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SOME NOTES ON LP-SASAKIAN MANIFOLDS WITH GENERALIZED SYMMETRIC METRIC CONNECTION

Oğuzhan Bahadir and Sudhakar K. Chaubey*

Abstract. The present study initially identify the generalized symmetric connections of type (α, β), which can be regarded as more generalized forms of quarter and semi-symmetric connections. The quarter and semi-symmetric connections are obtained respectively when $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$. Taking that into account, a new generalized symmetric metric connection is attained on Lorentzian para-Sasakian manifolds. In compliance with this connection, some results are obtained through calculation of tensors belonging to Lorentzian para-Sasakian manifold involving curvature tensor, Ricci tensor and Ricci semi-symmetric manifolds. Finally, we consider CR-submanifolds admitting a generalized symmetric metric connection and prove many interesting results.

1. Introduction

A particular metric connection with a torsion different from zero was introduced by Hayden on a Riemannian manifold [15]. The quarter-symmetric connections, being more generalized form of semi-symmetric connections, were suggested by Golab on a differentiable manifold [13]. These connections have been studied by many authors. For instance we cite ([1], [6]-[11], [14], [18], [30]) and the references therein. Tripathi [31] introduced and studied 17 types of connections which includes the semi-symmetric and quarter-symmetric connections. On the other hand, Matsumoto [21] introduced Lorentzian para-contact manifolds. Later, many geometers ([2], [3], [12], [16], [22]-[25], [29], [32]) have published different papers in this context.

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*Corresponding author

A linear connection on a (semi-)Riemannian manifold M is suggested to be a generalized symmetric connection if its torsion tensor T is presented as follows:

$$T(U,V)=\alpha\{u(V)U-u(U)V\}+\beta\{u(V)\varphi U-u(U)\varphi V\}, \tag{1}$$

for all vector fields U and V on M, where α and β are smooth functions on M. φ can be viewed as a tensor of type $(1,1)$ and u is regarded as a 1-form connected with the vector field which has a non-vanishing smooth non-null unit. Furthermore, the connection mentioned is suggested to be a generalized metric one when a Riemannian metric g in M is available as $\overline{\nabla}g=0$; or else, it is non-metric.

In equation (1), if $\alpha=0$, $\beta\neq 0$; $\alpha\neq 0$, $\beta=0$, then the generalized symmetric connection is called β-quarter-symmetric connection; α-semi-symmetric connection, respectively. Additionally, the generalized symmetric connection reduces to a semi-symmetric, and quarter-symmetric when $(\alpha,\beta)=(1,0)$, and $(\alpha,\beta)=(0,1)$, respectively. Thus, it can be suggested that the generalizing semi-symmetric and quarter-symmetric connections paves the way for a generalized symmetric metric connection. These two connections are of great significance both for the study of geometry and applications in physics. For instance, Pahan, Pal and Bhattacharyya studied generalized Robertson-Walker space-time with respect to a quarter-symmetric connection [26]. Furthermore, many authors investigated the geometrical and physical aspect of different spaces [4], [5], [17], [19], [20], [27], [28].

In the present paper, we define a new connection on Lorentzian para-Sasakian manifold, which is the generalization of semi-symmetric and quarter-symmetric connection. The preliminaries are presented in Section 2. Section 3 illustrates the generalized symmetric metric connection on a Lorentzian para-Sasakian manifold. As for Section 4, we calculate curvature tensor and the Ricci tensor of Lorentzian para-Sasakian with respect to a generalized symmetric metric connection. Besides, it is found that if a Lorentzian para-Sasakian manifold is Ricci semi-symmetric with respect to a generalized symmetric metric connection, then the manifold is a generalized η-Einstein manifold with respect to the generalized symmetric metric connection. In Section 5, we study the properties of CR-submanifold of a Lorentzian para-Sasakian manifold with respect to a generalized symmetric metric connection. Furthermore, we get integrability conditions of distributions on CR-submanifolds.

2. Preliminaries

Let M be a differentiable manifold of dimension n endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and Lorentzian metric g, which satisfies

$$\eta(\xi) = -1, \quad \phi^2(U) = U + \eta(U)\xi, \tag{2}$$

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \quad g(U,\xi) = \eta(U), \tag{3}$$

$$\nabla_U \xi = \phi U, \quad (\nabla_U \phi)(V) = g(U,V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi \tag{4}$$

for all vector fields U, V on M, where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Such manifold (M, ϕ, ξ, η, g) is called Lorentzian para-Sasakian (shortly, LP-Sasakian) manifold [21, 23]. The following are provided for LP-Sasakian manifold:

$$\phi\xi = 0, \ \eta(\phi U) = 0, \ \operatorname{rank}\phi = n-1. \tag{5}$$

If we write $\Phi(U,V) = g(\phi U, V)$ for all vector fields U, V on M, then the tensor field Φ is a symmetric $(0,2)$ tensor field [21]. In addition, if η is closed on an LP-Sasakian manifold then we have

$$(\nabla_U \eta)V = \Phi(U,V), \ \Phi(U,\xi) = 0, \tag{6}$$

for any vector fields U and V on M [21, 24]. An LP-Sasakian manifold provides the following relations [24, 22]:

$$R(\xi, U)V = g(U,V)\xi - \eta(V)U, \tag{7}$$

$$R(U,V)\xi = \eta(V)U - \eta(U)V, \tag{8}$$

$$S(U,\xi) = (n-1)\eta(U), \tag{9}$$

$$S(\phi U, \phi V) = S(U,V) + (n-1)\eta(U)\eta(V) \tag{10}$$

for all vector fields U and V on M, in which R and S can be viewed as the curvature tensor and the Ricci tensor of M, respectively.

