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COUPLED FIXED POINT THEOREMS ON FLM ALGEBRAS

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Abstract. This paper considers coupled fixed point theorems on unital without of order semi-simple fundamental locally multiplicative topological algebras (abbreviated by FLM algebras)

1. Introduction

Ansari in [1] introduced the notion of fundamental topological spaces and algebras, and proved Cohen's factorization theorem for these algebras. A topological linear space $\mathcal{A}$ is said to be fundamental if there exists $b > 1$ such that for every sequence (x_n) of $\mathcal{A}$, the convergence of $b^n(x_n - x_{n-1})$ to zero in $\mathcal{A}$ implies that (x_n) is Cauchy. A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

A fundamental topological algebra is called to be locally multiplicative, if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V. The fundamental locally multiplicative topological algebras (abbreviated by FLM) introduced by Ansari in [2]. Some celebrated theorems in Banach algebras generalized to FLM algebras in [10], and authors investigated some fixed points theorems for holomorphic functions on these algebras (see Theorems 3.5, 3.6 and 3.7 of [10]).

Recall that a ring $\mathcal{A}$ is prime if for $a, b \in \mathcal{A}$, $a\mathcal{A}b = 0$ implies that either $a = 0$ or $b = 0$. It is clear when A is unital if $ab = 0$ then either $a = 0$ or $b = 0$.

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In [5], Bhaskar and Lakshmikantham introduced notions of a mixed monotone mapping and a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mapping and discussed the existence and uniqueness of solution for periodic boundary value problem and some new results around this notion are obtained in [6, 7, 8, 9].

Let $\mathcal{A}$ be a metric space and let $F : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ be a function. An element $(x, y) \in \mathcal{A} \times \mathcal{A}$ is said to be a coupled fixed point of the mapping F, if $F(x, y) = x$ and $F(y, x) = y$.

In this paper at first (Section 2) we obtain some basic results for FLM algebras, and next we consider coupled fixed point theorems on FLM algebras.

2. Some Generalizations

By $\Omega_{\mathcal{A}}$ we mean the set of all elements $a \in \mathcal{A}$ such that $\rho_{\mathcal{A}}(a) < 1$, where $\rho_{\mathcal{A}}(a)$ is the spectral radius of $a \in \mathcal{A}$. We denote the center of topological algebra $\mathcal{A}$, by $Z(\mathcal{A})$, such that

$$Z(\mathcal{A}) = \{a \in \mathcal{A} : ax = xa, \quad for\ all\ x \in \mathcal{A}\}.$$

We recall the following definition from [10]:

Definition 2.1. *Let $(\mathcal{A}, d)$ be a metrizable topological algebra. We say $\mathcal{A}$ is a sub-multiplicatively metrizable topological algebra if*

$$d(0, xy) \leq d(0, x)d(0, y), \text{ and } d(0, \lambda x) < |\lambda| d(0, x),$$

for each $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. For abbreviation we denote $d_{\mathcal{A}}(0, x)$ by $D_{\mathcal{A}}(x)$ for any $x \in \mathcal{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be metric spaces with meters $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$, respectively. Then $\mathcal{A} \times \mathcal{B}$ becomes a metric space with the following meter

$$d\big((a_1, b_1), (a_2, b_2)\big) = d_{\mathcal{A}}(a_1, a_2) + d_{\mathcal{B}}(b_1, b_2), \tag{2.1}$$

for every $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. When $\mathcal{A}$ and $\mathcal{B}$ are algebras, then by the usual point-wise definitions for addition, scalar multiplication and product, $\mathcal{A} \times \mathcal{B}$ becomes an algebra.

Proposition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be complete metrizable FLM algebras with submultiplicative meters $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$, respectively. Then $\mathcal{A} \times \mathcal{B}$ is a complete metrizable FLM algebra with submultiplicative meter d.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be FLM algebras with meters $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$, respectively. By the definition of FLM algebras, obviously, $\mathcal{A} \times \mathcal{B}$ is a complete metrizable FLM algebra with meter d (the meter that defined in (2.1)). For submultiplicativity, we have

