A NEW CLASS OF GENERALIZED APOSTOL-TYPE FROBENIUS-EULER-HERMITE POLYNOMIALS

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Abstract. In this paper, we introduce a new class of generalized Apostol-type Frobenius-Euler-Hermite polynomials and derive some explicit and implicit summation formulae and symmetric identities by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Frobenius-Euler type polynomials and Hermite-based Apostol-Euler and Apostol-Genocchi polynomials studied by Pathan and Khan, Kurt and Simsek.

1. Introduction

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \cdots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Furthermore, $(\lambda)_0 = 1$ and

$$(\lambda)_k = \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + k + 1),$$

where $k \in \mathbb{N}, \lambda \in \mathbb{C}$.

The classical Frobenius-Euler polynomial $\mathbb{H}_n^{(\alpha)}(x; u)$ of order α is defined by means of the following generating function

$$\left(\frac{1 - u}{e^t - u}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathbb{H}_n^{(\alpha)}(x) \frac{t^n}{n!}, \qquad (1.1)$$

where u is an algebraic number and $\alpha \in \mathbb{Z}$.

Observe that $\mathbb{H}_n^{(1)}(x, u) = \mathbb{H}_n(x, u)$, which denotes the Frobenius-Euler polynomials and $\mathbb{H}_n^{(\alpha)}(0; u) = \mathbb{H}_n^{(\alpha)}(u)$, which denotes the Frobenius-Euler numbers of order α. $\mathbb{H}_n(x; -1) = E_n(x)$, which denotes the Euler polynomials, (see [1-11, 13-16]).

Recently, Kurt and Simsek [6, 7] and Simsek [14, 15] introduced the Apostol type Frobenius-Euler polynomials defined as follows.

Let $a, b, c \in \mathbb{R}^+$, $a \neq b, x \in \mathbb{R}$. The generalized Apostol type Frobenius-Euler polynomials are defined by means of the following generating function:

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathbb{H}_n^{(\alpha)}(x; u, a, b, c, \lambda) \frac{t^n}{n!}. \tag{1.2}$$

For $x = 0$ and $\alpha = 1$ in (1.2), we get

$$\frac{a^t - u}{\lambda b^t - u} = \sum_{n=0}^{\infty} \mathbb{H}_n(u, a, b; \lambda) \frac{t^n}{n!}, \tag{1.3}$$

where $\mathbb{H}_n(u, a, b; \lambda)$ denotes the generalized Apostol type Frobenius-Euler numbers (see [14]).

Pathan and Khan [9] introduced the generalized Hermite-Bernoulli polynomials of two variables ${}_H B_n^{(\alpha)}(x, y)$ defined by

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt + yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{(\alpha)}(x, y) \frac{t^n}{n!}, \tag{1.4}$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_H B_n(x, y)$ introduced by Dattoli et al [3, p.386(1.6)] in the form

$$\left(\frac{t}{e^t - 1}\right) e^{xt + yt^2} = \sum_{n=0}^{\infty} {}_H B_n(x, y) \frac{t^n}{n!}. \tag{1.5}$$

Definition 1.1. *Let $c > 0$. The generalized 2-variable 1-parameter Hermite Kampé de Fériet polynomials $H_n(x, y; c)$ polynomials for non-negative integer n are defined by*

$$c^{xt + yt^2} = \sum_{n=0}^{\infty} H_n(x, y; c) \frac{t^n}{n!}. \tag{1.6}$$

This is an extended 2-variable Hermite Kampé de Fériet polynomials $H_n(x, y)$ (see [1]) defined by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}.$$ (1.7)

Note that

$$H_n(x, y; e) = H_n(x, y).$$

In order to collect the powers of t we expand the left hand side of (1.6) to the representation

$$H_n(x, y; c) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \left(\begin{array}{c} n \\ 2j \end{array} \right) (\ln c)^{n-j} x^{n-2j} y^j.$$ (1.8)

There are various applications of the classical Euler and Frobenius numbers in many branches of mathematics, statistics and mathematical physics. Frobenius-Euler numbers and polynomials appear in number theory, summability, statistics, control theory, splines and combinatorics (see [4]). Generating functions for q-Apostol type Frobenius-Euler numbers and polynomials, generalized Frobenius Euler polynomials, normalized polynomials, array polynomials, Stirling numbers of the second kind and functional equation are studied by Kurt and Simsek ([6] and [7]) and Simsek ([14] and [15]). Janson [4] studied the Frobenius-Euler numbers and polynomials in detail (with applications to number theory, geometric formulation, rounding errors and sums of uniform random variables and the probability distribution). Janson also gave references to earlier authors as well as historical notes.

In the present paper, we introduce a new class of generalized Apostol-type Frobenius-Euler-Hermite polynomials which has quite distinct formulation from the work proposed by Kurt and Simsek ([6] and [7]) and Simsek ([14] and [15]) and derive some explicit and implicit summation formulae. This paper is organized as follows. We give a brief review of generalized Apostol type Frobenius-Euler-Hermite polynomials $_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$ and their properties. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions.

2. Definitions and properties of the generalized Apostol type Frobenius-Euler-Hermite polynomials $_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$

To facilitate the presentation of the material that follows, we present in this section some background on the generalized Apostol type Frobenius-Euler-Hermite polynomials and investigate its properties. First, we present the following definition.