An LP-Sasakian manifold M is said to be a generalized η-Einstein if the non-vanishing Ricci tensor S of M satisfies the relation

$$S(U,V) = ag(U,V) + b\eta(U)\eta(V) + cg(\phi U, V)$$

for every $U, V \in \Gamma(TM)$, in which a, b and c are viewed as scalar functions on M. If $c = 0$, then M is regarded as an η-Einstein manifold.

The Gauss and Weingarten formulas are given by

$$\begin{aligned} \nabla_U V &= \nabla'_U V + h(U,V), \ \forall\, U, V \in \Gamma(TM'), \\ \nabla_X N &= -A_N X + \nabla^{\perp}_X N, \ \forall\, N \in \Gamma(T^{\perp}M'), \end{aligned} \tag{11}$$

where $\{\nabla_X Y, A_N X\}$ and $\{h(X,Y), \nabla_X^{\perp} N\}$ belong to $\Gamma(TM^{'})$ and $\Gamma(T^{\perp}M^{'})$, respectively. For details, we refer [25].

3. Generalized Symmetric Metric Connection in an LP-Sasakian Manifold

When we view $\overline{\nabla}$ as a linear connection and ∇ as a Levi-Civita connection of Lorentzian para-contact metric manifold M in such a way that

$$\overline{\nabla}_U V = \nabla_U V + H(U,V)$$

for all vector fields X and Y on M. The following is obtained so that $\overline{\nabla}$ is a generalized symmetric connection of ∇, in which H is viewed as a tensor of type $(1,2)$;

$$H(U,V) = \frac{1}{2}[T(U,V) + T^{'}(U,V) + T^{'}(V,U)], \tag{12}$$

where T is viewed as the torsion tensor of $\overline{\nabla}$ and

$$g(T^{'}(U,V),W) = g(T(W,U),V). \tag{13}$$

Thanks to (1) and (13), we obtain the following;

$$T^{'}(U,V) = \alpha\{\eta(U)V - g(U,V)\xi\} + \beta\{\eta(U)\phi V - g(\phi U,V)\xi\}. \tag{14}$$

Using (1), (12) and (14) we obtain

$$H(U,V) = \alpha\{\eta(V)U - g(U,V)\xi\} + \beta\{\eta(V)\phi U - g(\phi U,V)\xi\}.$$

Thus we can write the proposition as:

Proposition 3.1. *For an LP-Sasakian manifold, the generalized symmetric metric connection* $\overline{\nabla}$ *of type* (α,β) *is given by*

$$\overline{\nabla}_U V = \nabla_U V + \alpha\{\eta(V)U - g(U,V)\xi\} + \beta\{\eta(V)\phi U - g(\phi U,V)\xi\}. \tag{15}$$

If $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (0,1)$ are chosen, the generalized symmetric metric connection is diminished to a semi-symmetric metric and a quarter-symmetric metric one as presented in the following;

$$\overline{\nabla}_U V = \nabla_U V + \eta(V)U - g(U,V)\xi,$$

$$\overline{\nabla}_U V = \nabla_U V + \eta(V)\phi U - g(\phi U,V)\xi.$$

From (4), (6) and (15) we have the following proposition.

Proposition 3.2. *The following relations are obtained when M is an LP-Sasakian manifold with generalized symmetric metric connection $\overline{\nabla}$:*

$$\begin{aligned}(\overline{\nabla}_U\phi)V &= [(1-\beta)g(U,V)+(2-2\beta)\eta(U)\eta(V)-\alpha\Phi(U,V)]\xi \\ &\quad +(1-\beta)\eta(V)U-\alpha\eta(V)\phi U,\\ \overline{\nabla}_U\xi &= (1-\beta)\phi U-\alpha U-\alpha\eta(U)\xi,\\ (\overline{\nabla}_U\eta)V &= (1-\beta)\Phi(U,V)-\alpha g(\phi U,\phi V)\end{aligned}$$

for every U, $V \in \Gamma(TM)$.

Example 3.3. *A 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$ is considered, in which (x,y,z) are regarded as the standard coordinates in R^3. Suppose that ν_1,ν_2,ν_3 are linearly independent global frame on M as presented below*

$$\nu_1 = e^z\frac{\partial}{\partial y},\quad \nu_2 = e^z\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big),\quad \nu_3 = \frac{\partial}{\partial z}.$$

Consider that g is a Lorentzian metric defined as

$$\begin{aligned}&g(\nu_1,\nu_2)=g(\nu_1,\nu_3)=g(\nu_2,\nu_3)=0,\\ &g(\nu_1,\nu_1)=g(\nu_2,\nu_2)=-g(\nu_3,\nu_3)=1.\end{aligned}$$