$$\begin{aligned} d\big((0,0),(a_1a_2,b_1b_2)\big) &= d_{\mathcal{A}}(0,a_1a_2)+d_{\mathcal{B}}(0,b_1b_2) \\ &\leq d_{\mathcal{A}}(0,a_1)d_{\mathcal{A}}(0,a_2)+d_{\mathcal{B}}(0,b_1)d_{\mathcal{B}}(0,b_2) \\ &\leq d_{\mathcal{A}}(0,a_1)d_{\mathcal{A}}(0,a_2)+d_{\mathcal{A}}(0,a_1)d_{\mathcal{B}}(0,b_2) \\ &\quad +d_{\mathcal{A}}(0,a_2)d_{\mathcal{B}}(0,b_1)+d_{\mathcal{B}}(0,b_1)d_{\mathcal{B}}(0,b_2) \\ &= d\big((0,0),(a_1,b_1)\big)d\big((0,0),(a_2,b_2)\big), \end{aligned} \tag{2.2}$$

for every $a_1,a_2\in\mathcal{A}$ and $b_1,b_2\in\mathcal{B}$. Also,

$$\begin{aligned} d\big((0,0),(\lambda a,\lambda b)\big) &= d_{\mathcal{A}}(0,\lambda a)+d_{\mathcal{B}}(0,\lambda b) \\ &< |\lambda|d_{\mathcal{A}}(0,a)+|\lambda|d_{\mathcal{B}}(0,b)=|\lambda|(d_{\mathcal{A}}(0,a)+d_{\mathcal{B}}(0,b)) \\ &= |\lambda|(d\big((0,0),(a,b)\big)). \end{aligned} \tag{2.3}$$

Therefore (2.2) and (2.3), show that d is submultiplicative. □

Similar to Definition 2.1, we write $D_{\mathcal{A}\times\mathcal{B}}(a,b)$ as an abbreviation for $d\big((0,0),(a,b)\big)$.

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be complete metrizable FLM algebras with submultiplicative meters $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$, respectively. Then*

$$\rho(x,y)\leq\rho_{\mathcal{A}}(x)+\rho_{\mathcal{B}}(y),$$

for any element $(x,y)\in\mathcal{A}\times\mathcal{B}$.

Proof. For given $a\in\mathcal{A}$ and $b\in\mathcal{B}$, we have $\rho_{\mathcal{A}}(a)=\lim_{n\to\infty}D_{\mathcal{A}}(a^n)^{1/n}$ and $\rho_{\mathcal{B}}(b)=\lim_{n\to\infty}D_{\mathcal{B}}(b^n)^{1/n}$ ([10, Theorem 3.3]). Proposition 2.2, follows that $\mathcal{A}\times\mathcal{B}$ is a complete metrizable FLM algebra with submultiplicative meter d. Then, again Theorem 3.3 of [10], implies that

$$\begin{aligned} \rho(x,y) &= \lim_{n\to\infty} D_{\mathcal{A}\times\mathcal{B}}\big((x,y)^n\big)^{\frac{1}{n}}=\lim_{n\to\infty} D_{\mathcal{A}\times\mathcal{B}}\big((x^n,y^n)\big)^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\big(D_{\mathcal{A}}(x^n)+D_{\mathcal{B}}(y^n)\big)^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} D_{\mathcal{A}}(x^n)^{\frac{1}{n}}+\lim_{n\to\infty} D_{\mathcal{B}}(y^n)^{\frac{1}{n}} \\ &= \rho_{\mathcal{A}}(x)+\rho_{\mathcal{B}}(y), \end{aligned} \tag{2.4}$$

for every $x\in\mathcal{A}$ and $y\in\mathcal{B}$. □

Similar to $\Omega_{\mathcal{A}}$ and $Z(A)$, we define these sets for $\mathcal{A}\times\mathcal{A}$ as follows

$$\Omega_{\mathcal{A}\times\mathcal{A}}=\{(x,y)\in\mathcal{A}\times\mathcal{A}\ :\ \rho(x,y)<1\},$$

and

$$\begin{aligned} Z(\mathcal{A}\times\mathcal{A}) &= \{(x,y)\in\mathcal{A}\times\mathcal{A} : (x,y)(a,b)=(a,b)(x,y), \\ &\qquad\qquad\qquad\qquad \text{for every } a,b\in\mathcal{A}\} \\ &= \{(x,y)\in\mathcal{A}\times\mathcal{A} : (xa,yb)=(ax,by), \text{ for every } a,b\in\mathcal{A}\}. \end{aligned}$$

