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ALMOST QUASI-YAMABE SOLITONS ON LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS-[$(LCS)_n$]

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Abstract. The object of the present paper is to study of Almost Quasi-Yamabe solitons and gradient almost quasi-Yamabe solitons on an Lorentzian concircular structure manifolds briefly say $(LCS)_n$ -manifolds under infinitesimal CL-transformations and obtained sufficient conditions for such solitons to be expanding, steady and shrinking. Also we obtained a necessary and sufficient condition of an almost quasi-Yamabe soliton with respect to the CL-connection to be an almost quasi-Yamabe soliton on $(LCS)_n$-manifolds with respect to Levi-Civita connection. Finally, we construct an example of steady almost quasi-Yamabe soliton on 3-dimensional $(LCS)_n$-manifolds.

1. Introduction

The Yamabe problem in differential geometry concerns the existence of Riemannian metrics with constant scalar curvature, and takes its name from the mathematician Hidehiko Yamabe in 1960. In differential geometry, the Yamabe flow is an intrinsic geometric flow in a process which deforms the metric of a Riemannian manifold. The fixed points of the Yamabe flow are metrics of constant scalar curvature in the given conformal class which is first introduced by R. S. Hamilton [7] by the following equation

$$\frac{\partial}{\partial t}g(t) = -r(t)g(t), \tag{1}$$

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where $r(t)$ denotes the scalar curvature of the metric $g(t)$. Yamabe soliton corresponds to self-similar solution of the Yamabe flow.

In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow defined by equation (5). However in minimum dimension ($n > 2$) the Yamabe and Ricci flow do not agree, since the first one preserve the conformal class of the metric but the Ricci flow does not so in general.

A Riemannian manifold (M, g) is a *Yamabe soliton* if it admits a vector field X such that

$$\mathcal{L}_X g = 2(r - \lambda)g, \tag{2}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector field X, r is the scalar curvature of the metric g and λ is a real number. Moreover, a vector field X is called a *soliton field*. In the following, we denotes the Yamabe soliton satisfying (2) by (M, g, X, λ). A Yamabe soliton is said to be *shrinking*, *steady* or *expanding* if it admits a soliton field for which, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

Recently, in [3] B.Y. Chen introduced the notion of quasi-Yamabe soliton which we shall consider in the present paper more general case, when the constant are let to be functions. If λ is a smooth function on M then the metric satisfying (2) is called almost Yamabe soliton (see also [6], [9]).

Let (M, g) be an n-dimensional Riemannian manifold ($n > 2$), ξ a vector field and η a 1-form on M.

Definition 1.1. ([2]). An *almost quasi-Yamabe soliton* on M is a data (g, V, λ, μ) which satisfy the equation

$$\frac{1}{2}\mathcal{L}_V g + (\lambda - r)g + \mu\eta \otimes \eta = 0, \tag{3}$$

where $\mathcal{L}_V$ is the Lie derivative operator along the vector field V and λ, μ are smooth functions on M.

When the potential vector field of (3) is of gradient type, i.e. $\xi = grad(f)$, then (g, ξ, λ, μ) is said to be a *gradient almost quasi-Yamabe soliton*(see [1], [3], [9]) and the equation satisfied by it becomes

$$Hess(f) = (r - \lambda)g - \mu df \otimes df. \tag{4}$$

In, 2018, Mandal and Hui, [13] also, studied the existence of Yamabe gradient solitons. In 1963, Y. Tashiro and S. Tachibana [20] introduced a transformation, called *CL-transformation*, on a Sasakian manifold under which C-loxodrome remains invariant. It is known that loxodrome is a curve on the unit sphere that intersects the meridians at a fixed angle and C-loxodrome is a loxodrome cutting geodesic trajectories of the characteristic vector field ξ of the manifold with constant angle. Here 'CL' stands for C-loxodrome. CL-transformation have been studied by several authors in different context such as Koto and Nagao [8], Takamatsu and Mizusawa [19], Shaikh et al. [16].

Hui and Chakarborty [5] studied CL-transformations on Kenmotsu manifolds with Ricci soliton. Moreover, Erken also study Yamabe solitons on three-dimensional normal almost para-contact metric manifolds [4].