When we consider that η is a 1-form represented as $\eta(Z)=g(Z,\nu_3)$ for every $Z\in\Gamma(TM)$ and ϕ is the $(1,1)$ tensor field presented as $\phi\nu_1 = -\nu_1, \phi\nu_2 = -\nu_2$ and $\phi\nu_3 = 0$, we thereby get $\eta(\nu_3) = -1$, $\phi^2Z = Z+\eta(Z)\nu_3$ and $g(\phi Z,\phi W) = g(Z,W)+\eta(Z)\eta(W)$ for all $Z,W\in\Gamma(TM)$ through use of linearity of ϕ and g. Therefore for $\nu_3 = \xi$, (ϕ,ξ,η,g) describes a Lorentzian para-contact structure on M. Considering that ∇ is the Levi-Civita connection concerning the Lorentzian metric g. The following is obtained;

$$[\nu_1,\nu_2]=0,\qquad [\nu_1,\nu_3]=-\nu_1,\qquad [\nu_2,\nu_3]=-\nu_2.$$

By means of using Koszul's formula, the following can be calculated in an easy way

$$\begin{aligned}&\nabla_{\nu_1}\nu_1=-\nu_3,\qquad \nabla_{\nu_1}\nu_2=0.\qquad \nabla_{\nu_1}\nu_3=-\nu_1,\qquad \nabla_{\nu_3}\nu_1=0,\\ &\nabla_{\nu_2}\nu_1=0,\ \ \nabla_{\nu_2}\nu_2=-\nu_3,\ \ \nabla_{\nu_2}\nu_3=-\nu_2,\ \ \nabla_{\nu_3}\nu_2=0,\ \ \nabla_{\nu_3}\nu_3=0.\end{aligned}$$

The relations presented above remark that (ϕ,ξ,η,g) is an LP-Sasakian structure on M [29].

Now, we can make similar calculations for a generalized symmetric metric connection. Using (15) in the above equations, we get

$$\overline{\nabla}_{\nu_1}\nu_1 = (-1-\alpha+\beta)\nu_3, \quad \overline{\nabla}_{\nu_1}\nu_2 = 0, \quad \overline{\nabla}_{\nu_1}\nu_3 = (-1-\alpha+\beta)\nu_1,$$
$$\overline{\nabla}_{\nu_2}\nu_1 = 0, \quad \overline{\nabla}_{\nu_2}\nu_2 = (-1-\alpha+\beta)\nu_3, \quad \overline{\nabla}_{\nu_2}\nu_3 = (-1-\alpha+\beta)\nu_2,$$
$$\overline{\nabla}_{\nu_3}\nu_1 = 0, \quad \overline{\nabla}_{\nu_3}\nu_2 = 0, \quad \overline{\nabla}_{\nu_3}\nu_3 = 0.$$

From the above equations, we can easily see that the relation (1) holds for all ν_1, ν_2 and ν_3. Also, we obtain $\overline{\nabla} g = 0$. Thus, $\overline{\nabla}$ is a generalized symmetric metric connection on M.

4. Curvature Tensor

Consider that M is an n-dimensional LP-Sasakian manifold, then the following can define the curvature tensor $\overline{R}$ of the generalized metric connection $\overline{\nabla}$ on M as

$$\overline{R}(U,V)W = \overline{\nabla}_U\overline{\nabla}_V W - \overline{\nabla}_V\overline{\nabla}_U W - \overline{\nabla}_{[U,V]}W. \tag{16}$$

When Proposition 3.2 is used, through (15) and (16), we obtain

$$\begin{aligned}\overline{R}(U,V)W &= R(U,V)W + K_1(V,W)U - K_1(U,W)V + K_2(V,W)\phi U \\ &\quad - K_2(U,W)\phi V + \{K_3(U,V)W - K_3(V,U)W\}\xi,\end{aligned} \tag{17}$$

where

$$K_1(V,W) = (\alpha\beta - \alpha)\Phi(V,W) + \alpha^2 g(V,W) + (\alpha^2 + \beta - \beta^2)\eta(V)\eta(W), \tag{18}$$

$$K_2(V,W) = (\beta^2 - 2\beta)\Phi(V,W) - \alpha(1-\beta)g(V,W), \tag{19}$$

$$K_3(U,V)W = \{(\alpha^2 + \beta)g(V,W) + \alpha\beta\Phi(V,W)\}\eta(U). \tag{20}$$

From (2)-(4), (7), (8) and (17)-(20), we have the following lemma.

Lemma 4.1. *If M is an n-dimensional LP-Sasakian manifold with a generalized symmetric metric connection, then we have*

$$\begin{aligned}\overline{R}(U,V)\xi &= (1-\beta+\beta^2)(\eta(V)U - \eta(U)V) \\ &\quad + \alpha(1-\beta)(\eta(U)\phi V - \eta(V)\phi U),\\ \overline{R}(\xi,V)W &= \{-a\Phi(V,W) + (1-\beta)g(V,W) - \beta^2\eta(V)\eta(W)\}\xi \\ &\quad -(1-\beta+\beta^2)\eta(W)V + \alpha(1-\beta)\eta(W)\phi V,\\ \overline{R}(\xi,V)\xi &= (1-\beta+\beta^2)(\eta(V)\xi + V) + \alpha(\beta-1)\phi V\end{aligned}$$

for every U, V, $W \in \Gamma(TM)$.

