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MODULE AMENABILITY OF MODULE LAU PRODUCT OF BANACH ALGEBRAS

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Abstract. Let A, B, $\mathfrak{U}$ be Banach algebras and B be a Banach $\mathfrak{U}$-bimodule also A be a Banach B-$\mathfrak{U}$-module. In this paper we study the relation between module amenability, weak module amenability and module approximate amenability of module Lau product $A \times_\alpha B$ and that of Banach algebras A, B.

1. Introduction

From this section, contents of the article can be written.

The notation of amenability of Banach algebras was introduced by B.Johnson in [14]. A Banach algebra A is amenable if every bounded derivation from A into any dual Banach A-bimodule is inner, equivalently if $H(A, X^*) = \{0\}$ for any Banach A-bimodule X, where $H(A, X^*)$ is the first Hochschild cohomology group of A with coefficient in X^*. Also, a Banach algebra A is weakly amenable if $H(A, A^*) = \{0\}$. Bade, Curtis and Dales introduced the notion of weak amenability on Banach algebras in [4]. They considered this concept only for commutative Banach algebras. After that Johnson defined the weak amenability for arbitrary Banach algebras [13] and showed that for a locally compact group G, $L^1(G)$ is weakly amenable [12]. Permanent weak module amenability of semigroup algebras is studied by Bodaghi, Amini and Jabbari in [8]. Also, the authors can use [8] for finding more non-trivial examples of module Lau algebras. Bodaghi and Jabbari have studied module amenability, n-weak module amenability of the triangular Banach algebras (which are the generalization of module extension Banach algebras) and application to the semigorup algebras of inverse semigroups in [9].

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Also, other notions of module amenability which are pertinent to the current paper are introduced in [6] and [7].
In [1] the notion of module amenability was introduced for a Banach algebra. Amini showed that for an inverse semigroup with set of idempotents E_S, the semigroup algebra $l^1(S)$ is $l^1(E_S)$ - module amenable if and only if S is amenable. The concept of weak module amenability was introduced in [3]. There the authores showed that for a commutative invers semigroup S, $l^1(S)$ is always weak module amenable as a Banach module over $l^1(E_S)$. In [15] the notion of module approximate amenability and contractibility was introduced for Banach algebras that are modules over another Banach algebra.
Let A and B be Banach algebras. Consider a bounded homomorphism $T : B \longrightarrow A$ with $\|T\| \leq 1$. Define T-Lau product $A \times_T B$ as the space $A \times B$ equipped with the norm $\|(a, b)\| = \|a\| + \|b\|$ and the multiplication $(a, b) \cdot (c, d) = (ac + aT(d) + T(b)c, bd)$ $(a, c \in A, b, d \in B)$. This product was first introduced and investigated by Bhatt and Dabhi in [5] when A is commutative. This product has been extended by Javanshiri and Nemati [11] to the general Banach algebras and studied amenability and n-weak amenability of $A \times_T B$ in the general case; for more information see [10]. Note that when $T = 0$, this multiplication is the usual coordinatewise product and so $A \times_T B$ is in fact the direct product $A \oplus B$.
Now let A and B be Banach algebras and A be Banach B-bimodule with module actions $\alpha_1 : B \times A \longrightarrow A$; $(b, a) \mapsto b \cdot a$ and $\alpha_2 : A \times B \longrightarrow A$; $(a, b) \mapsto a \circ b$ such that $a\alpha_1(b, a') = \alpha_2(a, b)a'$, for a, $a' \in A$ and $b \in B$. we define the module Lau product on $A \times B$ as follows $(a, b)(a', b') = (aa' + a \circ b' + b \cdot a', bb')$, for $(a, b), (a', b') \in A \times B$. Set $\alpha = (\alpha_1, \alpha_2)$. The cartesian product $A \times B$ with the above multiplication is an associative, not necessarily commutative, algebra, denoted by $A \times_\alpha B$ and is called a module Lau product. Note that $A \times_\alpha B$ is Banach algebra with the following norm $\|(a, b)\| = \|a\| + \|b\|$, $(a, b) \in A \times_\alpha B$. In the peresent paper we study the module amenability, weak module amenability and module approximate amenability of $A \times_\alpha B$.