Definition 2.1. *Let* $a, b, c > 0$ *and* $a \neq b$. *The generalized Apostol type Frobenius-Euler-Hermite polynomials* $_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$ *for nonnegative integer* n *are defined by*

$$\left(\frac{a^t - u}{\lambda b^t - u} \right)^\alpha c^{xt + yt^2} = \sum_{n=0}^\infty {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda) \frac{t^n}{n!}, \qquad (2.1)$$

so that

$$_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda) = \sum_{m=0}^n \binom{n}{m} \mathbb{H}_m^{(\alpha)}(u, a, b; \lambda) H_{n-m}(x, y; c). \tag{2.2}$$

For $\alpha = 1$, we obtain from (2.1) the generating function

$$\left(\frac{a^t - u}{\lambda b^t - u} \right) c^{xt + yt^2} = \sum_{n=0}^\infty {}_H E_n(x, y; u, a, b, c; \lambda) \frac{t^n}{n!}. \tag{2.3}$$

Special case of (2.1) for $y = 0$ leads to the extension of the generalized Apostol type Frobenius-Euler polynomials $\mathbb{H}_n^{(\alpha)}(x; u, a, b, c; \lambda)$ for nonnegative integer n defined by (1.2).

Setting $c = e$ in (2.1), we get

Definition 2.2. *Let* $a, b > 0$ *and* $a \neq b$. *The generalized Apostol type Frobenius-Euler-Hermite polynomials* $_H E_n^{(\alpha)}(x, y; u, a, b, e; \lambda)$ *for nonnegative integer* n *are defined by*

$$\left(\frac{a^t - u}{\lambda b^t - u} \right)^\alpha e^{xt + yt^2} = \sum_{n=0}^\infty {}_H E_n^{(\alpha)}(x, y; u, a, b, e; \lambda) \frac{t^n}{n!}. \tag{2.4}$$

It is easy to prove that

$$_H E_n^{(\alpha+\beta)}(x, y; u, a, b, c; \lambda) = \sum_{m=0}^n \binom{n}{m} {}_H E_m^{(\alpha)}(x, y; u, a, b, c; \lambda)$$

$$\times _H E_{n-m}^{(\beta)}(x, y; u, a, b, c; \lambda).$$
(2.5)

Further setting $\lambda, a = 1$, $u = -1$, $b = e$ in (2.4), the result reduces to known result of Pathan and Khan [11].

By using generalized Apostol type Frobenius-Euler-Hermite polynomials $_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$ defined by (2.1), we have the following properties which are stated as theorems below.

Theorem 2.3. Let $a, b, c > 0$ and $a \neq b$. For $x \in \mathbb{R}$ and $n \geq 0$. Then

$$_H E_n^{(\alpha)}(0, 0; u, a, b, 1; \lambda) = \mathbb{H}_n^{(\alpha)}(u, a, b; \lambda), \quad _H E_n^{(\alpha)}(x, y; -1, 1, e, c; \lambda)$$

$$= {}_H E_n^{(\alpha)}(x, y; c; \lambda).$$
(2.6)

$$_H E_n^{(\alpha+\beta)}(x + y, z + w; u, a, b, c; \lambda)$$

$$= \sum_{m=0}^{n} \binom{n}{m} {}_H E_m^{(\beta)}(y, w; u, a, b, c; \lambda) {}_H E_{n-m}^{(\alpha)}(x, z; u, a, b, c; \lambda). \quad (2.7)$$

$$_H E_n^{(\alpha)}(x + z, y; u, a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathbb{H}_{n-m}^{(\alpha)}(x; u, a, b, c; \lambda) H_m(z, y; c).$$
(2.8)

Proof. The formula in (2.6) is obvious. Applying definition (2.1), we have

$$I = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha+\beta)}(x + y, z + w; a, b, c; \lambda) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, z; u, a, b, c; \lambda) \frac{t^n}{n!} \sum_{m=0}^{\infty} {}_H E_m^{(\beta)}(y, w; u, a, b, c; \lambda) \frac{t^m}{m!}.$$

Replacing n by $n - m$ in above equation, we get

$$I = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} {}_H E_m^{(\beta)}(y, w; u, a, b, c; \lambda) {}_H E_{n-m}^{(\alpha)}(x, z; u, a, b, c; \lambda). \frac{t^n}{n!}.$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.7). Again by definition (2.1) of generalized Apostol type Frobenius-Euler-Hermite polynomials, we have

$$\left(\frac{a^t - u}{\lambda b^t - u} \right)^{\alpha} c^{(x+z)t+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x + z, y; u, a, b, c; \lambda) \frac{t^n}{n!}, \quad (2.9)$$

which can be written as

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^{\alpha} c^{xt} c^{zt + yt^2} = \sum_{n=0}^{\infty} \mathbb{H}_n^{(\alpha)}(x; u, a, b, c; \lambda) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(z, y, c) \frac{t^m}{m!}.$$

(2.10)

Replacing n by $n - m$ in R.H.S of above equation, we have

$$\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x + z, y; u, a, b, c; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \mathbb{H}_{n-m}^{(\alpha)}(x; u, a, b, c; \lambda)$$

$$\times H_m(z, y; c) \frac{t^n}{n!}.$$

Equating their coefficients of t^n leads to formula (2.8). □

Theorem 2.4. *The following relationship holds true:*

$$(2u - 1) \sum_{k=0}^{n} \binom{n}{k} \mathbb{H}_k(x; u, a, b, c; \lambda) {}_H E_{n-k}(y, z; 1 - u, a, b, c; \lambda)$$

$$= (u - 1) {}_H E_n(x + y, z; u, a, b, c; \lambda) + u {}_H E_n(x + y, z; 1 - u; a, b, c; \lambda)$$

$$+ \sum_{k=0}^{n} \binom{n}{k} (\ln a)^k {}_H E_{n-k}(x + y, z; u, a, b, c; \lambda) - \sum_{k=0}^{n} \binom{n}{k} (\ln a)^k$$

$$\times {}_H E_{n-k}(x + y, z; 1 - u, a, b, c; \lambda).$$

(2.11)