In the following, the Ricci tensor $\overline{S}$ and the scalar curvature $\overline{r}$ of an LP-Sasakian manifold are presented with $\overline{\nabla}$ as:

$$\overline{S}(U,V)=\sum_{i=1}^{n}\varepsilon_i g(\overline{R}(\nu_i,U)V,\nu_i),\quad \overline{r}=\sum_{i=1}^{n}\varepsilon_i\overline{S}(\nu_i,\nu_i),\quad \varepsilon_i=g(\nu_i,\nu_i)$$

in which $U,V\in\Gamma(TM)$, $\{\nu_1,\nu_2,...,\nu_n\}$ is viewed as orthonormal frame. Then by using (4) and (17) we obtain

$$\begin{aligned}\overline{S}(V,W)=&\sum_{i=1}^{n}\varepsilon_i\{g(R(\nu_i,V)W,\nu_i)-K_1(\nu_i,W)g(V,e_i)\\&+K_1(V,W)\varepsilon_i\}+K_2(V,W)g(\phi\nu_i,\nu_i)-K_2(\nu_i,W)g(\phi V,\nu_i)\\&+\{K_3(\nu_i,V)W-K_3(V,\nu_i)W\}\eta(\nu_i\}.\end{aligned}\tag{21}$$

Then by using (18)-(21) we obtain

$$\begin{aligned}\overline{S}(V,W)=&\,S(V,W)+\{-\alpha\beta+(n-2)(\alpha\beta-\alpha)\\&+(\beta^2-2\beta)trace\Phi\}\Phi(V,W)+\{-2\alpha^2+\beta-\beta^2+n\alpha^2\\&+(\alpha\beta-\alpha)trace\Phi\}g(V,W)+\{-2\alpha^2+n(\alpha^2+\beta-\beta^2)\}\eta(V)\eta(W).\end{aligned}\tag{22}$$

Due to the fact that Ricci tensor S of the Levi-Civita connection is symmetric, (22) provides us the following:

Corollary 4.2. *Consider that M is an n-dimensional LP-Sasakian manifold equipped with a generalized symmetric metric connection $\overline{\nabla}$. The Ricci tensor $\overline{S}$ with respect to the generalized symmetric metric connection $\overline{\nabla}$ is symmetric.*

Lemma 4.3. *Let an n-dimensional LP-Sasakian manifold M admit a generalized symmetric metric connection $\overline{\nabla}$. Then we have*

$$\overline{S}(V,\xi)=\{(n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\Phi\}\eta(V),\tag{23}$$

$$\overline{S}(\phi V,\phi W)=\overline{S}(V,W)+\{(n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\Phi\}\eta(V)\eta(W)\tag{24}$$

for any V, $W\in\Gamma(TM)$.

Proof. Using (2), (5) and (9) in the equation (22), we get (23). By using (4), (5) and (10) in the equation (22), we have (24). □

Theorem 4.4. *Consider that M is an n-dimensional LP-Sasakian manifold endowed with a generalized symmetric metric connection $\overline{\nabla}$. If M is Ricci semi-symmetric with respect to $\overline{\nabla}$. Then we have the following statements:*

(i) *M is a generalized η-Einstein manifold with respect to the generalized symmetric metric connection of type (α, β).*
(ii) *M is an η-Einstein manifold with respect to the generalized symmetric metric connection of type $(0, \beta)$.*
(iii) *M is an Einstein manifold with respect to the generalized symmetric metric connection of type $(\alpha, 0)$ $(\alpha \neq 1)$.*

Proof. Let $\overline{R}(X,Y) \cdot \overline{S} = 0$ on an n-dimensional LP-Sasakian manifold M for any $X, Y, Z, U \in \Gamma(TM)$, then we have

$$\overline{S}(\overline{R}(X,Y)Z, U) + \overline{S}(Z, \overline{R}(X,Y)U) = 0. \tag{25}$$

If we choose $Z = \xi$ and $X = \xi$ in (25), we get

$$\overline{S}(\overline{R}(\xi,Y)\xi, U) + \overline{S}(\xi, \overline{R}(\xi,Y)U) = 0. \tag{26}$$

Using Lemma 4.1 and Lemma 4.3 in (26), we obtain

$$\begin{aligned}(1-\beta+\beta^2)\overline{S}(Y,U) + \alpha(\beta-1)\overline{S}(\Phi Y,U) = \{(n-1)(1-\beta+\beta^2) \\ +\alpha(\beta-1)trace\Phi\}\{-a\Phi(Y,U) + (1-\beta)g(Y,U) - \beta^2\eta(Y)\eta(U)\}.\end{aligned} \tag{27}$$

If one substitutes $Y = \phi Y$ in the equation (27) and using (23), we get

$$\begin{aligned}(1-\beta+\beta^2)\overline{S}(\phi Y,U) + \alpha(\beta-1)\overline{S}(Y,U) = \{(n-1)(1-\beta+\beta^2) \\ +\alpha(\beta-1)trace\Phi\}\{(1-\beta)\Phi(Y,U) - \alpha g(Y,U) - \alpha\beta\eta(Y)\eta(U)\}.\end{aligned} \tag{28}$$

From the (27) and (28), we obtain

$$\begin{aligned}&\{(1-\beta+\beta^2)^2 - (\alpha\beta-\alpha)^2\}\overline{S}(Y,U) = \{(n-1)(1-\beta+\beta^2) \\ &+\alpha(\beta-1)trace\Phi\}\{\alpha\beta\Phi(Y,U) - (1-\beta)(1-\beta+\beta^2-\alpha^2)g(Y,U) \\ &+(-\beta^4+\beta^3-\beta^2+\alpha^2\beta^2-\beta\alpha^2)\eta(Y)\eta(U)\}.\end{aligned} \tag{29}$$

Thus, for $\alpha = 0$, $\beta \neq 0, 1$ and $\alpha \neq 0, 1$, $\beta = 0$ we get the following equations:

$$\overline{S}(Y,U) = (n-1)(1-\beta)g(Y,V) - (n-1)\beta^2\eta(Y)\eta(U), \tag{30}$$

and

$$(1-\alpha^2)\overline{S}(Y,U) = (1-\alpha^2)(n-1-\alpha\, trace\Phi)g(Y,U), \tag{31}$$

respectively. Equations (29), (30) and (31) tell us the statement of the Theorem 4.4. $\square$