2. Basic properties

Let $\mathfrak{U}$ be a Banach algebra and A be a Banach $\mathfrak{U}$-bimodule such that it has an associative product which makes it a Banach algebra which is compatible with the module action, in the sense that $\alpha \cdot (am) =$

$(\alpha \cdot a)m, (\alpha\beta) \cdot m = \alpha \cdot (\beta \cdot m), (\alpha, \beta \in \mathfrak{U}, a, m \in A)$. And the same for the right action.

Definition 2.1. The bounded map $D : A \longrightarrow X^*$ with $D(a + b) = D(a) + D(b)$, $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$, and $D(\alpha \cdot a) = \alpha \cdot D(a)$, $D(a \cdot \alpha) = D(a) \cdot \alpha$ ($\alpha \in \mathfrak{U}, a \in A$), is called module derivation.

Note that X^* is also Banach module over A and $\mathfrak{U}$ with compatible actions under the canonical actions of A and $\mathfrak{U}$,
$\alpha\cdot(a\cdot f) = (\alpha\cdot a)\cdot f, (a \in A, \alpha \in \mathfrak{U}, f \in X^*)$, and the same for right action. Here the canonical actions of A and $\mathfrak{U}$ on X^* are defined by $(\alpha \cdot f)(x) = f(x \cdot \alpha), (a \cdot f)(x) = f(x \cdot a), (\alpha \in \mathfrak{U}, a \in A, f \in X^*, x \in X)$ and same for right actions. As in [1] we call A - module X which have a compatible $\mathfrak{U}$-action as above, a A-$\mathfrak{U}$ modules, above assertion is to say that if X is an A-$\mathfrak{U}$- module, then so is X^*. Also we use the notation $Z_{\mathfrak{U}}(A, X^*)$ for the set of all module derivations $D : A \longrightarrow X^*$, and $N_{\mathfrak{U}}(A, X^*)$ for those which are inner and $H_{\mathfrak{U}}(A, X^*)$ for the quotient group. Hence A is module amenable if and only if $H_{\mathfrak{U}}(A, X^*) = \{0\}$, for each A-$\mathfrak{U}$-module X. Throughout we assume that A, B are Banach $\mathfrak{U}$-bimodule with actions $A \times \mathfrak{U} \longrightarrow A, (a, \alpha) \longmapsto a \cdot \alpha$, $\mathfrak{U} \times A \longrightarrow A, (\alpha, a) \longmapsto \alpha \circ a$, $B \times \mathfrak{U} \longrightarrow B, (b, \alpha) \longmapsto b \star \alpha$, $\mathfrak{U} \times B \longrightarrow B, (\alpha, b) \longmapsto \alpha * b$.
For the rest of this paper, we assume that $\mathfrak{U}$ is a Banach algebra, B is a Banach $\mathfrak{U}$-module and A is a Banach B-$\mathfrak{U}$-module.

Lemma 2.2. *1)$A \times_\alpha B$ is a Banach A-bimodule,*
2)$A \times_\alpha B$ is a Banach B-$\mathfrak{U}$-module.

Lemma 2.3. *If X is an A-$\mathfrak{U}$-module, Y is a Banach B-$\mathfrak{U}$-module and A is unital and symmetric (as B-module) then $X \times Y$ is a $A \times_\alpha B$-$\mathfrak{U}$-module.*

Let A is unital and symmetric B-module. For X, Y and A as above, we have
$(f, g)\cdot(a, b) = (f\cdot a+f\cdot(b\cdot 1), g\cdot b)$ and $(a, b)\cdot(f, g) = (a\cdot f+(b\cdot 1)\cdot f, b\cdot g)$, for all $(f, g) \in X^* \times Y^*$, $(a, b) \in A \times_\alpha B$.