Proof. We set

$$(2u - 1) \left(\frac{a^t - u}{\lambda b^t - u}\right) c^{xt} \left(\frac{a^t - (1 - u)}{\lambda b^t - (1 - u)}\right) c^{yt + zt^2}$$

$$= (a^t - u)(a^t - (1 - u)) c^{(x+y)t + zt^2} \left(\frac{1}{\lambda b^t - u} - \frac{1}{\lambda b^t - (1 - u)}\right).$$

From the above equation, we see that

$$(2u - 1) \left(\sum_{k=0}^{\infty} \mathbb{H}_k(x; u, a, b, c; \lambda) \frac{t^k}{k!}\right) \left(\sum_{n=0}^{\infty} {}_H E_n(y, z; 1 - u; a, b, c; \lambda) \frac{t^n}{n!}\right)$$

$$= (a^t - 1 + u) \sum_{n=0}^{\infty} {}_H E_n(x + y, z; u, a, b, c; \lambda) \frac{t^n}{n!}$$

$$- (a^t - u) \sum_{n=0}^{\infty} {}_H E_n(x + y, z; 1 - u; a, b, c; \lambda) \frac{t^n}{n!}$$

$$(2u-1)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{H}_k(x;u,a,b,c;\lambda)_HE_n(y,z;1-u,a,b,c;\lambda)\frac{t^{n+k}}{n!k!}$$

$$= (u-1)\sum_{n=0}^{\infty}{}_HE_n(x+y,z;u,a,b,c;\lambda)\frac{t^n}{n!}$$

$$+u\sum_{n=0}^{\infty}{}_HE_n(x+y,z;1-u,a,b,c;\lambda)\frac{t^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(\ln a)^k{}_HE_n(x+y,z;u,a,b,c;\lambda)\frac{t^{n+k}}{n!k!}$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(\ln a)^k{}_HE_n(x+y,z;1-u,a,b,c;\lambda)\frac{t^{n+k}}{n!k!}.$$

Replacing n by $n-k$ in r.h.s above equation, we get

$$(2u-1)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathbb{H}_k(x;u,a,b,c;\lambda)_HE_{n-k}(y,z;1-u,a,b,c;\lambda)\frac{t^n}{n!}$$

$$= (u-1)\sum_{n=0}^{\infty}{}_HE_n(x+y,z;u,a,b,c;\lambda)\frac{t^n}{n!}$$

$$+u\sum_{n=0}^{\infty}{}_HE_n(x+y,z;1-u,a,b,c;\lambda)\frac{t^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\ln a)^k{}_HE_{n-k}(x+y,z;u,a,b,c;\lambda)\frac{t^n}{n!}$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\ln a)^k{}_HE_{n-k}(x+y,z;1-u,a,b,c;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.   □

Remark 2.5. For $z = 0$ in equation (2.11), the result reduces to known result of Kurt and Simsek [6, Eq.9,p.3].

Remark 2.6. By substituting $z = 0, a = \lambda = 1$ and $b = c = e$ into Theorem (2.2), we get Carlitz's result (for details see [2, Eq.2.19]) as follows:

$$(2u-1)\sum_{k=0}^{n}\binom{n}{k}\mathbb{H}_k(x;u)\mathbb{H}_{n-k}(y;1-u)$$

$$= (u-1)\mathbb{H}_n(x+y;u) + u\mathbb{H}_n(x+y;1-u) + \mathbb{H}_n(x+y;u) - \mathbb{H}_n(x+y;1-u).$$
$$(2.12)$$

We recall the following generating function of the polynomials $Y_n(x;\lambda;a)$

$$\frac{t}{\lambda a^t - 1}a^{xt} = \sum_{n=0}^{\infty} Y_n(x;\lambda;a)\frac{t^n}{n!}, \quad (a \geq 1) \tag{2.13}$$

(c.f.[14, 15]). We also note that

$$Y_n(0;\lambda;a) = Y_n(\lambda;a).$$

If we substitute x = 0 and a = 1 into (2.13), we see that

$$Y_n(\lambda;1) = \frac{1}{\lambda - 1}.$$

Theorem 2.7. *The generalized Apostol type Frobenius Euler-Hermite polynomials holds true as follows:*

$$n\left({}_H E_{n+1}(x,y;u,a,b,b;\lambda) - \ln(b^x){}_H E_n(x,y;u,a,b,b;\lambda)\right)$$

$$= \ln(a^{\frac{1}{u}})\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}(1;\frac{1}{u},a){}_H E_k(x,y;u,a,b,b;\lambda)$$

$$+ \ln(b^{\frac{\lambda}{u}})\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}(\frac{1}{u};a){}_H E_k^{(2)}(x,y;u,a,b,b;\lambda). \tag{2.14}$$

Proof. Substituting $c = b, \alpha = 1$, in equation (2.1) and taking derivative with respect to t, we obtain

$$\sum_{n=0}^{\infty}{}_H E_{n+1}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!} = \frac{a^t \ln a}{a^t - u}\left(\frac{a^t - u}{\lambda b^t - u}\right)b^{xt+yt^2}$$

$$+ \frac{\lambda b^t \ln b}{a^t - u}\left(\frac{a^t - u}{\lambda b^t - u}\right)^2 b^{xt+yt^2} + \ln(b^x)\left(\frac{a^t - u}{\lambda b^t - u}\right)b^{xt+yt^2}.$$