5. CR-submanifolds of an LP-Sasakian manifold with generalized symmetric metric connection

Definition 5.1. [30] *An n-dimensional Riemannian manifold M of an LP-Sasakian manifold M' is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that*
(i) *D is invariant under ϕ, i.e., $\phi D \subset D$.*
(ii) *The orthogonal complement distribution $D^\perp : x \to D_x^\perp \subset T_xM$ of the distribution D on M is totally real , i.e., $\phi D^\perp \subset T^\perp M$.*

Definition 5.2. [30] *The distribution D (resp., $D^\perp$) is called horizontal (resp., vertical) distribution. The pair $(D, D^\perp)$ is called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ (resp., $\xi \in \Gamma(D^\perp)$). The CR-submanifold is also called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ (resp., $\xi \in \Gamma(D^\perp)$).*

The orthogonal complement $\phi D^\perp$ in $T^\perp M$ is given by

$$TM = D \oplus D^\perp,\ T^\perp M = \phi D^\perp \oplus \mu,$$

where $\phi\mu = \mu$.

Let M be a CR-submanifold of an LP-Sasakian manifold M' with a generalized symmetric metric connection $\overline{\nabla}$. For any $X \in \Gamma(TM')$ and $N \in \Gamma(T^\perp M')$ we can write

$$X = PX + QX,\ PX \in \Gamma(D),\ QX \in \Gamma(D^\perp), \tag{32}$$

$$\phi N = BN + CN,\ BN \in \Gamma(D^\perp),\ CN \in \Gamma(\mu). \tag{33}$$

The Gauss and Weingarten formulas with respect to $\overline{\nabla}$ are given, respectively,

$$\begin{aligned} \overline{\nabla}_X Y &= \overline{\nabla}'_X Y + \overline{h}(X, Y), \\ \overline{\nabla}_X N &= -\overline{A}_N X + \overline{\nabla}^\perp_X N \end{aligned} \tag{34}$$

for any $X, Y \in \Gamma(TM')$, where $\overline{\nabla}'_X Y$, $\overline{A}_N X \in \Gamma(TM')$. Here, $\overline{\nabla}'$, $\overline{h}$ and $\overline{A}_N$ are called the induced connection on M, the second fundamental form and the Weingarten mapping with respect to $\overline{\nabla}$. From (11), (15) and (34) we have

$$\begin{aligned} \overline{\nabla}'_X Y + \overline{h}(X, Y) = \nabla'_X Y + h(X, Y) &+ \alpha\{\eta(Y)X - g(X, Y)\xi\} \\ &+ \beta\{\eta(Y)\phi X - g(\phi X, Y)\xi\}. \end{aligned}$$

Using (32) and (33) in the above equation and comparing the tangential and normal components on both sides, we obtain

(35)
$$P\overline{\nabla}^{'}_X Y = P\nabla^{'}_X Y + \alpha\eta(Y)PX - \alpha g(X,Y)P\xi + \beta\eta(Y)\phi PX - \beta g(\phi X,Y)P\xi,$$

$$\overline{h}(X,Y) = h(X,Y) + \beta\eta(Y)\phi QX, \tag{36}$$

$$Q\overline{\nabla}^{'}_X Y = Q\nabla^{'}_X Y + \alpha\eta(Y)QX - \alpha g(X,Y)Q\xi - \beta g(\phi X,Y)Q\xi \tag{37}$$

for any $X,\ Y \in \Gamma(TM^{'})$.

Theorem 5.3. *Let M be a CR-submanifold of an LP-Sasakian manifold $M^{'}$ with a generalized symmetric metric connection $\widetilde{\nabla}$. Then we have the following expressions:*

(i) *If M is ξ-horizontal, $X,Y \in \Gamma(D)$ and D is parallel with respect to $\overline{\nabla}^{'}$, then the induced connection $\overline{\nabla}^{'}$ is a generalized symmetric metric connection.*

(ii) *If M is ξ-vertical, $X,Y \in \Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\overline{\nabla}^{'}$, then the induced connection $\overline{\nabla}^{'}$ is a generalized symmetric non-metric connection.*

(iii) *The Gauss formula with respect to generalized symmetric metric connection is of the form*

$$\overline{\nabla}_X Y = \overline{\nabla}^{'}_X Y + h(X,Y) + \beta\eta(Y)\phi QX. \tag{38}$$

(iv) *The Weingarten formula with respect to generalized symmetric metric connection is of the form*

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \alpha\eta(N)X + \beta\eta(N)\phi X - \beta g(\phi X,N)\xi. \tag{39}$$

Proof. Using (34) and (36) we have (iii). Moreover, from (15) and Weingarten formula, we get (iv). In view of (35), if M is ξ-horizontal, $X,Y \in \Gamma(D)$ and D is parallel with respect to $\overline{\nabla}^{'}$, we obtain

$$\overline{\nabla}^{'}_X Y = \nabla^{'}_X Y + \alpha\eta(Y)X - \alpha g(X,Y)\xi + \beta\eta(Y)\phi X - \beta g(\phi X,Y)\xi.$$

This equation is verifying (i). In view of (37) if M is ξ-vertical, $X,Y \in D^{\perp}$ and $D^{\perp}$ is parallel with respect to $\overline{\nabla}^{'}$, we have

$$\overline{\nabla}^{'}_X Y = \nabla^{'}_X Y + \alpha\eta(Y)X - \alpha g(X,Y)\xi - \beta g(\phi X,Y)\xi. \tag{40}$$