Proposition 2.4. *With above notation, let A be unital and symmetric (as B-module), and X be an A-$\mathfrak{U}$-module, Y be a Banach B-$\mathfrak{U}$-module*
$D \in Z_{\mathfrak{U}}(A \times_\alpha B, X^* \times Y^*)$ if and only if there are $D_1 \in Z_{\mathfrak{U}}(A, X^*)$, $D_2 \in Z_{\mathfrak{U}}(B, Y^*)$, $D_3 \in Z_{\mathfrak{U}}(B, X^*)$ such that
1) $D(a, b) = (D_1(a) + D_3(b), D_2(b))$,
2) $D_1(a \circ b) = D_1(a) \circledcirc b + a \cdot D_3(b)$,

3) $D_1(b \cdot c) = b \circledcirc D_1(c) + D_3(b) \cdot c$,
4) $D_3(bd) = D_3(b) \circledcirc d + b \circledcirc D_3(d)$, *where* $D_1(a) \circledcirc b = D_1(a) \cdot (b \cdot 1)$.

Proof. Suppose that $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X^* \times Y^*)$ then there are $d_1 : A \times_\alpha B \longrightarrow X^*$, $d_2 : A \times_\alpha B \longrightarrow Y^*$ such that D = (d_1, d_2), Set
$D_1 : A \longrightarrow X^*$; $D_1(a) = d_1(a, 0)$, $D_2 : B \longrightarrow Y^*$; $D_2(b) = d_2(0, b)$,
$D_3 : B \longrightarrow X^*$; $D_3(b) = d_1(0, b)$, $R : A \longrightarrow Y^*$; $R(a) = d_2(a, 0)$, now

$$\begin{aligned} D(a,b) &= (d_1,d_2)((a,0)+(0,b)) = (d_1,d_2)(a,0)+(d_1,d_2)(0,b) \\ &= (d_1(a,0),d_2(a,0)) + ((d_1(0,b),d_2(0,b)) \\ &= \Big(d_1(a,0)+d_1(0,b)\Big) + \Big(d_2(a,0)+d_2(0,b)\Big) \\ &= (D_1(a)+D_3(b), R(a)+D_2(b)), \end{aligned} \tag{2.1}$$

Since $D(a, 0) = D((a, 0)(1, 0)) = D(a, 0) \cdot (1, 0) + (a, 0) \cdot D(1, 0)$, we have $(D_1(a), R(a)) = (D_1(a), R(a)) \cdot (1, 0) + (a, 0) \cdot (D_1(1), R(1))$ thus $(D_1(a), R(a)) = (D_1(a) \cdot 1, 0)$ thus

$$R(a) = 0, \forall a \in A. \tag{2.2}$$

Comparing (2.1) and (2.2), we get $D(a, b) = (D_1(a) + D_3(b), D_2(b))$. Next

$$\begin{aligned} D((a,b)(m,n)) &= D(am + a \circ n + b \cdot m, bn) \\ &= (D_1(am) + D_1(a \circ n) + D_1(b \cdot m) + D_3(bn), D_2(bn)), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &(a,b) \cdot D(m,n) + D(a,b) \cdot (m,n) \\ =&(a,b) \cdot (D_1(m)+D_3(n), D_2(n))+(D_1(a)+D_3(b), D_2(b)) \cdot (m,n) \\ =&\Big(a \cdot (D_1(m)+D_3(n))+(b \cdot 1) \cdot (D_1(m)+D_3(n)), b \cdot D_2(n)\Big) \\ &+ \Big((D_1(a)+D_3(b)) \cdot m+(D_1(a)+D_3(b)) \cdot (n \cdot 1), D_2(b) \cdot n\Big) \\ =&\Big(a \cdot D_1(m) + D_1(a) \cdot m + a \cdot D_3(n) + D_3(b) \cdot m \\ &+ (b \cdot 1) \cdot D_1(m) + (b \cdot 1) \cdot D_3(n) \\ &+ D_1(a) \cdot (n \cdot 1) + D_3(b) \cdot (n \cdot 1), b \cdot D_2(n) + D_2(b) \cdot n\Big), \end{aligned} \tag{2.4}$$