Using equation (2.13), we have

$$\sum_{n=0}^{\infty}{}_H E_{n+1}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!} = \frac{\ln(a^{\frac{1}{u}})}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}(1;\frac{1}{u};a)$$

$$\times{}_H E_k(x,y;u;a,b,b;\lambda)\frac{t^n}{n!}$$

$$+ \frac{\ln(b^{\frac{\lambda}{u}})}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}(\frac{1}{u};a){}_H E_k^{(2)}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!}$$

$$+ \ln(b^x) \sum_{n=0}^{\infty} {}_H E_n(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

Thus after some elementary calculations, we arrive at (2.14). $\square$

Remark 2.8. *For y=0 in equation (2.14), the result reduces to a known result of Kurt and Simsek [6, Eq.11.,p.14].*

Theorem 2.9. *Let $\alpha \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{\alpha} \binom{\alpha}{k} \binom{n}{m} (-u)^{\alpha-k} (k \ln a)^m H_{n-m}(x, y; c)$$

$$= \sum_{k=0}^{\alpha} \binom{\alpha}{k} \binom{n}{m} \lambda^k (-u)^{\alpha-k} (k \ln b)^m {}_H E_{n-m}(x, y; u.a, b, c; \lambda).$$

(2.15)

Proof. From (2.1), we have

$$\left(\frac{a^t - u}{\lambda b^t - u} \right)^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda) \frac{t^n}{n!}$$

$$(a^t - u)^{\alpha} c^{xt+yt^2} = (\lambda b^t - u)^{\alpha} \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda) \frac{t^n}{n!}. \qquad (2.16)$$

$$L.H.S = \sum_{k=0}^{\alpha} \binom{\alpha}{k} (-u)^{\alpha-k} a^{tk} c^{xt+yt^2}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\alpha} \binom{\alpha}{k} \binom{n}{m} (-u)^{\alpha-k} (k \ln a)^m H_{n-m}(x, y; c) \right) \frac{t^n}{n!}. \quad (2.17)$$

$$R.H.S = \sum_{k=0}^{\alpha} \binom{\alpha}{k} (-u)^{\alpha-k} \lambda^k b^{tk} \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\alpha} \binom{\alpha}{k} \binom{n}{m} \lambda^k (-u)^{\alpha-k} (k \ln b)^m {}_H E_{n-m}(x, y; u.a, b, c; \lambda) \right) \frac{t^n}{n!}.$$

(2.18)

From equations (2.17) and (2.18), we get (2.15). $\square$

Theorem 2.10. *For $n \geq 0$, $p, q \in \mathbb{R}$, the following formula for generalized Apostol type Frobenius-Euler-Hermite polynomials holds true:*

$$_H E_n^{(\alpha)}(px, qy; u, a, b, c; \lambda)$$

$$= n! \sum_{k=0}^{n} \sum_{j=0}^{[\frac{k}{2}]} {}_H E_{n-k}^{(\alpha)}(x, y; u, a, b, c; \lambda)((p-1)x \ln c)^{k-2j}((q-1)y \ln c)^j$$

$$\times \frac{t^n}{(n-k-2j)!j!k!}. \qquad (2.19)$$

Proof. Rewriting the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(px, qy; u, a, b, c; \lambda)\frac{t^n}{n!} = \left(\frac{a^t - u}{\lambda b^t - u}\right)^{\alpha} c^{xt+yt^2} c^{(p-1)xt} c^{(q-1)yt^2}$$

$$= \left(\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)x \ln c)^k \frac{t^k}{k!}\right)$$

$$\times \left(\sum_{j=0}^{\infty}((q-1)y \ln c)^j \frac{t^{2j}}{j!}\right)$$

$$= \left(\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)\frac{t^n}{n!}\right)$$

$$\times \left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x \ln c)^k((q-1)y \ln c)^j \frac{t^{k+2j}}{n!k!j!}\right).$$

Replacing k by $k-2j$ in above equation, we have

$$L.H.S. = \left(\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)\frac{t^n}{n!}\right)$$

$$\times \left(\sum_{k=2j}^{\infty}((p-1)x \ln c)^{k-2j}((q-1)y \ln c)^j \frac{t^k}{(k-2j)!j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=2j}^{\infty} {}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)((p-1)x \ln c)^{k-2j}$$

$$((q-1)y \ln c)^j \frac{t^{n+k}}{(k-2j)!j!n!}.$$

Again replacing n by $n - k$ in above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{[\frac{k}{2}]} {}_H E_{n-k}^{(\alpha)}(x, y; u, a, b, c; \lambda)((p - 1)x \ln c)^{k-2j}$$

$$\times ((q - 1)y \ln c)^j \frac{t^n}{(n - k - 2j)! j! k!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result (2.19). $\qquad\square$

Remark 2.11. *By taking $c = e$ in (2.19), we get the following corollary.*

Corollary 2.12. *For $p, q \in \mathbb{R}$, $x, y \in \mathbb{C}$ and $n \geq 0$, we have*

$${}_H E_n^{(\alpha)}(px, qy; u, a, b; \lambda)$$

$$= n! \sum_{k=0}^{n} \sum_{j=0}^{[\frac{k}{2}]} {}_H E_{n-k}^{(\alpha)}(x, y; u, a, b; \lambda)((p - 1)x)^{k-2j}$$

$$((q - 1)y)^j \frac{t^n}{(n - k - 2j)! j! k!}. \qquad (2.20)$$

Theorem 2.13. *For $n \geq 0$, $p, q \in \mathbb{R}$ and $x, y \in \mathbb{C}$. Then we have*

$${}_H E_n^{(\alpha)}(px, qy; u, a, b, c; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} {}_H E_{n-k}^{(\alpha)}(x, y; u, a, b, c; \lambda) H_k((p - 1)x, (q - 1)y; c). \quad (2.21)$$

3. Summation formulae for generalized Apostol type Frobenius-Euler-Hermite polynomials

We give here implicit formulae for generalized Apostol type Frobenius-Euler-Hermite polynomials ${}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$. First, we begin with the following theorem.