Using (40) we get

$$(\overline{\nabla}^{'}_X g)(Y,Z) = \beta\{\eta(Y)g(\phi X,Z) + \eta(Z)g(\phi X,Y)\}.$$

Thus, we have (ii). □

Lemma 5.4. *Let M be a CR-submanifold of an LP-Sasakian manifold $M^{'}$ with a generalized symmetric metric connection. Then*

$$h(X,\phi PY)+\beta\eta(Y)\phi QX+\nabla^{\perp}_{X}\phi QY=Ch(X,Y)-\alpha\eta(Y)\phi QX+\phi Q\overline{\nabla}^{'}_{X}Y, \tag{41}$$

$$\begin{aligned} &P\overline{\nabla}^{'}_{X}\phi PY-PA_{\phi QY}X-\beta g(\phi X,\phi QY)P\xi=K(X,Y)P\xi\\ &\quad+(1-\beta)\eta(Y)PX-\alpha\eta(Y)\phi PX+\phi P\nabla^{'}_{X}Y+\beta\eta(Y)\eta(QX)P\xi, \end{aligned}\tag{42}$$

$$\begin{aligned} &Q\overline{\nabla}^{'}_{X}\phi PY-QA_{\phi QY}X-\beta g(\phi X,\phi QY)Q\xi=K(X,Y)Q\xi\\ &\quad+(1-\beta)\eta(Y)QX+Bh(X,Y)+\beta\eta(Y)QX+\beta\eta(Y)\eta(QX)Q\xi \end{aligned}\tag{43}$$

for any X, $Y\in\Gamma(TM)$, where $K(X,Y)=(1-\beta)g(X,Y)+(2-2\beta)\eta(Y)\eta(Y)-\alpha g(X,\phi Y)$ and B, C are defined in (33).

Proof. We know that $\overline{\nabla}_X\phi Y=(\overline{\nabla}_X\phi)Y+\phi(\overline{\nabla}_XY)$. By virtue of Proposition 3.2, (38) and (39), we get

$$\begin{aligned} &\overline{\nabla}^{'}_{X}\phi PY+h(X,\phi PY)+\beta\eta(Y)\phi QX-A_{\phi QY}X+\nabla^{\perp}_{X}\phi QY-\beta g(\phi X,\phi QY)\xi\\ &\quad=(1-\beta)\eta(Y)X-\alpha\eta(Y)\phi X+\{(1-\beta)g(X,Y)+(2-2\beta)\eta(Y)\eta(Y)\\ &\quad-\alpha g(X,\phi Y)\}\xi+\phi\overline{\nabla}^{'}_{X}Y+\phi h(X,Y)+\beta\eta(Y)(QX+\eta(QX)\xi). \end{aligned}$$

Using (32) and (33) and the above equation, comparing the normal, horizontal and vertical components, we have (41)-(43). □

Lemma 5.5. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $M^{'}$ with a generalized symmetric metric connection. Then*

$$\phi P[Y,Z]=A_{\phi Y}Z-A_{\phi Z}Y+(\beta-1)\{\eta(Z)Y-\eta(Y)Z\}$$

for any Y, $Z\in\Gamma(D^{\perp})$.

Proof. We know that $\overline{\nabla}_X\phi Y=(\overline{\nabla}_X\phi)Y+\phi(\overline{\nabla}_XY)$, $\forall\ Y,\ Z\in\Gamma(D^{\perp})$. Using Proposition 3.2, (38) and (39), we get

$$\begin{aligned} &-A_{\phi Z}Y+\nabla^{\perp}_{Y}\phi Z-\beta g(\phi Y,\phi Z)\xi=\{(1-\beta)g(Y,Z)+(2-2\beta)\eta(Y)\eta(Z)\\ &-\alpha g(Y,\phi Z)\}\xi+(1-\beta)\eta(Z)Y-\alpha\eta(Z)\phi Y+\phi\overline{\nabla}^{'}_{Y}Z+\phi h(Y,Z)+\beta\eta(Z)\phi^2QY. \end{aligned}$$

By using (41), we obtain

$$\begin{aligned} \phi\overline{\nabla}^{'}_{Y}Z=&-A_{\phi Z}Y+\{-g(Y,Z)+(\beta-2)\eta(Y)\eta(Z)+\alpha g(Z,\phi Y)\}\xi\\ &-Bh(Y,Z)+(\beta-1)\eta(Z)Y-\beta\eta(Z)(\phi QY+\phi^2QY). \end{aligned}$$

Interchanging Y and Z, we have

$$\begin{aligned} \phi\overline{\nabla}^{'}_{Z}Y=&-A_{\phi Y}Z+\{-g(Y,Z)+(-2+\beta)\eta(Y)\eta(Z)+\alpha g(Z,\phi Y)\}\xi\\ &-Bh(Y,Z)+(\beta-1)\eta(Y)Z-\beta\eta(Y)(\phi QZ+\phi^2QZ). \end{aligned}$$

By subtracting above equations, we get the statement of the Lemma 5.5. □

This Lemma is verifying the following theorem.

Theorem 5.6. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $M^{'}$ with a generalized symmetric metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = (\beta - 1)\{\eta(Y)Z - \eta(Z)Y\}$$

for any Y, $Z \in \Gamma(D^{\perp})$.