Lorentzian manifold is one of the most important sub-class of pseudo Riemannian manifolds. It plays a crucial role in mathematical physics (specially in the development of the theory of relativity and cosmology). K. Matsumoto [11] gave the idea of Lorentzian para-Sasakian manifolds (briefly LP-Sasakian manifolds). On the other hand in 2003, A. A. Shaikh [15] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds), which generalizes the notion of LP-Sasakian manifolds introduced by K. Matsumoto [11], I. Mihai and R. Rosca [10].

In ([17], [18]) authors investigated applications of $(LCS)_n$-manifolds in applied mathematics and theoretical physics. In [12] author proved that 4-dimensional Lorentzian concircular structure [in brief $(CS)_4$] spacetime coincides with *generalized Robertson Walker [in brief GRW] spacetime*. Consequently, to study $(CS)_4$-spacetime obeying Einstein equation under some curvature restrictions are equivalent to study that of GRW-spacetime obeying Einstein equation.

Motivated by the above studies, the present paper explores the study of infinitesimal CL-transformations on Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with almost quasi-Yamabe soliton. The paper organized as follows. Section 2 is concerned with preliminary. In Section 3, we study the infinitesimal CL-transformation and almost quasi-Yamabe soliton on $(LCS)_n$-manifolds. Also, we have studied almost quasi-Yamabe solitons on $(LCS)_n$-manifolds with respect to CL-connection which was introduced by Koto and Nagao [8]. In Section 4 we have given an example of an $(LCS)_n$-manifolds admitting an almost quasi-Yamabe soliton. In last Section 5, we have discussed about the gradient almost quasi Yamabe Soliton on $(LCS)_n$-manifolds.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdroff manifold with a Lorentzian metric g, that is,

M admits a smooth symmetric tensor field of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,....,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (*resp.* non-sapcelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ (*resp.* $\leq 0, = 0, > 0$) [14]. The category to which a given vector field is called its causal character.

Definition 2.1. In a Lorentzian manifold (M,g) [22] a vector field P defined by

$$g(X,P) = A(X)$$

for any $X \in \chi(M)$ is said to be *concircular vector field* if

$$(\nabla_X A)Y = \alpha \{g(X,Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form.

Let M be a Lorentzian manifold [22] admitting a unit timelike concircular vector field ξ, called the characteristics vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{5}$$

Since ξ is a unit concircular vector filed, it follows that there exists a non-zero 1-form η such that

$$g(X,\xi) = \eta(X). \tag{6}$$

The equations of the following form hold.

$$(\nabla_X \eta)Y = \alpha[g(X,Y) + \eta(X)\eta(Y)], \qquad (\alpha \neq 0), \tag{7}$$

$$\nabla_X \xi = \alpha \{X + \eta(X)\xi\}, \qquad (\alpha \neq 0) \tag{8}$$

for all vector fields X, Y, where ∇ denotes the operator of the covariant differentiation with respect to the Lorentzian metric g and α is non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{9}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{10}$$

then from (7) and (10) we have

$$\phi X = X + \eta(X)\xi, \tag{11}$$

$$g(\phi X, Y) = g(X, \phi Y), \tag{12}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with unit timelike concircular vector field ξ, it associated 1-form η and $(1,1)$ tensor field ϕ is said to be *Lorentzian concircular structure manifold* or briefly called $(LCS)_n$-manifold [15]. Especially if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure [21] of the following relations hold [15].

$$\phi^2 X = X + \eta(X)\xi, \tag{13}$$

$$\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta(\phi) = 0, \tag{14}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{15}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{16}$$

$$R(X,Y)Z = \phi R(X,Y)Z + (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi, \tag{17}$$

$$\begin{aligned} S(X,Y) = &\left[\frac{r}{n-1} - (\alpha^2 - \rho)\right] g(X,Y) \\ &- \left[\frac{r}{n-1} - n(\alpha^2 - \rho)\right] \eta(X)\eta(Y), \end{aligned} \tag{18}$$