Theorem 3.1. *The following implicit summation formulae for generalized Apostol type Frobenius-Euler-Hermite polynomials ${}_H E_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$ holds true:*

$${}_H E_{k+l}^{(\alpha)}(z, y; u, a, b, c; \lambda)$$

$$= \sum_{n,m=0}^{k,l} \binom{l}{m}\binom{k}{n} (\ln c)^{n+m} (z-x)^{n+m} {}_H E_{k+l-n-m}^{(\alpha)}(x,y;u,a,b,c;\lambda).$$

(3.1)

Proof. We replace t by $t+w$ and rewrite the generating function (2.1) as, (see [5, 10, 11])

$$\left(\frac{a^{t+w}-u}{\lambda b^{t+w}-u}\right)^{\alpha} c^{y(t+w)^2} = c^{-x(t+w)} \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}.$$

(3.2)

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$c^{(z-x)(t+w)} \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}$$

$$= \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(z,y;u,a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}.$$

(3.3)

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty} \frac{[(z-x)(t+w)]^N}{N!} \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}$$

$$= \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(z,y;u,a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!},$$

(3.4)

which on using formula [9]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!},$$

(3.5)

in the left hand side becomes

$$\sum_{n,m=0}^{\infty} \frac{(\ln c)^{m+n}(z-x)^{n+m}t^n w^m}{n!m!} \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}$$

$$= \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(z,y;u,a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}.$$

(3.6)

Now replacing k by $k - n$, and l by $l - m$ in the left hand side of (3.6), we get

$$\sum_{k,l=0}^{\infty} \sum_{n,m=0}^{k,l} \frac{(\ln c)^{m+n}(z-x)^{n+m}}{n!m!}$$

$$_H E_{k+l-n-m}^{(\alpha)}(x,y;u,a,b,c;\lambda) \frac{t^k}{(k-n)!} \frac{w^l}{(l-m)!}$$

$$= \sum_{k,l=0}^{\infty} {}_H E_{k+l}^{(\alpha)}(z,y;u,a,b,c;\lambda) \frac{t^k}{k!} \frac{w^l}{l!}. \qquad (3.7)$$

Finally on equating the coefficients of the like powers of t and w in the above equation, we get the required result. $\qquad \square$

Remark 3.2. *By taking $l = 0$ in Eq. (3.1), we immediately deduce the following result.*

Corollary 3.3. *The following implicit summation formula for Apostol type Frobenius-Euler-Hermite polynomials $_H \mathbb{H}_n^{(\alpha)}(x,y;u,a,b,c;\lambda)$ holds true:*

$$_H E_{k+l}^{(\alpha)}(z,y;u,a,b,c;\lambda)$$

$$= \sum_{n=0}^{k} \binom{k}{n} (\ln c)^n (z-x)^n {}_H E_{k-n}^{(\alpha)}(x,y;u,a,b,c;\lambda). \qquad (3.8)$$

Remark 3.4. *On replacing z by $z + x$ and setting $y = 0$ in Theorem (3.1), we get the following result involving Apostol type Frobenius-Euler-Hermite polynomials of one variable*

$$_H E_{k+l}^{(\alpha)}(z+x;u,a,b,c;\lambda)$$

$$= \sum_{n,m=0}^{k,l} \binom{l}{m} \binom{k}{n} (\ln c)^{n+m} z^{n+m} {}_H E_{k+l-n-m}^{(\alpha)}(x;u,a,b,c;\lambda),$$

$$(3.9)$$

whereas by setting $z = 0$ in Theorem 3.1, we get another result involving Apostol type Frobenius-Euler-Hermite polynomials of one and two variables

$$_H E_{k+l}^{(\alpha)}(y;u,a,b,c;\lambda)$$

$$= \sum_{n,m=0}^{k,l} \binom{l}{m} \binom{k}{n} (\ln c)^{n+m} (-x)^{n+m} {}_H E_{k+l-n-m}^{(\alpha)}(x,y;u,a,b,c;\lambda).$$

$$(3.10)$$

Theorem 3.5. *Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, we have*

$$_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) = \sum_{k=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \binom{n-2j}{k} (\ln c)^{n-k-j}$$

$$\times \mathbb{H}_k^{(\alpha)}(u; a, b; \lambda) x^{n-k-2j} y^j. \tag{3.11}$$

Proof. Applying the definition (2.1) to the term $\left(\frac{a^t - u}{\lambda b^t - u}\right)^\alpha$ and expanding the exponential function $c^{xt + yt^2}$ at $t = 0$ yields

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^\alpha c^{xt + yt^2} = \left(\sum_{k=0}^\infty \mathbb{H}_k^{(\alpha)}(u; a, b; \lambda) \frac{t^k}{k!}\right) \left(\sum_{n=0}^\infty x^n (\ln c)^n \frac{t^n}{n!}\right)$$

$$\times \left(\sum_{j=0}^\infty y^j (\ln c)^j \frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^\infty \left(\sum_{k=0}^n \binom{n}{k} (\ln c)^{n-k} \mathbb{H}_k^{(\alpha)}(u; a, b; \lambda) x^{n-k}\right) \frac{t^n}{n!} \left(\sum_{j=0}^\infty y^j (\ln c)^j \frac{t^{2j}}{j!}\right).$$

