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ON COFINITELY CLOSED WEAK δ-SUPPLEMENTED MODULES

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Abstract. A module M is called cofinitely closed weak δ-supplemented (briefly δ-ccws-module) if for any cofinite closed submodule N of M has a weak δ-supplement in M. In this paper we investigate the basic properties of δ-ccws modules. In the light of this study, we can list the main facts obtained as following: (1) Any cofinite closed direct summand of a δ-ccws module is also a δ-ccws module; (2) Let R be a left δ-V-ring. Then R is a δ-ccws module iff R is a ccws-module iff R is extending; (3) Any nonsingular homomorphic image of a δ-ccws-module is also a δ-ccws-module; (4) We characterize nonsingular δ-V-rings in which all nonsingular modules are δ-ccws.

1. Introduction

Throughout the paper, we assume that R is an associative ring with identity and all modules are unitary left R-modules, unless otherwise stated. A *(proper) submodule* K of M is denoted by $(K < M)$ $K \leq M$. A submodule K of M is said to be cofinite if $\frac{M}{K}$ is finitely generated as in [1]. The notation $K \ll M$ means that K is a *small submodule* of M, that is, M is itself the only submodule satisfying $K + T = M$ for a submodule T of M, i.e., for any proper submodule T of M, $K + T \neq M$. Dually, K is called an *essential submodule* of M if the intersection of K with any nonzero submodule of M is different from zero and denoted by $K \trianglelefteq M$ [13]. A submodule K of M is called *closed* in M if K has no proper essential extension in M, that is, if for a submodule L of M such that $K \trianglelefteq L$ then $K = L$, denoted by $K \leq_c M$. It is well known that $K \leq_c L$ and $L \leq_c M$ then $K \leq_c M$. If any submodule K of M is essential in a direct summand of M then M is called an *extending*

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module. Semisimple modules, injective modules and uniform modules are examples of extending modules [3]. If K is both closed and cofinite in M then we use the notation $K \leq_{cc} M$.

The *singular submodule* of a module M is denoted by $Z(M)$ containing the elements of M whose annihilators are essential in R. A module M is called singular (respectively, nonsingular) if $Z(M) = M$ (respectively, $Z(M) = 0$) [4]. Zhou introduced the concept of δ-small submodules as a generalization of small submodules in [15]. A submodule K of M is called *δ-small* in M and indicated by $K \ll_{\delta} M$ if $M \neq K + N$ for every proper submodule N of M with $\frac{M}{N}$ singular. Accordingly we denote by $\delta(M)$ the sum of all δ-small submodules of M. A submodule L of M is said to be a *δ-supplement* of K in M if $K + L = M$ and $K \cap L \ll_{\delta} L$ [6]. It is natural to say that a module M is *δ-supplemented* if for any submodule of M has a δ-supplement in M. It is clear that every supplemented module is also δ-supplemented but the converse is not true in general. A module M is *weakly δ-supplemented* if any submodule K of M has a *weak δ-supplement* in M, i.e., there exists a submodule L of M provided that $K + L = M$ and $K \cap L \ll_{\delta} M$ [9]. Al-Takhman introduced *cofinitely δ-supplemented modules* and he also introduced *cofinitely $\oplus$-δ-supplemented modules* in [2] as a generalization of δ-supplemented modules. A module M is called *cofinitely δ-supplemented ($\oplus$-cofinitelyδ-supplemented)* if every cofinite submodule K of M has a δ-supplement (that is a direct summand of M) in M. Following, *cofinitely weak δ-supplemented modules* (briefly δ-cws modules) are introduced in [5] and [8] such that every cofinite submodule of M has a weak δ-supplement in M. In [9] Talebi and Hamzekolaei introduced *closed weak δ-supplemented modules.* A module M is called *closed weak δ-supplemented module if* every closed submodule of M has a weak δ-supplement in M.

Extending modules are of an important role in module theory as a generalization of injective modules. It is possible to say that extending property is preserved on direct summands for a module but this idea can not be true for submodules or homomorphic images unless suitable conditions satisfied. Especially in recent years much work has been done to determine that the necessary and sufficient conditions to verify that the extending property is preserved under special conditions. In the light of these studies to reach to new generalizations of extending modules such as ccws- modules and closed weak δ-supplemented modules introduced in [12] and [9] respectively, are inevitable.