$$QX = \left[\frac{r}{n-1} - (\alpha^2 - \rho)\right] X - \left[\frac{r}{n-1} - n(\alpha^2 - \rho)\right] \eta(X)\xi, \tag{19}$$

$$(\nabla_X \phi)Y = \alpha \left\{ g(X,Y)\xi + 2\eta(X)\eta(Y) + \eta(Y)X \right\}, \tag{20}$$

$$(X\rho) = d\rho(X) = \beta\eta(X). \tag{21}$$

Using (17) and (18), for constants α and ρ, we have

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{22}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)[g(Y,Z)\xi - \eta(Z)Y], \tag{23}$$

$$S(X,\xi) = [(n-1)(\alpha^2 - \rho)]\eta(X), \tag{24}$$

$$Q\xi = [(n-1)(\alpha^2 - \rho)]\xi, \tag{25}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Again, from the definition of Lie derivative, we have

$$\begin{aligned} (\mathcal{L}_\xi g)(X,Y) &= (\nabla_X g)(X,Y) + 2\alpha g(X,Y) + 2\alpha\eta(X)\eta(Y) \\ &= 2\alpha[g(X,Y) + \eta(X)\eta(Y)]. \end{aligned} \tag{26}$$

Definition 2.2. A vector field V in an $(LCS)_n$-manifold M is said to be an *infinitesimal* CL*-transformation* ([1], [19]) if it satisfies

$$\mathcal{L}_V \left\{ {h \atop ji} \right\} = \omega_j \delta_i^h + \omega_i \delta_j^h + a(\eta_j \phi_i^h + \eta_i \phi_j^h) + b\phi_{ji}\xi^h, \qquad \phi_{ji} = \phi_j^l g_{li} \tag{27}$$

for certain constants a and b, where ω_i are components of the 1-form ω, $\mathcal{L}_V$ denotes the Lie derivative with respect to V and $\left\{ {h \atop ji} \right\}$ is the Chirstoffel symbol of the Riemmaian metric g.

Now, we recall the following useful results:

Proposition 2.3. ([1]). *If V is an infinitesimal CL-transformation on an $(LCS)_n$-manifold M, then 1-form ω is closed.*

Theorem 2.4. ([1]). *If V is an infinitesimal CL-transformation on an $(LCS)_n$-manifold M, then the relation*

$$\begin{aligned}(\alpha^2 - \rho)(\mathcal{L}_V g)(Y, Z) = &-(\nabla_Y \omega)Z + \{\alpha(a+b) - (2\alpha\rho - b)\eta(V)\} g(Y, Z) \\ &+ \alpha(3a+b)\eta(Y)\eta(Z)\end{aligned} \tag{28}$$

holds for any vector fields Y and Z on M.

Definition 2.5. ([1]). A transformation f on an $(LCS)_n$-manifold M, $n > 3$ with structure (ϕ, ξ, η, g) is said to be a CL*-transformation* if the Levi-Civita connection ∇ and a symmetric affine connection ∇^f, called CL-connection, induced from ∇ by f are related by

$$\nabla_X^f Y = \nabla_X Y + \omega(X)Y + \omega(Y)X + c\{\eta(X)\phi Y + \eta(Y)\phi X\} + 2g(\phi X, Y)\xi, \tag{29}$$

where ω is a 1-form and c is a constant.

If R and R^f are the curvature tensor with respect to Levi-Civita connection ∇ and CL-connection ∇^f, respectively in an $(LCS)_n$-manifold M, then we have

$$\begin{aligned}&R^f(X, Y)Z \\ &= R(X, Y)Z + \{B(X, Y)Z - B(Y, X)Z + B(X, Z)Y - B(Y, Z)X\} \\ &\quad - 2c(\alpha + 2)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi\end{aligned} \tag{30}$$

for any vector fields X, Y, Z on M, where the symmetric tensor field $B(X,Y)$ is defined as [1]

$$
\begin{aligned}
B(X,Y) =& (\nabla_X \omega)(Y) + [c(\alpha+2) - 2(\alpha + \omega(\xi))]g(X,Y) \\
& + [(c+2)(c-\alpha) - 2\omega(\xi)(c+1)]\eta(X)\eta(Y) \\
& - \omega(X)\omega(Y) - c[\omega(X)\eta(Y) + \eta(X)\omega(Y)].
\end{aligned} \tag{31}
$$