Clearly, if $(x,y)\in Z(\mathcal{A}\times\mathcal{A})$, then $x,y\in Z(\mathcal{A})$, and $Z(\mathcal{A})\subseteq Z(\mathcal{A}\times\mathcal{A})$. Also, if $(x,y)\in\Omega_{\mathcal{A}\times\mathcal{A}}$, then $(x,0),(0,y)$ are in $\Omega_{\mathcal{A}\times\mathcal{A}}$, and by Lemma 2.3 and it's proof, we have $x,y\in\Omega_{\mathcal{A}}$.

Let $E(\mathcal{A})$ be the set of all elements $x\in\mathcal{A}$ for which $E(x)=\sum_{n=1}^{\infty}\frac{x^n}{n!}$, can be defined. If $\mathcal{A}$ is a complete metrizable FLM algebra, then $E(\mathcal{A})=\mathcal{A}$ ([3, Theorem 5.4]). Therefore, in light of Theorem 5.4 of [3] and Proposition 2.2, we have the following Theorem.

Theorem 2.4. *Let $\mathcal{A}$ be a complete metrizable FLM algebra, then $E(\mathcal{A}\times\mathcal{A})=\mathcal{A}\times\mathcal{A}$.*

3. Coupled Fixed Point Theorems

In this section we consider some results about coupled fixed point Theorems on unital complete semi-simple metrizable FLM algebras.

Theorem 3.1. *Let $\mathcal{A}$ be a unital prime complete semi-simple metrizable FLM algebra with submultiplicative meter $d_{\mathcal{A}}$. If $F:\Omega_{\mathcal{A}\times\mathcal{A}}\subseteq\mathcal{A}\times\mathcal{A}\longrightarrow\Omega_{\mathcal{A}}$ is a holomorphic map that satisfies the conditions $F(0,0)=0$, $\frac{\partial F}{\partial x}(0,0)=id_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0)=0$, $\frac{\partial^2 F}{\partial x^2}(0,0)=0$, $\frac{\partial^2 F}{\partial y^2}(0,0)=0$, and $\frac{\partial^2 F}{\partial y\partial x}(0,0)=0$ then every $(a,b)\in\Omega_{\mathcal{A}\times\mathcal{A}}\cap Z(\mathcal{A}\times\mathcal{A})$ is a coupled fixed point for F.*

Proof. Fix $(a,b)\in\Omega_{\mathcal{A}\times\mathcal{A}}\cap Z(\mathcal{A}\times\mathcal{A})$, and consider the map $f:\mathbb{C}\times\mathbb{C}\longrightarrow\Omega_{\mathcal{A}}$ with $f(\lambda,\alpha)=F(\lambda a,\alpha b)$. Clearly, f is a holomorphic function on

$$\Big\{(\lambda,\alpha)\in\mathbb{C}\times\mathbb{C} : \\ \frac{|\beta|}{2}<\frac{1}{\rho(a,b)},\ |\beta|=\min\{|\lambda|,|\alpha|\},\rho_{\mathcal{A}}(a)<\frac{1}{|\lambda|},\rho_{\mathcal{A}}(b)<\frac{1}{|\alpha|}\Big\}.$$

Since $F(0,0)=0$, $\frac{\partial F}{\partial x}(0,0)=id_{\mathcal{A}}$ and $\frac{\partial F}{\partial y}(0,0)=0$, F has a Taylor expansion about 0:

$$F(x,y)=x+\sum_{j=3}^{\infty}\frac{1}{j!}\Big(\sum_{i=0}^{j}\binom{j}{i}x^iy^{j-i}\frac{\partial^j F}{\partial y^{j-i}\partial x^i}(0,0)\Big), \tag{3.1}$$

for every $(x,y) \in \Omega_{\mathcal{A}\times\mathcal{A}} \cap Z(\mathcal{A}\times\mathcal{A})$. Therefore

$$(3.2)\quad F(\lambda a, \alpha b) = \lambda a + \sum_{j=3}^{\infty} \frac{1}{j!}\Big(\sum_{i=0}^{j} \left(\begin{array}{c} j \\ i \end{array}\right) \lambda^i a^i \alpha^{j-i} b^{j-i} \frac{\partial^j F}{\partial y^{j-i}\partial x^i}(0,0)\Big).$$