In this paper we replace the condition of ccws-modules whose cofinite closed submodules has a weak supplement by the condition as δ-ccws modules whose cofinite closed submodule has a weak δ-supplement as a generalization of both extending modules and closed weak δ-supplemented modules. And we investigate the preserved properties of these δ-ccws-modules. We show that every cofinite direct summand of a δ-ccws-module is δ-ccws. An example is presented for a module that is δ-ccws but not cofinitely weak δ-supplemented. We investigate the finite (direct) sums and nonsingular homomorphic image of δ-ccws-modules. Moreover a characterization is obtained for a δ-V-ring in aspect of being a δ-ccws-module as an R-module. The relations between δ-ccws-modules and other types of δ-supplemented modules are proved under special conditions for cofinitely refinable modules.

2. Cofinitely Closed Weak δ-supplemented Modules

In [9] the authors defined closed weak δ-supplemented modules as a generalization of closed weak supplemented modules introduced by [14]. Now we give a new generalization for closed weak δ-supplemented modules and present various properties of them.

Definition 2.1. *A module M is called cofinitely closed weak δ-supplemented (briefly, δ-ccws-module) if any cofinite closed submodule of M has a weak δ-supplement in M. That is for every $N \leq_{cc} M$, there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll_\delta M$.*

By the help of this definition, it is clear that every extending module is a δ-ccws-module. By taking into consideration of known definitions, it is natural to compose following diagram between related modules.

cof. δ-supp. module $\Longrightarrow$ cof. weak δ-supp. module $\Longrightarrow$ δ-ccws-module

Since the only closed submodules of $\mathbb{Z}$, 0 and $\mathbb{Z}$, $\mathbb{Z}$ is extending as a $\mathbb{Z}$-module and so it is a δ-ccws-module which is not (cofinitely) weak δ-supplemented.

Proposition 2.2. *Let M be a δ-ccws-module. Then any cofinite direct summand of M is a δ-ccws-module.*

Proof. Let N be any cofinite direct summand of M and $K \leq_{cc} N$. Since N is closed in M, by using the isomorphism $\frac{\frac{M}{K}}{\frac{N}{K}} \cong \frac{M}{N}$,we have that $K \leq_{cc} M$. Then there exists a submodule L of M such that $M = K + L$

and $K \cap L \ll_\delta M$. Thus $N = (K + L) \cap N = K + (L \cap N)$ by the modular law. Since N is a direct summand of M and $K \cap L \leq N$, we have $K \cap (L \cap N) = K \cap L \ll_\delta N$ by [9, Lemma 1.2]. Thus N is a δ-ccws-module. □

Proposition 2.3. *Let $M = M_1 \oplus M_2$ be a Noetherian distributive module. Then M is cofinitely closed weak δ-supplemented if and only if each M_i $(i = 1, 2)$ is cofinitely closed weak δ-supplemented.*

Proof. $(\Longrightarrow)$: It is clear by Proposition 2.1.

$(\Longleftarrow)$: Let L be any cofinite closed submodule of M and M_i be a δ-ccws-module for each i $(i = 1, 2)$. Firstly, we will show that $M_i \cap L$ cofinite closed in submodule M_i for each i $(i = 1, 2)$. Assume that $M_1 \cap L \trianglelefteq K \trianglelefteq M_1$. We have that $L = (M_1 \cap L) \oplus (M_2 \cap L) \trianglelefteq K \oplus (M_2 \cap L)$, as M is distributive. Therefore $L = (M_1 \cap L) \oplus (M_2 \cap L) = K \oplus (M_2 \cap L)$, since L is closed in M. Hence, $K = M_1 \cap L$ is obtained and so $M_1 \cap L$ is closed in M_1 even $M_1 \cap L \leq_{cc} M_1$ as M is Noetherian. By the hypothesis, there exists a weak δ-supplement K_i of $L \cap M_i$ in M_i provided that $M_i = K_i + (L \cap M_i)$ and $L \cap K_i \ll_\delta M_i$ for each $i = 1, 2$. Following this we have $M = M_1 \oplus M_2 = K_1 \oplus K_2 \oplus L$ and $L \cap (K_1 \oplus K_2) = (L \cap K_1) \oplus (L \cap K_2) \ll_\delta M_1 \oplus M_2 = M$. □