From (30) we get

$$
S^f(Y,Z) = S(Y,Z) - (n-1)B(Y,Z) + 2c(\alpha+2)[g(Y,Z) + \eta(Y)\eta(Z)], \tag{32}
$$

where S^f and S are respectively the Ricci tensor of an $(LCS)_n$-manifold M with respect to the CL-connection ∇^f and Levi-Civita connection. Also, from (32), we get

$$
r^f = r - (n-1)tr.B + 2(n+1)c(\alpha+1), \tag{33}
$$

where r^f and r are respectively the scalar curvature of an $(LCS)_n$-manifold M with respect to the CL-connection ∇^f and Levi-Civita connection.

3. Infinitesimal CL-transformation and almost quasi-Yamabe solitons

In this section, we deal with the infinitesimal CL-transformations on an $(LCS)_n$-manifold M with almost quasi-Yamabe soliton.

Let (g, V, λ, μ) be an almost quasi-Yamabe soliton on an $(LCS)_n$-manifold M. Then from (3) and (28), we obtain

$$
\begin{aligned}
(r-\lambda)&g(Y,Z) - \mu\eta(Y)\eta(Z) \\
=& \frac{1}{2(\alpha^2-\rho)}\left\{-(\nabla_Y\omega)(Z) - (2\alpha\rho - b)\eta(V)\right\}g(Y,Z) \\
& + \frac{1}{2(\alpha^2-\rho)}\left\{\alpha(3a+b)\eta(Y)\eta(Z)\right\}.
\end{aligned} \tag{34}
$$

Since the metric g is symmetric and the 1-form ω is closed by Proposition 2.3, so interchanging Y and Z in (34) and subtracting the obtained results from (34), we get $\mu = 0$ and $(3a+b) = 0$, which implies that the infinitesimal CL-transformation V is a projective Killing vector field. Also yields

$$
(r-\lambda)g(Y,Z) = \frac{1}{2(\alpha^2-\rho)}\left\{-(\nabla_Y\omega)(Z) - (2\alpha\rho - b)\eta(V)\right\}g(Y,Z). \tag{35}
$$

This leads to the following:

Theorem 3.1. *If (g, V, λ, μ) is an almost quasi-Yamabe soliton in an $(LCS)_n$-manifold M such that V is an infinitesimal CL-transformation, then V is a projective Killing vector field and (35) holds.*

Since $\mu = 0$ in (3), then almost quasi-Yamabe soliton is a Yamabe soliton. Therefore, we also have the following.

Theorem 3.2. *If (g, V, λ) is a Yamabe soliton in an $(LCS)_n$-manifold M such that V is an infinitesimal CL-transformation, then V is a projective Killing vector field. Moreover, an almost quasi-Yamabe soliton coincide with Yamabe soliton.*

We now consider an almost quasi-Yamabe soliton (g, V, λ, μ) on an $(LCS)_n$-manifold M with respect to the CL-connection ∇^f. Then we have

$$\frac{1}{2}(\mathcal{L}^f_V g)(Y, Z) = (r^f - \lambda)g(Y, Z) - \mu\eta(Y)\eta(Z), \tag{36}$$

where $\mathcal{L}^f_V$ is the Lie derivative along the vector field V on M with respect to the CL-connection ∇^f.

By virtue of (26) we have

$$\begin{aligned}(\mathcal{L}^f_V g)(Y, Z) &= g(\nabla^f_Y V, Z) + g(Y, \nabla^f_Z V)\\ &= g(\nabla_Y Z + \omega(Y)V + \omega(V)Y + c\{\eta(Y)\phi V + \eta(V)\phi Y\}\\ &\quad + 2g(\phi Y, V)\xi, Z) + g(Y, \nabla_Z Y + \omega(Z)V + \omega(V)Z\\ &\quad + c\{\eta(Z)\phi V + \eta(V)\phi Z\} + 2g(\phi Z, V)\xi, Z)\\ &= (\mathcal{L}_V g)(Y, Z) + \omega(Y)g(V, Z) + \omega(Z)g(Y, V) + 2\omega(V)g(Y, Z)\\ &\quad + (2 + c)\{\omega(V)g(\phi V, Z) + \omega(Y)g(Y, \phi V) + \omega(Z)g(\phi V, Y)\}.\end{aligned} \tag{37}$$