By (2.3), (2.4),

$$
\begin{aligned}
&(D_1(am) + D_1(a \circ n) + D_1(b \cdot m) + D_3(bn), D_2(bn)) \\
&= \Big(a \cdot D_1(m) + D_1(a) \cdot m + a \cdot D_3(n) + D_3(b) \cdot m + (b \cdot 1) \cdot D_1(m) \\
&+ (b \cdot 1) \cdot D_3(n) + D_1(a) \cdot (n \cdot 1) + D_3(b) \cdot (n \cdot 1), b \cdot D_2(n) + D_2(b) \cdot n\Big) \\
&= \Big(a \cdot D_1(m) + D_1(a) \cdot m + a \cdot D_3(n) + D_3(b) \cdot m + b \circledcirc D_1(m) \\
&+ b \circledcirc D_3(n) + D_1(a) \circledcirc n + D_3(b) \circledcirc n, b \cdot D_2(n) + D_2(b) \cdot n\Big).
\end{aligned}
$$

Take a = n = 0 to get $D_1(b \cdot m) = D_3(b) \cdot m + b \circledcirc D_1(m)$, take b = m = 0 to get $D_1(a \circ n) = a \cdot D_3(n) + D_1(a) \circledcirc n$, and take a = m = 0 to get $(D_3(bn), D_2(bn)) = (b \circledcirc D_3(n) + D_3(b) \circledcirc n, b \cdot D_2(n) + D_2(b) \cdot n)$. Thus

$$
\begin{aligned}
D_3(bn) &= b \circledcirc D_3(n) + D_3(b) \circledcirc n \;\; i.e. \;\; D_3 \in Z(B, X^*), \\
D_2(bn) &= b \cdot D_2(n) + D_2(b) \cdot n \;\; i.e. \;\; D_2 \in Z(B, Y^*),
\end{aligned}
$$

Take b = n = 0 so $D_1(am) = a \cdot D_1(m) + D_1(a) \cdot m$ i.e. $D_1 \in Z(A, X^*)$. Also

$$D_1(\alpha \circ a) = d_1(\alpha \circ a, 0) = d_1(\alpha \cdot (a, 0)) = \alpha \cdot d_1(a, 0) = \alpha \cdot D_1(a).$$

Similarly $D_2(\alpha * b) = \alpha \cdot D_2(b)$, $D_3(\alpha * b) = \alpha \cdot D_3(b)$. Also since $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X^* \times Y^*)$, $D((a, b) + (m, n)) = D(a, b) + D(m, n)$ and $D(\alpha \cdot (a, b)) = \alpha \cdot D(a, b)$. Thus

$$
\begin{aligned}
D(\alpha \cdot (a, b)) &= D(\alpha \circ a, \alpha * b) = (d_1, d_2)(\alpha \circ a, \alpha * b) \\
&= (d_1(\alpha \circ a, \alpha * b), d_2(\alpha \circ a, \alpha * b)). \qquad (2.5)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\alpha \cdot D(a, b) &= \alpha \cdot ((d_1, d_2)(a, b)) = \alpha \cdot (d_1(a, b), d_2(a, b)) \\
&= (\alpha \cdot d_1(a, b), \alpha \cdot d_2(a, b)). \qquad (2.6)
\end{aligned}
$$

Comparing (2.5), (2.6), we get $d_1(\alpha \circ a, \alpha * b) = \alpha \cdot d_1(a, b)$, $d_2(\alpha \circ a, \alpha * b) = \alpha \cdot d_2(a, b)$. Also, $d_1((a, b) + (m, n)) = d_1(a, b) + d_1(m, n)$ and $d_2((a, b) + (m, n)) = d_2(a, b) + d_2(m, n)$. Hence

$$
\begin{aligned}
D_1(a + m) &= d_1(a + m, 0) = d_1((a, 0) + (m, 0)) \\
&= d_1(a, 0) + d_1(m, 0) = D_1(a) + D_1(m).
\end{aligned}
$$