Generally, it is wrong to think that a finite direct sum of δ-ccws-modules is also a δ-ccws-module. After a useful lemma given in [9, Lemma 2.3], we will prove a proposition to verify this idea under specific conditions.

Lemma 2.4. *Let N and L be cofinite submodules of M such that $N + L$ has a weak δ-supplement X in M and $N \cap (X + L)$ has a weak δ-supplement Y in N. Then $X + Y$ is a weak δ-supplement of L in M.*

Proposition 2.5. *Let $M = M_1 \oplus M_2$ for δ-ccws-modules M_1 and M_2. Suppose that $M_i \cap (M_j + L) \leq_{cc} M_i$ and $M_j \cap (L + K) \leq_{cc} M_j$, where K is a weak δ-supplement of $M_i \cap (M_j + L)$ in M_i, $i \neq j$, for any $L \leq_c M$. Then M is a δ-ccws-module.*

Proof. Let L be any cofinite closed submodule of M, then $M = M_1 + M_2 + L$ has a weak δ-supplement 0 in M. Since $M_1 \cap (M_2 + L)$ is a cofinite closed submodule in M_1 and M_1 is a δ-ccws-module, then there exists a submodule K of M_1 provided that $M_1 = M_1 \cap (M_2 + L) + K$ and $M_1 \cap (M_2 + L) \cap K = K \cap (M_2 + L) \ll_\delta M_1$. By Lemma 2.3, K is a weak δ-supplement of $M_2 + L$ in M. By the assumption $M_2 \cap (K + L)$ has a weak δ-supplement of L in M. So, $K + L$ is a weak δ-supplement of L in M. Hence, M is a δ-ccws-module. □

Proposition 2.6. *Let $M = M_1 + M_2$, where M_1 is a δ-ccws-module and M_2 be any module. Suppose that for any $N \leq_{cc} M$, $N \cap M_1 \leq_{cc} M_1$. Then M is a δ-ccws-module if and only if every $N \leq_{cc} M$ with $M_2 \nleq N$ has a weak δ-supplement.*

Proof. $(\Longrightarrow)$: It is obvious.

$(\Longleftarrow)$: Let $N \leq_{cc} M$ with $M_2 \leq N$. Then $M = M_1 + M_2 = M_1 + N$ and M_1+N has a trivial weak δ-supplement 0 in M. Since $N \cap M_1 \leq_{cc} M_1$ and M_1 is a δ-ccws-module, $N \cap M_1$ has a weak δ-supplement X in M_1. By Lemma 2.3, X is a weak δ-supplement of N in M. □

Recall that a module M is called *singular (nonsingular)* provided that $Z(M) = M$ $(Z(M) = 0)$ where $Z(M) = \{m \in M : Ann(m) \trianglelefteq R\}$. And the class of all nonsingular left R-modules is closed under submodules, direct products, essential extensions and module extensions. A ring R is called *left nonsingular (singular)* if ${}_RR$ is nonsingular (singular). Let R be a ring. Then R is left nonsingular iff all left projective modules are nonsingular [4]. Moreover, a submodule N of a module M is *φ-closed* in M if $\frac{M}{N}$ is nonsingular [4]. Generally, φ-closed submodules are always closed but closed submodules need not be φ-closed. For example, 0 is a closed submodule of a module M, but 0 is not φ-closed in M in general. If the module M is nonsingular, then every closed submodule of M is φ-closed in M. For more detailed information about these concepts given here we refer to [4].