In view of (33) and (36), (37) yields

$$\begin{aligned}&\frac{1}{2}(\mathcal{L}_V g)(Y, Z) - (r - \lambda)g(Y, Z)\\ &\quad - \mu\eta(Y)\eta(Z) - (n-1)tr.Bg(Y, Z) + 2(n+1)c(\alpha + 2)\\ &= -\frac{1}{2}\{\omega(Y)g(V, Z) + \omega(Z)g(Y, V)\} - \omega(V)g(Y, Z)\\ &\quad - \frac{(2 + c)}{2}\{\omega(V)g(\phi V, Z) + \omega(Y)g(Y, \phi V) + \omega(Z)g(\phi V, Y)\} = 0.\end{aligned} \tag{38}$$

If (g, V, ξ, μ) is an almost quasi-Yamabe soliton on an $(LCS)_n$-manifold M with respect to Levi-Civita connection, then (3) holds. Thus from (3) and (38), we can state the following:

Theorem 3.3. *An almost quasi-Yamabe soliton on an $(LCS)_n$- manifold M is invariant under CL-connection if and only if the relation*

$$\begin{aligned}&\omega(Y)g(V,Z)+\omega(Z)g(Y,V)+2\omega(V)g(Y,Z)\\ &+(2+c)\{\omega(V)g(\phi V,Z)+\omega(Y)g(Y,\phi V)+\omega(Z)g(\phi V,Y)\}\\ &+2(n-1)tr.Bg(Y,Z)+4(n+1)c(\alpha+2)=0\end{aligned}\tag{39}$$

holds for arbitrary vector fields Y and Z.

Now, let (g, ξ, λ, μ) be an almost quasi-Yamabe soliton on an $(LCS)_n$-manifold M with respect to the CL-connection. Then we have

$$\frac{1}{2}(\mathcal{L}^f_\xi g)(Y,Z)=(r^f-\lambda)g(Y,Z)-\mu\eta(Y)\eta(Z).\tag{40}$$

From (5), (8), (12) and (29) we have

(41)

$$\begin{aligned}&(\mathcal{L}^f_\xi g)(Y,Z)=g(\nabla^f_Y\xi,Z)+g(Y,\nabla^f_Z\xi)\\ &+g(\alpha\{Y+\eta(Y)\xi\}+\omega(Y)\xi+\omega(\xi)Y+c\{\eta(\xi)\phi Y+\eta(Y)\phi\xi\}+2g(\phi Y,\xi)\xi,Z)\\ &+g(Y,\alpha\{Z+\eta(Z)\xi\}+\omega(Z)\xi+\omega(\xi)Z+c\{\eta(\xi)\phi Z+\eta(Z)\phi\xi\}+2g(\phi Z,\xi)\xi)\\ &=2[\{\alpha+\omega(\xi)\}g(Y,Z)+\alpha\eta(Y)\eta(Z)+cg(\phi Y,Z)]+\omega(Y)\eta(Z)+\omega(Z)\eta(Y).\end{aligned}$$

Using (33) and (40) in (41), we get

$$\begin{aligned}&\{r-\lambda-\alpha-\omega(\xi)\}g(Y,Z)+(\alpha-\mu)\eta(Y)\eta(Z)+cg(\phi Y,Z)\\ &-(n-1)tr.Bg(Y,Z)+2c(n+1)(\alpha+1)-\frac{1}{2}\{\omega(Y)\eta(Z)+\omega(Z)\eta(Y)\}=0.\end{aligned}\tag{42}$$

This leads to the following.