And Similarly $D_2(b+n) = D_2(b)+D_2(n)$ and $D_3(b+d) = D_3(b)+D_3(d)$. Finally

$$\begin{aligned} D_1(\alpha \circ a) &= d_1(\alpha \circ a, 0) \\ &= d_1(\alpha \cdot (a, 0)) \\ &= \alpha \cdot d_1(a, 0) \\ &= \alpha \cdot D_1(a), \end{aligned}$$

and the same for D_2, D_3. Consequently $D_1 \in Z_{\mathfrak{U}}(A, X^*)$, $D_2 \in Z_{\mathfrak{U}}(B, Y^*)$, $D_3 \in Z_{\mathfrak{U}}(B, X^*)$. Conversely, Let $D_1 \in Z_{\mathfrak{U}}(A, X^*), D_2 \in Z_{\mathfrak{U}}(B, Y^*), D_3 \in Z_{\mathfrak{U}}(B, X^*)$ are as above

$$\begin{aligned} D((a, b)(m, n)) &= D(am + a \circ n + b \cdot m, bn) \\ &= (D_1(am) + D_1(a \circ n) + D_1(b \cdot m) + D_3(bn), D_2(bn)) \\ &= \Big(a \cdot D_1(m) + D_1(a) \cdot m + D_1(a) \circledcirc n \\ &+ a \cdot D_3(n) + D_3(b \cdot m + b \circledcirc D_1(m) \\ &+ b \circledcirc D_3(n) + D_3(b) \circledcirc n\ ,\ b \cdot D_2(n) + D_2(b) \cdot n\Big), \end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned} &(a, b) \cdot D(m, n) + D(a, b) \cdot (m, n) \\ &= (a, b) \cdot (D_1(m) + D_3(n), D_2(n)) \\ &+ (D_1(a) + D_3(b), D_2(b)) \cdot (m, n) \\ &= \Big(a \cdot D_1(m) + a \cdot D_3(n) + (b \cdot 1) \cdot D_1(m) + (b \cdot 1) \cdot D_3(n), b \cdot D_2(n)\Big) \\ &+ \Big(D_1(a) \cdot m + D_3(b) \cdot m + D_1(a) \cdot (n \cdot 1) + D_3(b) \cdot (n \cdot 1), D_2(b) \cdot n\Big) \\ &= \Big(a \cdot D_1(m) + D_1(a) \cdot m + D_1(a) \circledcirc n + a \cdot D_3(n) + D_3(b) \circledcirc n \\ &+ D_3(b) \cdot m + b \circledcirc D_1(m) + b \circledcirc D_3(n), b \cdot D_2(n) + D_2(b) \cdot n\Big). \end{aligned} \tag{2.8}$$

Comparing (2.7), (2.8), we get $D((a, b)(m, n)) = D(a, b) \cdot (m, n) + (a, b) \cdot D(m, n)$, for all $(a, b), (m, n) \in A \times_\alpha B$, thus

$$D \in Z(A \times_\alpha B, X^* \times Y^*). \tag{2.9}$$

$$D(\alpha\cdot(a,b)) = D(\alpha\circ a, \alpha * b) = (D_1(\alpha\circ a) + D_3(\alpha * b), D_2(\alpha * b))$$
$$= (\alpha\cdot D_1(a) + \alpha\cdot D_3(b), \alpha\cdot D_2(b))$$
$$= \alpha\cdot(D_1(a) + D_3(b), D_2(b)) = \alpha.D(a,b). \tag{2.10}$$

and

$$D((a,b)+(m,n)) = D(a+m, b+n)$$
$$= (D_1(a+m) + D_3(b+n), D_2(b+n))$$
$$= (D_1(a) + D_1(m) + D_3(b) + D_3(n), D_2(b) + D_2(n))$$
$$= (D_1(a) + D_3(b), D_2(b)) + (D_1(m) + D_3(n), D_2(n))$$
$$= D(a,b) + D(m,n) \tag{2.11}$$