Replacing n by $n - 2j$, we have

$$\sum_{n=0}^\infty {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}$$

$$= \sum_{n=0}^\infty \left(\sum_{k=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \binom{n-2j}{k} (\ln c)^{n-k-j} \mathbb{H}_k^{(\alpha)}(u; a, b; \lambda) x^{n-k-2j} y^j\right) \frac{t^n}{n!}. \tag{3.12}$$

Combining (3.12) and (2.1) and equating their coefficients of t^n produce the formula (3.11). $\square$

Theorem 3.6. *Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, we have*

$$_H E_n^{(\alpha)}(x + 1, y; u; a, b, c; \lambda) = \sum_{j=0}^{[\frac{n}{2}]} \sum_{k=0}^{n-2j} \binom{n-2j}{k} y^j (\ln c)^{n-k-j}$$

$$\times \mathbb{H}_k^{(\alpha)}(x; u; a, b, c; \lambda). \tag{3.13}$$

Proof. By the Definition of generalized Apostol type Frobenius-Euler-Hermite polynomials, we have

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^\alpha c^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x+1, y; u; a, b, c; \lambda)\frac{t^n}{n!}, \quad (3.14)$$

which can be written as

$$\left(\sum_{k=0}^{\infty} \mathbb{H}_k^{(\alpha)}(x; u; a, b, c; \lambda)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty} (\ln c)^n \frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} y^j (\ln c)^j \frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k}\mathbb{H}_k^{(\alpha)}(x; u; a, b, c; \lambda)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty} y^j (\ln c)^j \frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} y^j (\ln c)^{n-k+j}\mathbb{H}_k^{(\alpha)}(x; u; a, b, c; \lambda)\frac{t^{n+2j}}{n!j!}.$$

Replacing n by $n - 2j$, we have

$$\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x+1, y; u; a, b, c; \lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{[\frac{n}{2}]}\sum_{k=0}^{n-2j} \binom{n-2j}{k} y^j (\ln c)^{n-k-j}$$

$$\times \mathbb{H}_k^{(\alpha)}(x; u; a, b, c; \lambda)\frac{t^n}{n!}. \quad (3.15)$$

Combining (3.14) and (3.15) and equating their coefficients of t^n leads to formula (3.13). $\square$

Theorem 3.7. *Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$,*

$$_H E_n^{(\alpha+1)}(x, y; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathbb{H}_{n-m}(u; a, b; \lambda)$$

$$\times {}_H E_m^{(\alpha)}(x, y; u; a, b, c; \lambda). \quad (3.16)$$

Proof. By the Definition of generalized Apostol type Frobenius-Euler-Hermite polynomials, we have

$$\frac{a^t - u}{\lambda b^t - u}\left(\frac{a^t - u}{\lambda b^t - u}\right)^\alpha c^{xt+yt^2} = \frac{a^t - u}{\lambda b^t - u}\sum_{m=0}^{\infty} {}_H E_m^{(\alpha)}(x, y; u; a, b, c; \lambda)\frac{t^m}{m!}$$

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^{\alpha+1} c^{xt+yt^2} = \frac{a^t - u}{\lambda b^t - u}\sum_{m=0}^{\infty} {}_H E_m^{(\alpha)}(x, y; u; a, b, c; \lambda)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \mathbb{H}_n(u; a, b; \lambda) \frac{t^n}{n!} \sum_{m=0}^{\infty} {}_H E_m^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^m}{m!}.$$

Now replacing n by $n - m$ and equating the coefficients of t^n leads to formula (3.16). □

Theorem 3.8. *For arbitrary real or complex parameter α, the following implicit summation formula involving generalized Apostol type Frobenius-Euler-Hermite polynomials ${}_H\mathbb{H}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$ holds true:*

$$_H E_n^{(\alpha)}(x+1, y; u; a, b, c; \lambda) = \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} {}_H E_k^{(\alpha)}(x, y; u; a, b, c; \lambda).$$

$$(3.17).$$

Proof. By the definition of generalized Apostol type Frobenius-Euler-Hermite polynomials, we have

$$\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x+1, y; u; a, b, c; \lambda) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}$$

$$= \left(\frac{a^t - u}{\lambda b^t - u} \right)^{\alpha} c^{xt + yt^2} (c^t - 1)$$

$$= \left(\sum_{k=0}^{\infty} {}_H E_k^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^k}{k!} \right) \left(\sum_{n=0}^{\infty} (\ln c)^n \frac{t^n}{n!} \right)$$

$$- \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\ln c)^{n-k} {}_H E_k^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{(n-k)!}$$

$$- \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.17). □

Theorem 3.9. *For arbitrary real or complex parameter α, the following implicit summation formula involving generalized Apostol type Frobenius-Euler-Hermite polynomials ${}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$ holds true:*

$$\sum_{m=0}^{n} \binom{n}{m} (\ln ab)^m (\alpha)^m {}_H E_{n-m}^{(\alpha)}(-x, y; u; a, b, c; \lambda)$$

$$= (-1)^n {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \qquad (3.18)$$

$${}_H E_n^{(\alpha)}(\alpha - x, y; u; \frac{c}{b}, \frac{c}{a}, c; \lambda) = (-1)^n {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda). \quad (3.19)$$