We claim that

$$(3.3)\qquad \sum_{i=0}^{j} \left(\begin{array}{c} j \\ i \end{array}\right) \lambda^i a^i \alpha^{j-i} b^{j-i} \frac{\partial^j F}{\partial y^{j-i}\partial x^i}(0,0),$$

is zero for every $j \geq 3$. Assume towards a contradiction that there exists $j \geq 2$ such that (3.3) is non-zero. Let k be the integer that

$$(3.4)\qquad \left(\begin{array}{c} k \\ i \end{array}\right) \lambda^i a^i \alpha^{k-i} b^{k-i} \frac{\partial^k F}{\partial y^{k-i}\partial x^i}(0,0) \neq 0.$$

Suppose that q is an element of $\mathcal{A}$ that $\rho_{\mathcal{A}}(q) = 0$. Now; we consider three cases (1) $i = k$, (2) $1 \leq i < k$, and (3) $i = 0$.

Case (1): In this case we have $\lambda^k a^k \frac{\partial^k F}{\partial x^k}(0,0) \neq 0$. Let $n \geq 1$, by (3.2) and (3.4) we have

$$\begin{aligned} F(n^{\frac{1}{k}}\lambda a + n\lambda^k q, \alpha b) &= n^{\frac{1}{k}}\lambda a + n\lambda^k q + \frac{1}{k!}(n^{\frac{1}{k}}\lambda a + n\lambda^k q)^k \frac{\partial^k F}{\partial x^k}(0,0) \\ &= n^{\frac{1}{k}}\lambda a + n\lambda^k q + \frac{1}{k!}(n^k \lambda^{k^2} q^k + kn^{\frac{1}{k}}\lambda a n^{k-1}\lambda^{k(k-1)} q^{k-1} \\ &\quad + \cdots + n\lambda^k a^k)\frac{\partial^k F}{\partial x^k}(0,0) \\ &= n^{\frac{1}{k}}\lambda a + n\lambda^k(q + \frac{1}{k!}a^k \frac{\partial^k F}{\partial x^k}(0,0)) + P(\lambda)\frac{\partial^k F}{\partial x^k}(0,0). \qquad (3.5) \end{aligned}$$

In (3.5), by $P(\lambda)$, we mean the remain part of $(n^{\frac{1}{k}}\lambda a + n\lambda^k q)^k$. Since $a \in Z(\mathcal{A})$, $aq = qa$. Then Lemma 2.3 and Lemma 3.6 of [10] imply

$$\begin{aligned} \rho(n^{\frac{1}{k}}\lambda a + n\lambda^k q, \alpha b) &\leq \rho_{\mathcal{A}}(n^{\frac{1}{k}}\lambda a + n\lambda^k q) + \rho_{\mathcal{A}}(\alpha b) \\ &< n^{\frac{1}{k}}|\lambda|\rho_{\mathcal{A}}(a) + |\alpha|\rho_{\mathcal{A}}(b) \\ &< \kappa(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b)), \end{aligned}$$

where $\kappa = \max\{n^{\frac{1}{k}}|\lambda|, |\alpha|\}$. Now, we define a holomorphic function H from $\{\lambda \in \mathbb{C} \,:\, 0 < |\lambda| < \frac{1}{\rho(a,b)}\}$ into $\mathcal{A}$, as follows

$$H(\lambda) = \frac{F(n^{\frac{1}{k}}\lambda a + n\lambda^k q, \alpha b) - n^{\frac{1}{k}}\lambda a}{n\lambda^k}.$$

Then by (3.5) we conclude that $H(0) = q + \frac{1}{k!}a^k \frac{\partial^k F}{\partial x^k}(0,0)$. Vesentini's Theorem ([4, Theorem 3.4.7]), implies that $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\lambda \in \mathbb{C} \ : \ 0 < |\lambda| < \frac{1}{\rho(a,b)}\}$. As well as, by the maximum principle we can write $\rho_{\mathcal{A}}(H(0)) \leq \max_{|\lambda|=1} \rho_{\mathcal{A}}(H(\lambda))$. Then Lemma 3.6 of [10] implies that

$$\begin{aligned} \rho_{\mathcal{A}}(q + \frac{1}{k!} a^k \frac{\partial^k F}{\partial x^k}(0,0)) &\leq \max_{|\lambda|=1} \rho(H(\lambda)) < \frac{1}{nk!} \rho_{\mathcal{A}}(a^k) \rho_{\mathcal{A}}(\frac{\partial^k F}{\partial x^k}(0,0)) \\ &< \frac{1}{nk!|\lambda|^k} \rho_{\mathcal{A}}(\frac{\partial^k F}{\partial x^k}(0,0)). \end{aligned} \tag{3.6}$$