Theorem 3.4. *If (g, ξ, λ, μ) is an almost quasi-Yamabe soliton on an $(LCS)_n$-manifold M with respect to the CL-connection, then (42) holds.*

Replacing Y and Z by ξ in (42), we obtain

$$\lambda=r-2\omega(\xi)-(n-1)tr.B-2(n+1)c(\alpha+2)-2\alpha-\mu.\tag{43}$$

This leads to the following.

Theorem 3.5. *An almost quasi-Yamabe soliton (g, ξ, λ, μ) on an $(LCS)_n$-manifold M with respect to CL-connection is shrinking, steady and expanding as*

1. $r < 2\omega(\xi) - (n-1)tr.B - 2(n+1)c(\alpha+2) - 2\alpha - \mu$,
2. $r = 2\omega(\xi) - (n-1)tr.B - 2(n+1)c(\alpha+2) - 2\alpha - \mu$,
3. $r > 2\omega(\xi) - (n-1)tr.B - 2(n+1)c(\alpha+2) - 2\alpha - \mu$

respectively.

Also, if $\mu = 0$ in (43), then an almost quasi-Yamabe soliton turns into a Yamabe soliton. Thus we have the following:

Theorem 3.6. *A Yamabe soliton (g, ξ, λ) on an $(LCS)_n$-manifold M with respect to CL-connection is shrinking, steady and expanding as*

1. $r < 2\omega(\xi) - (n-1)tr.B - 2(n+1)c(\alpha+2) - 2\alpha$,
2. $r = 2\omega(\xi) - (n-1)tr.B - 2(n+1)c(\alpha+2) - 2\alpha$,
3. $r > 2\omega(\xi) - (n-1)tr.B - 2(n+1)c(\alpha+2) - 2\alpha$

respectively.

Also, A. A. shaikh et al. [1] proved that a tensor field A of type $(1,3)$ is invariant under the CL-transformation f is given by

$$\begin{aligned} A(X,Y)Z =& R(X,Y)Z - \frac{1}{n-1}[\{S(Y,Z)X - S(X,Z)Y\} \\ &+ \{S(Y,Z)X\eta(X) - S(X,Z)\eta Y\}\xi \\ &+ \frac{(\alpha^2-\rho)}{n-2}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ (n-1)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi] \end{aligned} \tag{44}$$

and it is called the CL-curvature tensor field on M.

Definition 3.7. ([1]). An $(LCS)_n$-manifold M is said to be CL-*flat* if the CL-curvature tensor field A of type $(1,3)$ vanishes identically on M.

Definition 3.8. ([1]). An $(LCS)_n$-manifold M is said to be CL-*symmetric* if $(\nabla_U A)(X,Y) = 0$ for all X, Y and U on M.

Definition 3.9. ([1]). An $(LCS)_n$-manifold M is said to be CL-*semi-symmetric* if the $R(X,Y).A = 0$.

In [1], A. A. Shaikh et al. proved that in an $(LCS)_n$-manifold M, the concept of CL-semi-symmetric, CL-symmetric, CL-flatness and the manifold of constant curvature -1, i.e., manifold is *Einstein* are equivalent and its Ricci tensor is of the form

$$S(Y,Z) = (n-1)(\alpha^2-\rho)g(Y,Z). \tag{45}$$

From (45), we get

$$r = n(n-1)(\alpha^2-\rho). \tag{46}$$

Again in [5], Hui and Chakraborty studied second order parallel tensor in Kenmotsu manifold as a corollary of their result, so we have the following:

Theorem 3.10. *In an $(LCS)_n$-manifold M, every second order parallel symmetric metric tensor is a constant multiple of the metric tensor.*

Suppose that the $(0,2)$-type symmetric tensor field $\mathcal{L}_V g - 2rg - 2\mu\eta\otimes\eta$ is parallel for any vector field V on an $(LCS)_n$-manifold M. Then Theorem 3.10 yields $\mathcal{L}_V g - 2rg - 2\mu\eta\otimes\eta$ is a constant multiple of the metric tensor g, i.e.,

$$(\mathcal{L}_V g)(X,Y) - 2rg(X,Y) - 2\mu\eta(X)\eta(Y) = -2\lambda g(X,Y)$$

for all X, Y on M, where λ is a constant. Hence the relation (3) holds. This implies that (g, V, λ, μ) yields an almost quasi-Yamabe soliton. Hence we can state the following:

