k\varphi_e(x/2,(2k-1)x/2)+8\varphi_e(x/2,3kx/2)+|8k^2-1|\varphi_e(x,x)\\
&\quad+\varphi_e(x,kx)+2\varphi_e(x,(k+1)x)+2\varphi_e(x,(k-1)x)\\
&\quad+|k+1|\varphi_e(x,(2k+1)x)+|k-1|\varphi_e(x,(2k-1)x)+\varphi_e(x,3kx)\\
&\quad+2\varphi_e(2x,x)+k^2\varphi_e(2x,2x)+2\varphi_e(2x,kx)+\varphi_e(2x,2kx)\\
&\quad+\varphi_e(3x,x)\big)+\frac{1}{|k-1|}\big(\varphi_e(0,4x)+10\varphi_e(0,2x)+16\varphi_e(0,x)\big).
\end{aligned}$$

In particular, F *is represented by*

$$F(x)=\lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i\frac{(-1)^{n-i}10^i}{16^n}f(2^{2n-i}x) \tag{5}$$

for all $x \in V$.

Proof. Let S be the set of all functions $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g,h)=\inf\big\{K\in\mathbb{R}_+\,\big|\,\|g(x)-h(x)\|\le K\Phi_m(x)\text{ for all }\ x\in V\big\}.$$

It is easy to show that (S,d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := -\frac{g(4x)}{16} + \frac{10g(2x)}{16}$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}(10)^i}{16^n} g(2^{2n-i}x)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| \le& \frac{1}{16}\|g(4x) - h(4x)\| + \frac{10}{16}\|g(2x) - h(2x)\| \\ \le& K(\frac{1}{16}\Phi_m(4x) + \frac{10}{16}\Phi_m(2x)) \\ \le& K(\frac{(\sqrt{41}-5)^2}{16}L^2\Phi_m(x) + \frac{10(\sqrt{41}-5)}{16}L\Phi_m(x))\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g,h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Using (1) we obtain that

$$\|f(x) - Jf(x)\| = \left\|\frac{f(4x) - 10f(2x) + 16f(x)}{16}\right\| = \left\|\frac{E_m f(x)}{16}\right\| \le \frac{\Phi_m(x)}{16}$$

for all $x \in V$. It means that $d(f, Jf) \le \frac{1}{16} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f,g) < \infty\}$, which is represented by (5) for all $x \in V$. Notice that

$$d(f,F) \le \frac{1}{1-L}d(f,Jf) \le \frac{1}{16(1-L)},$$

which implies (4). By the definition of F, together with (2) and (3), we have

$$\begin{aligned}
\|D_m F(x,y)\| &= \lim_{n\to\infty} \|D_m J^n f(x,y)\| \\
&= \lim_{n\to\infty} \Big\| \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}(10)^i}{16^n} D_m f(2^{2n-i}x, 2^{2n-i}y) \Big\| \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{10^i}{16^n} \varphi(2^{2n-i}x, 2^{2n-i}y) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{(\sqrt{41}-5)^{n-i}10^i}{16^n} L^{n-i}\varphi(2^n x, 2^n y) \\
&\le \lim_{n\to\infty} \frac{(\sqrt{41}+5)^n}{16^n} \varphi(2^n x, 2^n y) \\
&\le \lim_{n\to\infty} \frac{(\sqrt{41}+5)^n(\sqrt{41}-5)^n}{16^n} L^n \varphi(x,y) \\
&\le \lim_{n\to\infty} L^n \varphi(x,y) = 0
\end{aligned}$$

for all $x, y \in V$ i.e., F is a solution of the functional equation $D_m F(x,y) = 0$. Notice that if F is a solution of the functional equation $D_m F(x,y) = 0$, then the equality $F(x) - JF(x) = \frac{E_m F(x)}{16}$ implies that F is a fixed point of J. □

We continue our investigation with the next result.

Theorem 2.4. *Let m be a fixed integer such that $m \in \{1,2,3\}$ and let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality (2) holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property*

$$L\varphi(2x, 2y) \ge \frac{16}{\sqrt{41}-5}\varphi(x,y) \tag{6}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying the functional equation $D_m F(x,y) = 0$ and the inequality

$$\|f(x) - F(x)\| \le \frac{(66-10\sqrt{41})L^2}{256(1-L)} \Phi_m(x) \tag{7}$$

for all $x \in V$. *In particular,* F *is represented by*

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 10^i (-16)^{n-i} f\left(\frac{x}{2^{2n-i}}\right) \tag{8}$$

for all $x \in V$.