The above inequality hold for every $n \geq 1$. Therefore if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}(q + \frac{1}{k!} a^k \frac{\partial^k F}{\partial x^k}(0,0)) = 0,$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, Theorem 3.4 of [10], implies that $a^k \frac{\partial^k F}{\partial x^k}(0,0)$ is in radical of $\mathcal{A}$. Since $\mathcal{A}$ is semi-simple, $a^k \frac{\partial^k F}{\partial x^k}(0,0) = 0$. Since $a \in \Omega_{\mathcal{A}}$, $a^k \neq 0$. Since $\mathcal{A}$ is without of order, $\frac{\partial^k F}{\partial x^k}(0,0) = 0$, a contradiction. Thus our claim is true, and from (3.2), we conclude that $F(a,b) = a$. Similarly we have $F(b,a) = b$.

Case (2): Now, let $1 \leq i \leq k$. Again by (3.2) and (3.4) we have

$$\begin{aligned} F(\lambda a + n\lambda^i q, n^{\frac{1}{k-i}} \alpha b) &= \lambda a + n\lambda^i q + \frac{1}{k!}(\lambda a + n\lambda^i q)^i n \alpha^{k-i} b^{k-i} \frac{\partial^k F}{\partial y^{k-i} \partial x^i}(0,0) \\ &= \lambda a + n\lambda^i q + \frac{1}{k!}(n^{i+1} \lambda^{i^2} q^i \alpha^{k-i} b^{k-i} \\ &\quad + i\lambda a n^i \lambda^{i(i-1)} q^{i-1} \alpha^{k-i} b^{k-i} + \cdots \\ &\quad + n\lambda^i a^i \alpha^{k-i} b^{k-i}) \frac{\partial^k F}{\partial y^{k-i} \partial x^i}(0,0) \\ &= \lambda a + n\lambda^i (q + \frac{1}{k!} a^i \alpha^{k-i} b^{k-i} \frac{\partial^k F}{\partial y^{k-i} \partial x^i}(0,0)) \\ &\quad + P(\lambda, \alpha) \frac{\partial^k F}{\partial y^{k-i} \partial x^i}(0,0). \end{aligned} \tag{3.7}$$

Again by Lemma 2.3 and Lemma 3.6 of [10] we have

$$\begin{aligned} \rho(\lambda a + n\lambda^k q, n^{\frac{1}{k-i}} \alpha b) &\leq \rho_{\mathcal{A}}(\lambda a + n\lambda^k q) + \rho_{\mathcal{A}}(n^{\frac{1}{k-i}} \alpha b) \\ &< |\lambda| \rho_{\mathcal{A}}(a) + n^{\frac{1}{k-i}} |\alpha| \rho_{\mathcal{A}}(b) \\ &< \kappa(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b)), \end{aligned}$$

where $\kappa = \max\{|\lambda|, n^{\frac{1}{k-i}} |\alpha|\}$. Now, we define a holomorphic function H from $\{\eta \in \mathbb{C} \ : \ \kappa < \frac{1}{\rho(a,b)}, \ \kappa = |\eta| = \max\{|\lambda|, n^{\frac{1}{k-i}} |\alpha|\}\}$ into $\mathcal{A}$, as follows

$$H(\lambda) = \frac{F(\lambda a + n\lambda^i q, n^{\frac{1}{k-i}}\alpha b) - \lambda a}{n\lambda^i}.$$