Comparing (2.9), (2.10), (2.11), we get $D \in Z_{\mathfrak{U}}(A\times_\alpha B, X^*\times Y^*)$ □

Proposition 2.5. *With the above notation,* $D = \delta_{(\varphi,\psi)}(\varphi\in X^*, \psi\in Y^*)$ *if and only if* $(D_1 = \delta_\varphi,\ D_2 = \delta_\psi,\ D_3 = \overline{\delta}_\varphi)$*, where* $\overline{\delta}_\varphi(b) = \varphi\circledcirc b - b\circledcirc\varphi = \varphi\cdot(b\cdot 1) - (b\cdot 1)\cdot\varphi$.

Proof. Let $D \in N_{\mathfrak{U}}(A\times_\alpha B, X^*\times Y^*)$. By the by previous Proposition, there exists D_1, D_2, D_3 as above such that $D(a,b) = (D_1(a) + D_3(b), D_2(b))$, also

$$D(a,b) = \delta_{(\varphi,\psi)}(a,b) = (\varphi,\psi)\cdot(a,b) - (a,b)\cdot(\varphi,\psi)$$
$$= (\varphi\cdot a + \varphi\circledcirc b, \psi\cdot b) - (a\cdot\varphi + b\circledcirc\varphi, b\cdot\psi)$$
$$= (\varphi\cdot a - a\cdot\varphi + \varphi\circledcirc b - b\circledcirc\varphi, \psi\cdot b - b\cdot\psi)$$

Take $b = 0$, $D_1(a) = \varphi\cdot a - a\cdot\varphi$, for all $a\in A$ to get $D_1 = \delta_\varphi$. Take $a = 0$, $(D_3(b), D_2(b)) = D(0,b) = (\varphi\circledcirc b - b\circledcirc\varphi, \psi\cdot b - b\cdot\psi)$ to get $D_2 = \delta_\psi, D_3 = \overline{\delta}_\varphi$. Let $D(a,b)= (D_1(a) + D_3(b), D_2(b))$ and $(D_1 = \delta_\varphi$, $D_2 = \delta_\psi$, $D_3 = \overline{\delta}_\varphi)$, then

$$D(a,b) = (D_1(a) + D_3(b), D_2(b)) = (\delta_\varphi(a) + \overline{\delta}_\varphi(b), \delta_\psi(b))$$
$$= (\varphi\cdot a - a\cdot\varphi + \varphi\circledcirc b - b\circledcirc\varphi, \psi\cdot b - b\cdot\psi)$$
$$= (\varphi,\psi)\cdot(a,b) - (a,b)\cdot(\varphi,\psi) = \delta_{(\varphi,\psi)}(a,b).$$

□

3. Module amenability

A is called module amenable (as an $\mathfrak{U}$-bimodule), if for any Banach space X which is at the same time a Banach A-module and a Banach $\mathfrak{U}$-module with compatible actions

$(a\cdot x)\cdot\alpha = a\cdot(x\cdot\alpha), \alpha\cdot(a\cdot x) = (\alpha\cdot a)\cdot x, (\alpha\in\mathfrak{U}, a\in A, x\in X)$, and the same for the other side actions, and each module derivation $D : A \longrightarrow X^*$, there is an $x \in X^*$ such that $D(a) = a\cdot x - x\cdot a = \delta_x(a)$,($a \in A$).