Theorem 3.11. *If the tensor field $\mathcal{L}_V g - 2rg - 2\mu\eta\otimes\eta$ on an $(LCS)_n$-manifold M is parallel for any vector field V, then (g, V, λ, μ) is an almost quasi-Yamabe soliton.*

Let us consider h be a $(0,2)$-type symmetric parallel tensor field on an $(LCS)_n$-manifold M such that

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) - 2rg(X,Y) - 2\mu\eta(X)\eta(Y). \tag{47}$$

From (26) we have

$$(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = 2\alpha[g(X,Y) + \eta(X)\eta(Y)]. \tag{48}$$

Using (46) and (48) in (47), we get

$$h(X,Y) = 2[\alpha - n(n-1)(\alpha^2 - \rho)]g(X,Y) + 2(\alpha - \mu)\eta(X)\eta(Y). \tag{49}$$

Putting $X = Y = \xi$ in (49), we obtain

$$h(\xi,\xi) = 2n(n-1)(\alpha^2 - \rho) - 2\mu. \tag{50}$$

If (g, ξ, λ, μ) is an almost quasi-Yamabe soliton on an $(LCS)_n$-manifold M, then from (3) we have

$$h(X,Y) = -2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y) \tag{51}$$

and hence

$$h(\xi,\xi) = 2(\lambda - \mu). \tag{52}$$

From (50) and (52) we get $\lambda = n(n-1)(\alpha^2 - \rho) > 0$ and consequently the almost quasi-Yamabe soliton (g, ξ, λ, μ) is expanding. Thus we can state the following:

Theorem 3.12. *If the tensor field $\mathcal{L}_\xi g - 2rg - 2\eta \otimes \eta$ on a CL-flat (respectively CL-symmetric, CL-semi-symmetric) $(LCS)_n$-manifold is parallel, then the almost quasi-Yamabe soliton (g, ξ, λ, μ) is always expanding.*

4. Example of an $(LCS)_3$-manifolds admitting the almost quasi-Yamabe solitons

Example 4.1. Consider the three dimensional manifold

$$M = \left\{(x, y, z) \in \mathbb{R}^3 | z \neq 0\right\},$$

where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$ and let the linearly independent vector fields are

$$e_1 = z^2 \frac{\partial}{\partial x}, \quad e_2 = z^2 \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Lorentzian metric defined by
$g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = 1$, $g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0$.

Let η be the 1-form defined by $\xi = e_3$, $\eta(X) = g(X, e_3)$ for any vector field X on M and ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = 0.$$

Then by using the linearity of ϕ and g, we have $\phi^2 X = X + \eta(X)\xi$ with $\xi = e_3$.

Further $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an $(LCS)_3$-structure in $\mathbb{R}^3$ [15].

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is know as Koszul's formula.

$$\begin{gathered} \nabla_{e_1} e_1 = -\frac{2}{z} e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\frac{2}{z} e_1, \\ \nabla_{e_2} e_2 = -\frac{2}{z} e_2, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_3 = -\frac{2}{z} e_2, \\ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \end{gathered}$$

Using the above relation, for any vector X on M, we have $\nabla_X\xi = \alpha[X+\eta(X)\xi)]$, where $\alpha = -\frac{2}{z}$. Hence (ϕ,ξ,η,g,α)-structure defines the $(LCS)_3$-structure in $\mathbb{R}^3$.

Example 4.2. On the $(LCS)_3$-manifold $(M,g,\xi,\eta,\phi,\alpha)$ considered in Example 4.1, the data (g,ξ,λ,μ) for $\lambda = \frac{2(z-12)}{z^2}$ and $\mu = \frac{2}{z}$ defines an almost quasi-Yamabe soliton.