Corollary 2.7. *Let $M = M_1 + M_2$ be a nonsingular module with M_1 δ-ccws and M_2 any R-module Then M is a δ-ccws module if and only if every $N \leq_{cc} M$ with $M_2 \nleq N$ has a weak δ-supplement in M.*

Proof. Since M is nonsingular and N is closed in M, N is also φ-closed in M, that means, $\frac{M}{N}$ is nonsingular. Then $\frac{M_1}{M_1 \cap N} \cong \frac{M_1+N}{N}$ is also nonsingular as a submodule of $\frac{M}{N}$. Finally $M_1 \cap N$ is φ-closed and so closed in M_1. Thus by Proposition 2.5 the proof is clear. □

Corollary 2.8. *Let M be a nonsingular δ-ccws-module. Then M is cofinitely weak δ-supplemented.*

Proposition 2.9. *Let M be a δ-ccws-module and the factor module $\frac{M}{L}$ be nonsingular for a submodule L of M. Then $\frac{M}{L}$ is a δ-ccws-module.*

Proof. Let $\frac{N}{L}$ be any cofinite closed submodule of $\frac{M}{L}$ and $\pi : M \longrightarrow \frac{M}{L}$ be the natural homomorphism. Since $\frac{M}{L}$ is nonsingular and $\frac{N}{L} \leq_c \frac{M}{L}$, $\frac{N}{L}$ is also φ-closed in $\frac{M}{L}$. That means $\frac{\frac{M}{L}}{\frac{N}{L}} \cong \frac{M}{N}$ is nonsingular.

Therefore N is φ-closed and so closed in M. Moreover as $\frac{\frac{M}{L}}{\frac{N}{L}} \cong \frac{M}{N}$ is finitely generated, we have $N \leq_{cc} M$. By the hypothesis there exists a submodule K of M provided $M = N + K$, $N \cap K \ll_{\delta} M$. Hence, it is clear to see that $\frac{K+L}{L}$ is a weak δ-supplement of $\frac{N}{L}$ in $\frac{M}{L}$. Finally $\frac{M}{L}$ is a δ-ccws-module. □

Proposition 2.10. *Let M be a module with $\delta(M) = 0$. Then M is a δ-ccws-module if and only if $N \leq_{cc} M$ is a direct summand of M.*

Proof. ($\Longrightarrow$) Let N be any submodule of M with $N \leq_{cc} M$. Then there exists a submodule L of M such that $M = N + L$ and $N \cap L \ll_{\delta} M$ since M is a δ-ccws-module. Hence, $N \cap L \leq \delta(M) = 0$ and $M = N \oplus L$ is obtained.

($\Longleftarrow$) : It is obvious. □

Corollary 2.11. *Let M be a finitely generated module with $\delta(M) = 0$. Then the following statements are equivalent.*

1. M is a ccws-module;
2. M is a δ-ccws-module;
3. M is extending.

Recall [10] that a ring R is a *left δ-V-ring* if for any left R-module M, $\delta(M) = 0$. Since every small submodule is δ-small, $Rad(M) \leq \delta(M)$ for any module M.

Corollary 2.12. *Let R be a left δ-V-ring. Then ${}_RR$ is a δ-ccws-module iff R is an extending ring.*

A Prüfer domain is a type of commutative ring that generalizes Dedekind domains defined as a commutative ring without zero divisors in which every nonzero finitely generated ideal is invertible [3]. The ring of integer valued polynomials with rational number coefficients is a Prüfer domain, although the ring $\mathbb{Z}[x]$ of integer polynomials is not, (see in [7]).

The following example shows that any finite direct sum of a δ-ccws module need not be a δ-ccws-module.

Example 2.13. *Let $R = \mathbb{Z}[x]$, where $\mathbb{Z}$ is the ring of integers and consider $M = R \oplus R$ as an R-module. Then the R-module M is not extending as $\mathbb{Z}[x]$ is not a Prüfer domain* [7]. *So, M is not a δ-ccws-module despite $\delta(M) = 0$, by Corollary 2.10.*

Recall that an epimorphism $f : M \longrightarrow N$ is called δ-small if $Ker(f) \ll_{\delta} M$. Then M is a δ-small cover of N together with f denoted by (M, f). Besides if $f : M \longrightarrow N$ is δ-small epimorphism, then $f^{-1}(K)$ is δ-small in M for any $K \ll_\delta N$ [9, Lemma 2.9]. Moreover it can be seen in [9, Lemma 4.6] that the composition of two δ-small epimorphism is again a δ-small epimorphism.