Proof. We replace t by $-t$ in (2.1) and then subtract the result from (2.1) itself finding

$$c^{yt^2} \left[\left(\frac{a^t - u}{\lambda b^t - u} \right)^\alpha (c^{xt} - (ab)^{\alpha t} c^{-xt}) \right]$$

$$= \sum_{n=0}^{\infty} [1 - (-1)^n] {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}, \qquad (3.20)$$

which is equivalent to

$$\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!} - \left(\sum_{m=0}^{\infty} (\alpha)^m (\ln ab)^m \frac{t^m}{m!} \right)$$

$$\times \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(-x, y; u; a, b, c; \lambda) \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!} - \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} (\alpha)^m (\ln ab)^m \right)$$

$$\times {}_H E_{n-m}^{(\alpha)}(-x, y; u; a, b, c; \lambda) \frac{t^n}{(n-m)!}$$

$$= \sum_{n=0}^{\infty} [1 - (-1)^n] {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!},$$

and thus by equating coefficients of like powers of t^n, we get (3.18). In order to get (3.19), we write (3.20) in the form

$$c^{yt^2} \left[\left(\frac{a^t - u}{\lambda b^t - u} \right)^\alpha c^{xt} - \left(\frac{(\frac{c}{a})^t - u}{\lambda (\frac{c}{b})^t - u} \right)^\alpha (c^{(\alpha - x)t}) \right]$$

$$= \sum_{n=0}^{\infty} [1 - (-1)^n] {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}, \qquad (3.21)$$

which is equivalent to

$$\sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(\alpha - x, y; \frac{c}{b}, \frac{c}{a}, c; \lambda) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} [1 - (-1)^n] {}_H E_n^{(\alpha)}(x, y; u; a, b, c; \lambda) \frac{t^n}{n!}. \qquad (3.22)$$

Now comparing the coefficients of t^n in above equation, we get the result (3.19). □

4. Identities for Apostol type Frobenius-Euler-Hermite polynomials

In this section, we give general symmetry identities for the Apostol type Frobenius-Euler polynomials $\mathbb{H}_n^{(\alpha)}(u; a, b, c; \lambda)$ and generalized Apostol type Frobenius-Euler-Hermite polynomials $_HE_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$ by applying the generating functions (1.2) and (2.1).

Theorem 4.1. *Let $a, b, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$. Then the following identity holds true:*

$$\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_HE_{n-k}^{(\alpha)}(bx, b^2y; A, B, c; \lambda)_HE_k^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_HE_{n-k}^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda)$$

$$\times {}_HE_k^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda). \tag{4.1}$$

Proof. Start with

$$A(t) = \left(\left(\frac{A^{at} - u}{\lambda B^{at} - u}\right)\left(\frac{A^{bt} - u}{\lambda B^{bt} - u}\right)\right)^{\alpha} c^{abxt + a^2b^2yt^2}. \tag{4.2}$$

Then the expression for $A(t)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain

$$A(t) = \sum_{n=0}^{\infty} {}_HE_n^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda)\frac{(at)^n}{n!}$$

$$\times \sum_{k=0}^{\infty} {}_HE_k^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda)\frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_HE_{n-k}^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda)\right.$$

$$\left. {}_HE_k^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda)\right) \frac{t^n}{n!}.$$

On the similar lines we can show that

$$A(t) = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(ax, a^2x; u; A, B, c; \lambda) \frac{(bt)^n}{n!}$$

$$\times \sum_{k=0}^{\infty} {}_H E_k^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda) \frac{(at)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_H E_{n-k}^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda) \right.$$

$$\left. {}_H E_k^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda) \right) \frac{t^n}{n!}.$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result (4.1). $\qquad \square$

Remark 4.2. *For $\alpha = 1$, the above result reduces to*

$$\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_H E_{n-k}(bx, b^2y; A, B, c; \lambda) {}_H E_k(ax, a^2y; u; A, B, c; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_H E_{n-k}(ax, a^2y; u; A, B, c; \lambda) {}_H E_k(bx, b^2y; u; A, B, c; \lambda).$$

(4.3)

Further by taking $c = e$ in Theorem 4.1, we immediately deduce the following result involving generalized Apostol type Frobenius-Euler-Hermite polynomials ${}_H E_n^{(\alpha)}(x, y; u; A, B, e; \lambda)$ for nonnegative integer n.

$$\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_H E_{n-k}^{(\alpha)}(bx, b^2y; A, B, e; \lambda) {}_H E_k^{(\alpha)}(ax, a^2y; u; A, B, e; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_H E_{n-k}^{(\alpha)}(ax, a^2y; u; A, B, e; \lambda)$$

$$\times {}_H E_k^{(\alpha)}(bx, b^2y; u; A, B, e; \lambda).$$

(4.4)

Remark 4.3. *By setting $b = 1$ in Theorem 4.1, we immediately following result*

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} {}_H E_{n-k}^{(\alpha)}(x, y; A, B, c; \lambda) {}_H E_k^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda)$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k {}_H E^{(\alpha)}_{n-k}(ax, a^2y; u; A, B, c; \lambda) {}_H E^{(\alpha)}_k(x, y; u; A, B, c; \lambda).$$

(4.5)

Theorem 4.4. *Let $a, b, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$. Then the following identity holds true:*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} a^{n-k} b^k {}_H E^{(\alpha)}_{n-k} \left(bx + \frac{b}{a}i + j, b^2z; u; A, B, c; \lambda \right)$$

$$\times \mathbb{H}^{(\alpha)}_k(ay; u; A, B, c; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} b^{n-k} a^k {}_H E^{(\alpha)}_{n-k} \left(ax + \frac{a}{b}i + j, a^2z; u; A, B, c; \lambda \right)$$

$$\times \mathbb{H}^{(\alpha)}_k(by; u; A, B, c; \lambda).$$

(4.6)