Corollary 5.7. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $M^{'}$ with a generalized symmetric metric connection of type $(\alpha, 1)$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z = A_{\phi Z}Y$$

for any $Y, Z \in \Gamma(D^{\perp})$.

Corollary 5.8. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $M^{'}$ with a semi-symmetric metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = \eta(Z)Y - \eta(Y)Z$$

for any $Y, Z \in \Gamma(D^{\perp})$.

Proposition 5.9. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $M^{'}$ with a generalized symmetric metric connection. Then*

$$\phi Ch(X,Y) = Ch(\phi X, Y) = Ch(X, \phi Y)$$

for any $X, Y \in \Gamma(D)$.

Proof. From (43) we get

$$Q\overline{\nabla}^{'}_{X}\phi Y = \{(1-\beta)g(X,Y) - \alpha g(X,\phi Y)\}Q\xi + Bh(X,Y) \tag{44}$$

and

$$Q\overline{\nabla}^{'}_{\phi X}\phi Y = \{(1-\beta)g(\phi X,Y) - \alpha g(X,Y)\}Q\xi + Bh(\phi X,Y). \tag{45}$$

Interchanging X and Y in (44) we have

$$Q\overline{\nabla}^{'}_{Y}\phi X = \{(1-\beta)g(X,Y) - \alpha g(Y,\phi X)\}Q\xi + Bh(X,Y).$$

Replacing X by ϕX in the above equation, we obtain

$$Q\overline{\nabla}^{'}_{Y}X = \{(1-\beta)g(\phi X,Y) - \alpha g(X,Y)\}Q\xi + Bh(\phi X,Y). \tag{46}$$

Subtracting (45) from (46) we lead

$$Q(\overline{\nabla}^{'}_{\phi X}\phi Y - \overline{\nabla}^{'}_{Y}X) = 0.$$

Thus, we get

$$\overline{\nabla}^{'}_{\phi X}\phi Y - \overline{\nabla}^{'}_{Y}X \in D. \tag{47}$$

Moreover, from (41), we find

$$h(X,\phi Y) = Ch(X,Y) + \phi Q\overline{\nabla}'_X Y. \tag{48}$$

Replacing X by ϕX and Y by ϕY in (48) we obtain

$$h(\phi X,Y) = Ch(\phi X,\phi Y) + \phi Q\overline{\nabla}'_{\phi X}\phi Y. \tag{49}$$

Interchanging X and Y in (48), we get

$$h(\phi X,Y) = Ch(X,Y) + \phi Q\overline{\nabla}'_Y X. \tag{50}$$

Subtracting (49) from (50) and using (47), we have

$$Ch(\phi X,\phi Y) = Ch(X,Y).$$

Replacing X by ϕX in the last equation we find

$$Ch(\phi^2 X,\phi Y) = Ch(\phi X,Y).$$

Thus, we obtain

$$Ch(X,\phi Y) = Ch(\phi X,Y).$$

Using (44) we obtain

$$Q\overline{\nabla}'_X\phi^2 Y = \{(1-\beta)g(X,\phi Y) - \alpha g(X,\phi^2 Y)\}Q\xi + Bh(X,\phi Y).$$

Thus,

$$Q\overline{\nabla}'_X Y = \{(1-\beta)g(X,\phi Y) - \alpha g(X,Y)\}Q\xi + Bh(X,\phi Y). \tag{51}$$

Using (51) in (48), we have

$$h(X,\phi Y) = Ch(X,Y) + \phi Bh(X,\phi Y).$$

Applying ϕ on both sides, we obtain

$$\phi h(X,\phi Y) = \phi Ch(X,Y) + \phi Bh(X,\phi Y). \tag{52}$$

Using (33) in (52), proof is completed. □

Theorem 5.10. *Let M be a ξ-horizontal CR-submanifold of an LP-Sasakian manifold M' with a generalized symmetric metric connection. Then the distribution D is integrable if and only if*

$$h(\phi X,Y) = h(\phi Y,X)$$

for any $X,Y \in \Gamma(D)$.

Proof. From (43) we get

$$Q\overline{\nabla}'_X\phi Y = Bh(X,Y).$$

Replacing X by ϕX we have

$$Q\overline{\nabla}'_{\phi X}\phi Y = Bh(\phi X,Y).$$

Interchanging X and Y we obtain

$$Q\overline{\nabla}'_{\phi Y}\phi X = Bh(\phi Y, X).$$

Subtracting last two equations, we find

$$Q[\phi X, \phi Y] = B\{h(\phi X, Y) - h(\phi Y, X)\}.$$

Proof is completed. □

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References

[1] N. S. Agashe and M. R. Chafle, *A semi symetric non-metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math. **23** (1992), 399-409.

[2] O. Bahadir, *Lorentzian para-Sasakian manifold with quartersymmetric non-metric connection*, Journal of Dynamical Systems and Geometric Theories **14(1)** (2016), 17-33.

[3] S. K. Chaubey, *Some properties of LP-Sasakian manifolds equipped with m-projective curvature tensor*, Bulletin of Math. Analysis and Applications **3(4)** (2011), 50-58.

[4] S. K. Chaubey and U. C. De, *Characterization of the Lorentzian para-Sasakian manifolds admitting a quarter-symmetric non-metric connection*, SUT Journal of Mathematics **55(1)** (2019), 53-67.

[5] S. K. Chaubey and U. C. De, *Lorentzian para-Sasakian manifolds admitting a new type of quarter-symmetric non-metric ξ-connection*, Int. Electron. J. Geom. **12(2)** (2019), 250-259.