Theorem 3.1. *If $A \times_\alpha B$ is module amenable then B is module amenable. In addition if A is unital and symmetric(as B-bimodule) then A is also module amenable.*

Proof. Assume that Y is a Banach B-$\mathfrak{U}$ module and d $\in Z_{\mathfrak{U}}(B, Y^*)$, $\{0\}\times Y^*$ is an $A\times_\alpha B$-$\mathfrak{U}$-module. Define $D : A\times_\alpha B \longrightarrow \{0\}\times Y^*$; $D(a, b) = (0, d(b))$. Clearly $D \in Z_{\mathfrak{U}}(A\times_\alpha B,\{0\}\times Y^*)$. Now since $A\times_\alpha B$ is module amenable there is $(0, \psi) \in \{0\}\times Y^*$ such that $D = \delta_{(0,\psi)}$ thus $d = \delta_\psi$.
Now assume that A is unital and symmetric as a B-module. Let X be a Banach A-$\mathfrak{U}$-module and Y be a Banach B-$\mathfrak{U}$-module, by Proposition 2.3, $X \times Y$ is a Banach $A \times_\alpha B$-$\mathfrak{U}$-module. Let $D_1 \in Z_{\mathfrak{U}}(A, X^*)$, $D_2 \in Z_{\mathfrak{U}}(B, Y^*)$. Define $D_3 : B \longrightarrow X^*$ b; $D_3(b) = D_1(b\cdot 1)$ then

$$\begin{aligned} D_3(bn) &= D_1(bn\cdot 1) = D_1((b\cdot 1)(n\cdot 1)) \\ &= D_1(b\cdot 1)\cdot(n\cdot 1) + (b\cdot 1)\cdot D_1(n\cdot 1) \\ &= D_3(b)\circledcirc n + b\circledcirc D_3(n). \end{aligned} \tag{3.1}$$

Note that X is a B-bimodule with module multiplications as $X\times B \longrightarrow X$; $x \triangleright b = x\circ(b\cdot 1)$, and $B\times X \longrightarrow X$; $b\triangleleft x = (b\cdot 1).x$. Also

$$\begin{aligned} D_1(a\circ b) &= D_1(a(1\circ b)) = D_1(a)\cdot(1\circ b) + a\cdot D_1(1\circ b) \\ &= D_1(a)\circledcirc b + a\cdot D_3(b), \end{aligned} \tag{3.2}$$

$$\begin{aligned} D_1(b\cdot m) &= D_1((b\cdot 1)m) = D_1(b\cdot 1)\cdot m + (b\cdot 1)\cdot D_1(m) \\ &= D_3(b)\cdot m + b\circledcirc D_1(m). \end{aligned} \tag{3.3}$$

Now Define D: $A\times_\alpha B \longrightarrow X^*\times Y^*$; $D(a, b) = (D_1(a) + D_3(b), D_2(b))$. Since $D_1 \in Z_{\mathfrak{U}}(A, X^*)$, $D_3 \in Z_{\mathfrak{U}}(B, X^*)$, also by assumption $D_2 \in Z_{\mathfrak{U}}(B, Y^*)$ and by (3.1), (3.2), (3.3) the conditions of Proposition 2.4 are satiesfed so $D \in Z_{\mathfrak{U}}(A\times_\alpha B, X^*\times Y^*)$. Since $A\times_\alpha B$ is module amenable, D is inner so there is $(\varphi, \psi) \in X^*\times Y^*$ such that $D = \delta_{(\varphi,\psi)}$. Thus by Proposition 2.5, $D_1 = \delta_\varphi$, $D_2 = \delta_\psi$. This means that A, B are module amenable. □

Theorem 3.2. *Let A be unital and symmetric(as B-bimodule). If A, B are module amenable then $A\times_\alpha B$ is module amenable.*