Then (3.7) follows that $H(0) = q + \frac{1}{k!}a^i\alpha^{k-i}b^{k-i}\frac{\partial^k F}{\partial y^{k-i}\partial x^i}(0,0)$. Then $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\eta \in \mathbb{C} \ : \ \kappa < \frac{1}{\rho(a,b)}, \ \kappa = |\eta| = \max\{|\lambda|, n^{\frac{1}{k-i}}|\alpha|\}\}$. As well as, Lemma 3.6 of [10] implies that

$$\begin{aligned} \rho_{\mathcal{A}}(q + \frac{1}{k!}a^i\alpha^{k-i}b^{k-i}\frac{\partial^k F}{\partial y^{k-i}\partial x^i}(0,0)) &\leq \max_{|\lambda|=1}\rho(H(\lambda)) \\ &< \frac{|\alpha|^{k-i}}{nk!}\rho_{\mathcal{A}}(a^k)\rho_{\mathcal{A}}(b^{k-i})\rho_{\mathcal{A}}(\frac{\partial^k F}{\partial x^k}(0,0)) \\ &< \frac{1}{nk!|\lambda|^k}\rho_{\mathcal{A}}(\frac{\partial^k F}{\partial x^k}(0,0)). \end{aligned} \tag{3.8}$$

The above inequality hold for every $n \geq 1$. Therefore if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}(q + \frac{1}{k!}a^i\alpha^{k-i}b^{k-i}\frac{\partial^k F}{\partial y^{k-i}\partial x^i}(0,0)) = 0,$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, Theorem 3.4 of [10], implies that $a^k b^{k-i}\frac{\partial^k F}{\partial x^k}(0,0)$ is in radical of $\mathcal{A}$. Since $\mathcal{A}$ is semi-simple, $a^k b^{k-i}\frac{\partial^k F}{\partial x^k}(0,0) = 0$. Since $a, b \in \Omega_{\mathcal{A}}$, $a^k \neq 0$ and $b^{k-i} \neq 0$. Again by using of that $\mathcal{A}$ is prime, we conclude that $\frac{\partial^k F}{\partial x^k}(0,0) = 0$, a contradiction. Thus our claim is true, and (3.2) implies that $F(a,b) = a$. Similarly we have $F(b,a) = b$.

Case (3): Let $i = 0$. Then we have $\alpha^k b^k \frac{\partial^k F}{\partial y^k}(0,0) \neq 0$. Similar to the previous two cases, we have

$$\begin{aligned} F(\lambda a + n\alpha^k q, n^{\frac{1}{k}}\alpha b) &= \lambda a + n\alpha^k q + \frac{1}{k!}n\alpha^k b^k \frac{\partial^k F}{\partial y^k}(0,0) \\ &= \lambda a + n\alpha^k(q + \frac{1}{k!}b^k\frac{\partial^k F}{\partial y^k}(0,0)). \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned} \rho(\lambda a + n\alpha^k q, n^{\frac{1}{k}}\alpha b) &\leq \rho_{\mathcal{A}}(\lambda a + n\alpha^k q) + \rho_{\mathcal{A}}(n^{\frac{1}{k}}\alpha b) \\ &< |\lambda|\rho_{\mathcal{A}}(a) + n^{\frac{1}{k}}|\alpha|\rho_{\mathcal{A}}(b) \\ &< \kappa(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b)), \end{aligned}$$

where $\kappa = \max\{|\lambda|, n^{\frac{1}{k}}|\alpha|\}$. Now, we define a holomorphic function H from $\{\eta \in \mathbb{C} : \kappa < \frac{1}{\rho(a,b)},\ \kappa = |\eta| = \max\{|\lambda|, n^{\frac{1}{k}}|\alpha|\}\}$ into $\mathcal{A}$, as follows

$$H(\lambda) = \frac{F(\lambda a + n\alpha^k q, n^{\frac{1}{k}}\alpha b) - \lambda a}{n\alpha^k}.$$

Then (3.9) follows that $H(0) = q + \frac{1}{k!}b^k \frac{\partial^k F}{\partial y^k}(0,0)$. Then $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\eta \in \mathbb{C} : \kappa < \frac{1}{\rho(a,b)},\ \kappa = |\eta| = \max\{|\lambda|, n^{\frac{1}{k}}|\alpha|\}\}$, and

$$\begin{aligned} \rho_{\mathcal{A}}(q + \frac{1}{k!}b^k \frac{\partial^k F}{\partial y^k}(0,0)) &\leq \max_{|\lambda|=1} \rho(H(\lambda)) \\ &< \frac{1}{nk!}\rho_{\mathcal{A}}(b^k)\rho_{\mathcal{A}}(\frac{\partial^k F}{\partial y^k}(0,0)) \\ &< \frac{1}{nk!|\alpha|^k}\rho_{\mathcal{A}}(\frac{\partial^k F}{\partial y^k}(0,0)). \end{aligned} \tag{3.10}$$