Proof. Let

$$B(t) = \left(\left(\frac{A^{at} - u}{\lambda B^{at} - u} \right) \left(\frac{A^{bt} - u}{\lambda B^{bt} - u} \right) \right)^\alpha \frac{1 + \lambda(-1)^{a+1} c^{abt}}{(\lambda c^{at} + 1)(\lambda c^{bt} + 1)} c^{ab(x+y)t + a^2b^2zt^2}$$

$$B(t) = \left(\frac{A^{at} - u}{\lambda B^{at} - u} \right)^\alpha c^{abxt + a^2b^2zt^2} \left(\frac{1 - \lambda(-c^{-bt})^a}{\lambda c^{bt} + 1} \right) \left(\frac{A^{bt} - u}{\lambda B^{bt} - u} \right)^\alpha$$

$$\times c^{abyt} \left(\frac{1 - \lambda(-c^{-at})^b}{\lambda c^{at} + 1} \right)$$

$$= \left(\frac{A^{at} - u}{\lambda B^{at} - u} \right)^\alpha c^{abxt + a^2b^2zt^2} \sum_{i=0}^{a-1} (-\lambda)^i c^{bti} \left(\frac{A^{bt} - u}{\lambda B^{bt} - u} \right)^\alpha c^{abyt} \sum_{j=0}^{b-1} (-\lambda)^j c^{atj}$$

$$= \left(\frac{A^{at} - u}{\lambda B^{at} - u} \right)^\alpha c^{a^2b^2zt^2} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} c^{(bx + \frac{b}{a}i + j)at}$$

$$\times \sum_{k=0}^{\infty} \mathbb{H}^{(\alpha)}_k(ay; u; A, B, c; \lambda) \frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} {}_H E^{(\alpha)}_n \left(bx + \frac{b}{a}i + j, b^2z; u; A, B, c; \lambda \right) \frac{(at)^n}{n!}$$

$$\times \sum_{k=0}^{\infty} \mathbb{H}^{\alpha}_k(ay; u; A, B, c; \lambda) \frac{(bt)^k}{(k)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} a^{n-k} b^k$$

$$\times {}_H E_{n-k}^{(\alpha)} \left(bx + \frac{b}{a} i + j, b^2 z; u; A, B, c; \lambda \right) \mathbb{H}_k^{(\alpha)}(ay; u; A, B, c; \lambda) \frac{t^n}{n!}.$$

$$(4.7)$$

On the other hand

$$B(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} b^{n-k} a^k$$

$$\times {}_H E_{n-k}^{(\alpha)} \left(ax + \frac{a}{b} i + j, a^2 z; u; A, B, c; \lambda \right) \mathbb{H}_k^{(\alpha)}(by; u; A, B, c; \lambda) \frac{t^n}{n!}. \quad (4.8)$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result (4.6).   □

5. Conclusion

In the previous sections we have touched on the problem of recognizing the algebraic structure underlying the generalized Apostol type Frobenius-Euler-Hermite polynomials as given by the definition (2.1). The analysis is aimed at accounting for the wealth of the properties exhibited by these polynomials within the context of the Frobenius-Euler numbers and polynomials which provides a unifying formalism where the theory of special functions can be framed in a natural way.

Some analogies with the theory of Frobenius-Euler numbers and polynomials can be recognized and usefully exploited to infer further properties of these polynomials and links with other special functions. Let us stress that the scheme suggested by the following properties of Frobenius-Euler numbers and polynomials $\theta_{n,\rho}(x)$ studied in details by Janson [4] can be applied to connect other special functions of relevance in mathematical physics, for instance, Laguerre and hypergeometric functions in place of Hermite polynomials considered in this paper.

Let us briefly comment on the following general polynomials $\theta_{n,\rho}(x)$ which were perhaps first introduced by Carlitz (see [2] and [4]) in the form

$$\frac{e^{\rho t}}{1 - xt} = \sum_{n=0}^{\infty} \frac{\theta_{n,\rho}(x)}{(1-t)^{n+1}} \frac{t^n}{n!}, \qquad (5.1)$$

or, equivalently

$$\sum_{n=0}^{\infty} \theta_{n,\rho}(x)\frac{t^n}{n!} = \frac{(1-x)e^{\rho t(1-x)}}{1-xe^{t(1-x)}}.$$

More generally, we can obtain [4]

$$\sum_{n=0}^{\infty} \frac{\theta_{n,1-\rho}(x)}{(x-1)^n}\frac{t^n}{n!} = e^{t\rho}\frac{(1-x)}{e^t - x}.$$

This gives an advantage that (5.1) (for all x, since we deal with polynomials) can be differentiated in ρ termwise for all ($\rho \in \mathbb{C}$), which for all $n \geq 1$, is

$$\frac{\partial}{\partial \rho}\frac{\theta_{n,\rho}(x)}{(1-x)^{n+1}} = \sum_{j=0}^{\infty} n(j+\rho)^{n-1}x^j = \frac{n\theta_{n-1,\rho}(x)}{(1-x)^n}. \qquad (5.2)$$

Furthermore in a classical case $\rho = 1$ and $x = i, i^2 = -1$, $\theta_{n,1}$ has a connection with the Bernoulli numbers B_{2n} [3]

$$\theta_{n,1}(i) = (-2i)^m\frac{(2^{n+1}-1)}{n+1}B_{n+1}, \; n = 2m + 1.$$

It is interesting to notice that (5.1) can be written as a *Rodrigues formula*

$$\theta_{n,\rho}(x) = (1-x)^{n+1}\left(\rho + x\frac{d}{dx}\right)^n \frac{1}{1-x}, \qquad (5.3)$$

which yields the recursion formula after expansion. Here ρ is a parameter that can be any complex number.

Many other combinatorial numbers and polynomials satisfy recursion and other formulas similar to (5.2) and (5.3), (see [16]) for a general version. This, however, will be the topic of future investigations.

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