Proposition 2.14. *Let (M, f) be a δ-small cover of a δ-ccws-module N. If $Ker(f) \leq K$ for any $0 \neq K \leq_{cc} M$, then M is also a δ-ccws-module.*

Proof. Let $f : M \longrightarrow N$ be a δ-small epimorphism and $0 \neq K \leq_{cc} M$. First we show that $f(K) \leq_{cc} N$. Assume that $f(K) \trianglelefteq L \leq N$. Then $K = K + Ker(f) = f^{-1}(f(K)) \trianglelefteq f^{-1}(L)$. Hence, we have that $K = f^{-1}(L)$ since K is closed in M. Thus, $f(K) = L \cap Im(f) = L \cap N = L$ is closed in N. Moreover, $\frac{N}{L} \cong \frac{M}{K}$ is finitely generated as f is an epimorphism from M to N and so $f(K) \leq_{cc} N$ is obtained. Then, there exists a weak δ-supplement of $f(K)$ in N since N is a δ-ccws-module. Thus, K has a weak δ-supplement in M by [9, Lemma 2.10]. □

Lemma 2.15. *Let $f : M \longrightarrow N$ be a epimorphism and $K \leq_{cc} N$ and N is nonsingular. Then $H = f^{-1}(K) \leq_{cc} M$.*

Proof. The proof is similar to that of [14, Lemma 4]. □

Theorem 2.16. *Let M be a δ-ccws-module. Then so is any nonsingular homorphic image of M.*

Proof. Let N be any arbitrary homomorphic image of M. So, there exists an epimorphism f from M to N with M is δ-ccws and N is nonsingular. Let $K \leq_{cc} N$. Then $f^{-1}(K) \leq_{cc} M$ by Lemma 2.13 and so there exists a weak δ-supplement L of $f^{-1}(K)$ in M such that $f^{-1}(K) + L = M$ and $f^{-1}(K) \cap L \ll_\delta M$. Thus $K + f(L) = N$ and $K \cap f(L) \ll_\delta N$, since $Ker(f) \leq f^{-1}(K)$. Hence, N is a δ-ccws-module. □

Corollary 2.17. *Let M be a δ-ccws-module with $\frac{M}{\delta(M)}$ nonsingular. Then so is $\frac{M}{\delta(M)}$.*

Remark 2.18. *In Theorem 2.14 the fact that N is nonsingular is not required. For example, $\mathbb{Z}$ is a δ-ccws-module but for any prime number p, $\mathbb{Z}_p \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$ is a δ-ccws-module since $\mathbb{Z}_p$ is singular.*

Theorem 2.19. *Let M be a δ-ccws-module and f be an epimorphism from M to N. If $Ann(m) = Ann(f(m))$ for every element m of $M \backslash Ker(f)$, then N is a δ-ccws-module.*

Proof. By [14, Lemma 5] there exists a submodule $U \leq_{cc} M$ containing $Ker(f)$ which corresponds to $L \leq_{cc} N$ such that $\frac{U}{Ker(f)} \cong L$. Since M

is a δ-ccws-module, U has a weak δ-supplement K in M and so $\frac{K+Ker(f)}{Ker(f)}$ is also a weak δ-supplement of $\frac{U}{Ker(f)}$ in $\frac{M}{Ker(f)}$. Since $\frac{M}{Ker(f)} \cong N$, then N is a δ-ccws-module. □

Lemma 2.20. *Let M be a δ-ccws-module and $N \leq_{cc} M$. For a δ-small submodule T of M there exists a submodule K of M provided $M = K + N = K + N + T$, $K \cap N \ll_\delta M$ and $K \cap (N + T) \ll_\delta M$.*