Indeed, the Riemann and the Ricci curvature tensor fields are computed as follows:

$$R(e_1,e_2)e_2 = \frac{4}{z^2}e_1, \quad R(e_1,e_3)e_3 = -\frac{6}{z^2}e_1, \quad R(e_2,e_1)e_1 = \frac{4}{z^2}e_2,$$

$$R(e_2,e_3)e_3 = -\frac{6}{z^2}e_2, \quad R(e_3,e_1)e_1 = \frac{6}{z^2}e_3, \quad R(e_3,e_2)e_2 = \frac{6}{z^2}e_3.$$

From the above expression of the curvature tensor we can also obtain Ricci tensor

$$S(e_1,e_1) = S(e_2,e_2) = -\frac{10}{z^2}, \quad S(e_3,e_3) = -\frac{12}{z^2}.$$

Also, $\alpha = -\frac{2}{z}$, $\rho = -\frac{2}{z^2}$.
Therefore,

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = -\frac{24}{z^2}. \tag{53}$$

Thus scalar curvature $r = -\frac{24}{z^2}$. By the definition of almost quasi-Yamabe soliton and using (3), we obtain

$$2\alpha[g(e_i,e_i) + \eta(e_i)\eta(e_i)] + 2(\lambda - r)g(e_i,e_i) + 2\mu\eta(e_i)\eta(e_i) = 0$$

for all $i \in \{1,2,3\}$, and we have

$$2(1+\delta_{i3}) - 2\frac{3}{z^2} + 2\lambda + \frac{48}{z^2} + 2\mu\delta_{i3} = 0$$

for all $i \in \{1,2,3\}$. Therefore $\lambda = \frac{2(z-12)}{z^2}$ and $\mu = \frac{2}{z}$. The data (g,ξ,λ,μ) is an almost quasi-Yamabe soliton on 3-dimensional (LCS)-manifolds. Since the $\lambda > 0$, i.e., expanding.

5. Gradient almost quasi Yamabe Soliton

Consider the vector field ξ is of the gradient type of a potential function f, we say that (M,g,f,μ) is of a gradient almost quasi Yamabe

soliton and for $\xi = \frac{1}{2}\nabla f$.
Equation (2) reduces to

$$Hess(f) = (r - \lambda)g - \mu df \otimes df, \tag{54}$$

where $Hessf = \nabla df$ is the *Hessian* tensor of f and ∇ denotes the Levi-Civita connection on manifold M.

As an application, we show the existence of self-dual gradient almost quasi-Yamabe soliton. We emphasize that the gradient almost quasi-Yamabe soliton equation (3) codifies geometric information about the structure of manifold M by means of the scalar curvature and the second fundamental form of the level sets of the potential function f.

The analysis of the level sets of f naturally splits into two following different cases:
(i) The first case corresponds to non-degenerate hypersurfaces i.e., $\|\nabla f\| \neq 0$ and the manifold M is called a *non-isotropic gradient Yamabe soliton.*
(ii) The second case corresponds to degenerate hypersurface i.e., $\|\nabla f\| \neq 0$ by $\nabla f \neq 0$, and the manifold is called *isotropic gradient Yamabe soliton.* When f is constant the Yamabe soliton to be trivial.

By contracting equation (54) we immediately obtain

$$\Delta f = n(r - \lambda) - \mu |\xi|^2, \tag{55}$$

$$Hessf = \frac{1}{n}\Delta g - div(\mu\eta \otimes \eta). \tag{56}$$

Equation (55) show that the almost quasi-Yamabe soliton equation is a special case of the more general *Mobius equation* if $\mu = 0$ (Yamabe soliton) i.e.,

$$Hessf = \frac{1}{n}\Delta g. \tag{57}$$

From the direct consequence of the equation (55), we have the following.

Theorem 5.1. *Let $(M, g, \varphi, \eta, \xi)$ be a 3-dimensional (LCS) manifold and η be the g-dual 1-form of the gradient vector field $\xi = grad(f)$ with $g(\xi, \xi) = -1$. If (54) define the gradient almost quasi-Yamabe soliton in M, then the Lapalacian equation satisfied by f becomes*

$$\Delta(f) = 3(r - \lambda) - \mu |\xi|^2. \tag{58}$$

As an application of (56) we get the level sets of the function f are totally umbilical and normalized gradient vector field $\frac{\nabla f}{\|\nabla f\|}$ is a non-null geodesic vector field.

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