Therefore if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}(q + \frac{1}{k!}b^k \frac{\partial^k F}{\partial y^k}(0,0)) = 0,$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, $b^k \frac{\partial^k F}{\partial y^k}(0,0)$ is in radical of $\mathcal{A}$. Therefore $b^k \frac{\partial^k F}{\partial y^k}(0,0) = 0$. Since $b \in \Omega_{\mathcal{A}}$, so $b^k \neq 0$, that $\frac{\partial^k F}{\partial x^k}(0,0) = 0$, a contradiction. Thus (3.2) implies that $F(a,b) = a$ and similarly $F(b,a) = b$.

By gathering the above three cases, we conclude (a,b) is a coupled fixed point for F, and since (a,b) was arbitrary, every point of $\Omega_{\mathcal{A}\times\mathcal{A}} \cap Z(\mathcal{A}\times\mathcal{A})$ is coupled fixed point for F. $\square$

Example 3.2. *Let $X = \mathbb{R}$ be the space of real numbers and let $F : X \times X \longrightarrow X$ be a function defined by $F(x,y) = x$ that satisfies on conditions of Theorem 3.1.*

Example 3.3. *Let X be a unital without of order complete semi-simple Banach algebra and let $F : X \times X \longrightarrow X$ be a function defined by $F(x,y) = e^{y^2}xe^{-y^2}$ that satisfies on conditions of Theorem 3.1. For example let $X = M(G)$ the measure space on a locally compact Hausdorff space G. Another algebra that we can choose it is $\ell^1(G)$, where G is a locally compact discrete group.*

In the following Theorem we characterize coupled fixed points of holomorphic functions on FlM algebras as follows.

Theorem 3.4. *Let $\mathcal{A}$ be a unital prime complete semi-simple metrizable FLM algebra. For given $(a,b) \in \Omega_{\mathcal{A}\times\mathcal{A}} \backslash Z(\mathcal{A}\times\mathcal{A})$, there is a holomorphic map $F : \Omega_{\mathcal{A}\times\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ satisfies the conditions $F(0,0) = 0$, $\frac{\partial F}{\partial x}(0,0) = id_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0) = 0$, $\frac{\partial^2 F}{\partial x^2}(0,0) = 0$, $\frac{\partial^2 F}{\partial y^2}(0,0) = 0$, and $\frac{\partial^2 F}{\partial y \partial x}(0,0) = 0$ such that $F(a,b) \neq a$ and $F(b,a) \neq b$.*

Proof. Let $(a,b) \in \Omega_{\mathcal{A}\times\mathcal{A}} \backslash Z(\mathcal{A}\times\mathcal{A})$. Then there exist $(u,u) \in \mathcal{A}\times\mathcal{A}$, such that

$$(ua, ub) \neq (au, bu).$$

Let $D_{\mathcal{A}\times\mathcal{A}}(u,u) < 1$, then $D_{\mathcal{A}}(u) < 1$. Define $U := \log(e-u)$, then

$$e^{-U} a e^{U} \neq a, \text{ and } e^{-U} b e^{U} \neq b.$$

Now define $F : \Omega_{\mathcal{A}\times\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ as follows

$$F(x,y) = e^{-\frac{x^2 U}{a^2}} x e^{\frac{y^2 U}{b^2}} \tag{3.11}$$

for every (x,y) in $\Omega_{\mathcal{A}\times\mathcal{A}}$. Clearly F is a holomorphic function, $F(0,0) = 0$, $\frac{\partial F}{\partial x}(0,0) = id_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0) = 0$, and $\frac{\partial^2 F}{\partial x^2}(0,0) = 0$, $\frac{\partial^2 F}{\partial y^2}(0,0) = 0$, and $\frac{\partial^2 F}{\partial y \partial x}(0,0) = 0$, but $F(a,b) \neq a$ and $F(b,a) \neq b$. $\square$

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