Proof. Let the set (S, d) be as in the proof of Theorem 2.3. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := 10g\left(\frac{x}{2}\right) - 16g\left(\frac{x}{4}\right)$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i 10^i (-16)^{n-i} g\left(\frac{x}{2^{2n-i}}\right)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| \le& 10\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + 16\left\|g\left(\frac{x}{4}\right) - h\left(\frac{x}{4}\right)\right\| \\ \le& 16K\Phi_m\left(\frac{x}{4}\right) + 10K\Phi_m\left(\frac{x}{2}\right) \\ \le& L^2\frac{(\sqrt{41}-5)^2}{16}K\Phi_m(x) + 10\frac{\sqrt{41}-5}{16}LK\Phi_m(x) \\ \le& LK\Phi_m(x)\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (1) and (2), we see that

$$\|f(x) - Jf(x)\| \le \Phi_m\left(\frac{x}{4}\right) \le \frac{(\sqrt{41}-5)^2L^2}{16^2}\Phi_m(x)$$

for all $x \in V$. It means that $d(f, Jf) \le \frac{(66-10\sqrt{41})L^2}{256} < \infty$ by the definition of d. Therefore according to Theorem 2.3, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by (8) for all $x \in V$. Notice that

$$d(f, F) \le \frac{1}{1-L}d(f, Jf) \le \frac{(66-10\sqrt{41})L^2}{256(1-L)},$$

which implies (7). By the definition of F, together with (2) and (8), we have

$$\begin{aligned}
\|D_mF(x,y)\| &= \lim_{n\to\infty} \left\| \sum_{i=0}^{n} {}_nC_i 10^i (-16)^{n-i} D_m f\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) \right\| \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 10^i 16^{n-i} \varphi\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 10^i (\sqrt{41}-5)^{n-i} L^{n-i} \varphi\left(\frac{x}{2^{n}}, \frac{y}{2^{n}}\right) \\
&\le \lim_{n\to\infty} (\sqrt{41}+5)^n \varphi\left(\frac{x}{2^{n}}, \frac{y}{2^{n}}\right) \\
&\le \lim_{n\to\infty} (\sqrt{41}+5)^n \frac{(\sqrt{41}-5)^n L^n}{16^n} \varphi(x,y) \\
&\le \lim_{n\to\infty} L^n \varphi(x,y) = 0
\end{aligned}$$

for all $x, y \in V$ i.e., F is a solution of the functional equation $D_mF(x,y) = 0$. Notice that if F is a solution of the functional equation $D_mF(x,y) = 0$, then the equality $F(x) - JF(x) = E_mF\left(\frac{x}{4}\right)$ implies that F is a fixed point of J. □

Since f is an additive-cubic mapping if $D_mf(x,y) = 0$, and $f - Jf = 0$ if f is an additive-cubic mapping, we obtain the following corollaries from Theorem 2.3 and Theorem 2.4.

Corollary 2.5. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality* (2) *holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property* (3) *for all $x, y \in V$, then there exists a unique additive-cubic mapping $F : V \to Y$ satisfying the inequality* (4) *for all $x \in V$.*

Corollary 2.6. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality* (2) *holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property* (6) *for all $x, y \in V$, then there exists a unique additive-cubic mapping $F : V \to Y$ satisfying the inequality* (7) *for all $x \in V$.*

Corollary 2.7. *Let X be a normed space and $p \in (0, \log_2(\sqrt{41}-5) \cup (4-\log_2(\sqrt{41}-5), \infty)$. If a function $f : X \to Y$ satisfies the inequality*

$$\|D_m f(x,y)\| \le \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$ and for some $\theta \ge 0$, then there exists a unique additive and cubic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{(\sqrt{41}-5)\Phi_m\theta\|x\|^p}{16(\sqrt{41}-5-2^p)} & \text{when } 2^p < \sqrt{41}-5, \\ \frac{(\sqrt{41}-5)\Phi_m\theta\|x\|^p}{(\sqrt{41}-5)4^p - 16\cdot 2^p} & \text{when } 2^p > \frac{16}{\sqrt{41}-5} \end{cases}$$

for all $x \in X$, where Φ_m are defined by

$$\begin{aligned} \Phi_1 =\Phi_2 :=& \frac{8k^4+12k^2+1+(5k^4+6k^2)2^p+3^p+k^2 4^p}{|k^4-k^2|} \\ &+ \frac{4(|k|+1)^p + 2(2|k|+1)^p)}{|k^4-k^2|} \\ \Phi_3 :=& \frac{1}{2^p|k^3-k|} \times (16|k|+1+8|k|^p+8|k||2k+1|^p+8|k||k-1|^p \\ &+ 8\cdot|3k|^p + 2^p(2|8k^2-1|+16k^2+|k+1|+|k-1|+9+3|k|^p \\ &+ 2|k+1|^p + 2|k-1|^p + |k+1||2k+1|^p \\ &+ |k-1||2k-1|^p + |3k|^p + |2k|^p + 3^p + (6+12k^2)2^p + k^2 4^p) \end{aligned}$$

Proof. If we put

$$\varphi(x,y) := \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$ and

$$L := \begin{cases} 2^{p-\log_2(\sqrt{41}-5)} & \text{if } p < \log_2(\sqrt{41}-5), \\ 2^{4-\log_2(\sqrt{41}-5)-p} & \text{if } p > 4-\log_2(\sqrt{41}-5) \end{cases}$$

then our assertions follow from Theorems 2.3 and 2.4. □

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