Proof. By the hypothesis, there exists a submodule K of M provided that $M = N + K$ and $N \cap K \ll_\delta M$. Let $f : M \longrightarrow \frac{M}{N} \oplus \frac{M}{K}$ and $g : \frac{M}{N} \oplus \frac{M}{K} \longrightarrow \frac{M}{N+T} \oplus \frac{M}{K}$ be epimorphisms via $f(m) = (m + N, m + K)$ and $g(m_1 + N, m_2 + K) = (m_1 + N + T,\ m_2 + K)$. It can be easily seen that f is a δ-small epimorphism. Now, let consider the canonical epimorphism $\pi : M \longrightarrow \frac{M}{N}$. In this case, $\pi(T) = \frac{N+T}{N} \ll_\delta \frac{M}{N}$ since $T \ll_\delta M$. Hence, $Ker(g) = \frac{N+T}{N} \oplus 0 \ll_\delta \frac{M}{N} \oplus \frac{M}{K}$. That means g is also a δ-small epimorphism and so $g \circ f$ is also a a δ-small epimorphism. Hence, we get $Ker(f) = (N + T) \cap K \ll_\delta M$. □

Recall from [11] that a module M is called cofinitely refinable if for any cofinite submodule U of M and any submodule V of M with $U+V = M$, there is a direct summand U' of M with $U' \leq U$ and $U' + V = M$.

Theorem 2.21. *Let M be a cofinitely refinable singular module. Suppose that for any cofinite submodule N of M, there exists a submodule L of M such that $L \leq_{cc} M$ and $N = L + D$ or $L = N + D'$ for some D, $D' \ll_\delta M$. Then the following statements are equivalent:*

1. M is $\oplus$-cofinitely δ-supplemented;
2. M is cofinitely δ-supplemented;
3. M is cofinitely weak δ-supplemented;
4. M is δ-ccws.

Proof. (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) is clear.
(4) $\Longrightarrow$ (1) : Let N be any cofinite submodule of M. By the hypothesis, there is a cofinite closed submodule L of M such that $N = L + D$ or $L = N + D'$ for some D, $D' \ll_\delta M$.

Case 1 : Let $N = L + D$, $D \ll_\delta M$. Since M is a δ-ccws-module and $L \leq_{cc} M$, there exists a submodule K of M such that $M = L + K$ and $L \cap K \ll_\delta M$. Following we have $M = N + K$ and $(L + D) \cap K = N \cap K \ll_\delta M$ by Lemma 2.17. As M is cofinitely refinable, there is a direct summand U of M such that $U \leq N$ and $M = U + K$. So, $U \cap K \leq N \cap K \ll_\delta M$ and so $U \cap K \ll_\delta U$ since $U \cap K \leq U \leq M$ and U is a direct summand of M. Hence, M is $\oplus$-cofinitely δ-supplemented.

Case 2 : Let $L = N + D'$, $D' \ll_\delta M$. As M is a δ-ccws-module there exists a weak δ-supplement K of L in M such that $M = L + K$ and $L \cap K \ll_\delta M$. From here $M = L + K = (N + D') + K = (N + K) + D'$ is obtained. Since M is cofinitely refinable, there is a direct summand U of M such that $U \leq N$ and $M = U + K$. Therefore as $U \cap K \leq N \cap K \ll_\delta M$ and U is a direct summand of M, we have $U \cap K \ll_\delta U$. That verifies M is $\oplus$-cofinitely δ-supplemented. □

The following theorem can be supplied via [9, Lemma 4.12].

Theorem 2.22. *Assume that for any cofinite submodule U of M, there exists a singular submodule K of M which is a weak δ-supplement of a maximal submodule P of M with $K + U \leq_{cc} M$. Then M is a cofinitely closed weak δ-supplemented module if and only if M is a cofinitely weak δ-supplemented module..*

Proof. Let M be a cofinitely weak δ-supplemented module. Then M is also a cofinitely closed weak δ-supplemented module. Conversely, if M is a δ-ccws-module, then there exists a singular submodule K of M which is a weak δ-supplement of a maximal submodule P of M with $K + U \leq_{cc} M$ for any cofinite submodule U of M As M is a δ-ccws module, then the submodule $K + U$ of M has a weak δ-supplement in M. So, U has a weak δ-supplement in M by [9, Lemma 4.12]. □

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