[6] S. K. Chaubey and A. Yildiz, *Riemannian manifolds admitting a new type of semi-symmetric non-metric connection*, Turk J Math. **43** (2019), 1887-1904.

[7] S. K. Chaubey and R. H. Ojha, *On a semi-symmetric non-metric connection*, Filomat **26(2)** (2012), 63-69.

[8] S. K. Chaubey and R. H. Ojha, *On semi-symmetric non-metric and quarter-symmetric metric connections*, Tensor N. S. **70(2)** (2008), 202-213.

[9] S. K. Chaubey, J. W. Lee and S. Yadav, *Riemannian manifolds with a semi-symmetric metric P-connection*, J. Korean Math. Soc. **56(4)** (2019), 1113-1129.

[10] M. Danish Siddiqi, M. Ahmad and J. P. Ojha, *CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection*, Afr. Diaspora J. Math. (N.S.) **17** (2014), 93-105.

[11] U. C. De and D. Kamilya, *Hypersurfaces of Rieamnnian manifold with semi-symmetric non-metric connection*, J. Indian Inst. Sci. **75** (1995), 707-710.

[12] U. C. De and A. K. Sengupta, *CR-submanifolds of a Lorentzian para-Sasakian manifold*, Bull. Malaysian Math. Sci. Soc. **23** (2000), 99-106.

[13] S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor N. S. **29** (1975), 249-254.

[14] A. Haseeb and R. Prasad, *η-Ricci solitons on ϵ-LP-Sasakian manifolds with a quarter-symmetric metric connection*, Honam Mathematical J. **41(3)** (2019), 539558.

[15] H. A. Hayden, *Subspaces of a space with torsion*, London Math. Soc. **34** (1932), 27-50.

[16] S. K. Hui, S. Uddin, C. Ozel and A. A. Mustafa, *Warped product submanifolds of LP-Sasakian manifolds*, Hindawi Publishing Corporation, Discrete Dynamics in Nature and Society **2012**, Article ID 868549, 11 pages.

[17] D. H. Jin, *Lightlike hypersurfaces of an indefinite Kaehler manifold with a symmetric metric connection of type* (l, m), Bull. Korean Math. Soc. **53(4)** (2016), 1171-1184.

[18] Y. Liang, *On semi-symmetric recurrent-metric connection*, Tensor N. S. **55** (1994), 107-112.

[19] J. W. Lee, C. W. Lee and D. W. Yoon, *Inequalities for generalized δ-Casorati curvatures of submanifolds in real space forms endowed with a semi-symmetric metric connection*, Revista De La Union Matematica Argentina **57(2)** (2016), 53-62.

[20] C. W. Lee, J. W. Lee and G. E. Vilcu and D. W. Yoon, *Optimal inequalities for the Casorati curvatures of submanifolds of generalized space forms endowed with semi-symmetric metric connections*, Bull. Korean Math. Soc. **52(5)** (2015), 1631-1647.

[21] K. Matsumoto, *On Lorentzian Paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. **12(2)** (1989), 151-156.

[22] K. Matsumoto and I. Mihai *On a certain transformation in a Lorentzian para Sasakian manifold*, Tensor N. S. **47** (1988), 189-197.

[23] I. Mihai and R. Rosca, *On Lorentzian P-Sasakian manifolds*, Clssical Analysis, World Scientific Publ. **1992**, 155-169.

[24] I. Mihai, A. A. Shaikh and U. C. De, *On Lorentzian para-Sasakian manifolds*, Rendiconti del Seminario Mat. di Messina, Serie II, 1999.

[25] C. Ozgur, M. Ahmad and A. Haseeb, *CR-submanifolds of a Lorentzian para-Sasakian manifold with a semi-symmetric metric connection*, Hacettepe Journal of Mathematics and Statistics **39(4)** (2010), 489-496.

[26] S. Pahan, B. Pal and A. Bhattacharyya, *On Einstein warped products with a quarter-symmetric connection*, International Journal of Geometric Methods in Modern Physics **14** (2017), 1750050.

[27] S. Pahan, B. Pal and A. Bhattacharyya, *On Ricci flat warped products with a quarter-symmetric connection*, J. Geom. **107** (2016), 627-634.

[28] Q. Qu and Y. Wang, *Multiply warped products with a quarter-symmetric connection*, Journal of Mathematical Analysis and Applications **431(2)** (2015), 955-987.

[29] A. A. Shaikh and K. K. Baishya, *On Φ-symmetric LP-Sasakian manifolds*, Yokohama Mathematical Journal **52** (2006), 97-112.

[30] B. G. Schmidt, *Conditions on a connection to be a metric connection*, Commun. Math. Phys. **29** (1973), 55-59.

[31] M. M. Tripathi, *A new connection in a Riemannian manifold*, Int. Electron. J. Geom. **1** (2006), 15-24.

[32] S. K. Yadav, S. K. Chaubey and S. K. Hui, *On the perfect fluid Lorentzian para-Sasakian spacetimes*, Bulg. J. Phys. **46(1)** (2018), 1-15.

Oğuzhan Bahadır
Department of Mathematics, Faculty of Arts and Sciences, K.S.U,
Kahramanmaras, TURKEY.
E-mail: oguzbaha@gmail.com

Sudhakar K. Chaubey
Section of Mathematics, Department of Information Technology,
Shinas College of Technology, Shinas, P.O. Box 77, Postal Code 324, Oman.
E-mail: sk22_math@yahoo.co.in