Proof. Suppose that $X \times Y$ is a Banach $A \times_\alpha B$-$\mathfrak{U}$-module and $D \in Z_{\mathfrak{U}}(A\times_\alpha B, X^*\times Y^*)$. It is easy to check that X is a Banach A-$\mathfrak{U}$-module and Y is B-$\mathfrak{U}$-module with module multiplications $X \times A \longrightarrow X$ defined by $x \cdot a = q_X((x, 0) \cdot (a, 0))$. $A \times X \longrightarrow X$; $a \cdot x = q_X((a, 0) \cdot (x, 0))$, $Y\times B \longrightarrow Y$; $y\cdot b = q_Y((0, y)\cdot(0, b))$, $B\times Y \longrightarrow Y$; $b\cdot y = q_Y((0, b)\cdot(0, y))$, where $q_X : X \times Y \longrightarrow X$; $q_X(x, y) = x$, and $q_Y : X \times Y \longrightarrow Y$; $q_Y(x, y) = y$, and $X\times\mathfrak{U} \longrightarrow X$; $x\circ\alpha = q_X((x, 0)\cdot\alpha)$, $\mathfrak{U}\times X \longrightarrow X$; $\alpha\cdot x = q_X(\alpha \cdot (x, 0))$, $Y \times \mathfrak{U} \longrightarrow Y$; $y\nabla\alpha = q_Y((0, y) \cdot \alpha)$, $\mathfrak{U} \times Y \longrightarrow Y$; $\alpha \triangle y = q_Y(\alpha \cdot (0, y))$ with compatible actions. Now Since $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X^* \times Y^*)$, by Proposition 2.4, there are $D_1 \in Z_{\mathfrak{U}}(A, X^*)$, $D_3 \in Z_{\mathfrak{U}}(B, X^*)$, $D_2 \in Z_{\mathfrak{U}}(B, Y^*)$ such that $D(a, b) = (D_1(a) + D_3(b), D_2(b))$. Since $D_1 \in Z_{\mathfrak{U}}(A, X^*)$ and A is module amenable there is $\varphi \in X^*$ such that $D_1 = \delta_\varphi$, also since $D_2 \in Z_{\mathfrak{U}}(B, Y^*)$ and B is module amenable there is $\psi \in Y^*$ such that $D_2 = \delta_\psi$. Since $D_1(b\cdot 1) = b \circledcirc D_1(1) + D_3(b)\cdot 1$, hence

$$\delta_\varphi(b \cdot 1) = D_3(b),$$

thus

$$\varphi \cdot (b \cdot 1) - (b \cdot 1) \cdot \varphi = D_3(b) \;\; i.e., \;\; D_3 = \overline{\delta}_\varphi.$$

By Proposition 2.5, $D = \delta_{(\varphi,\psi)}$. □

Example 3.3. Let $\mathbb{N}$ be the set of positive integers. Consider $S = (\mathbb{N}, \vee)$ with the maximum operation $m \vee n = \max\{m, n\}$, then S is a amenable countable, abelian inverse semigroup with the identity 1. Clearly $E_S = S$. This semigroup is denoted by $\mathbb{N}_\vee$. $l^1(\mathbb{N}_\vee)$ is unital with unit δ_1. Since $\mathbb{N}_\vee$ is amenable and $l^1(\mathbb{N}_\vee)$ is unital so $l^1(\mathbb{N}_\vee)$ is module amenable (as an $l^1(\mathbb{N}_\vee)$-$l^1(\mathbb{N}_\vee)$)-bimodule. Also $l^1(\mathbb{N}_\vee)$ is symmetric $l^1(\mathbb{N}_\vee)$-module. Hence $l^1(\mathbb{N}_\vee) \times_\alpha l^1(\mathbb{N}_\vee)$ is module amenable.

Example 3.4. Let S be an inverse semigroup with the set of idempotents $E_S = S$ such that $l^1(S)$ is unital and symmetric $l^1(E)$- bimodule with the multiplication, as the right action and the trivial left action. Then $l^1(S) \times_\alpha l^1(S)$ is module amenable if and only if $l^1(S)$ is module amenable. Also by [1, theorem 3.1], $l^1(S)$ is module amenable if and only if S is amenable. Thus for such S, $l^1(S) \times_\alpha l^1(S)$ is module amenable if and only if S is module amenable.

4. Weak module amenability

The Banach algebra A is called weak module amenable (as an $\mathfrak{U}$-bimodule), if $H_{\mathfrak{U}}(A, X) = \{0\}$, where X is a commutative $\mathfrak{U}$-submodule of A^*([2]). Let X be a commutative $\mathfrak{U}$-submodule of A^*, Y be a commutative $\mathfrak{U}$-submodule of B^* then $X \times Y$ is commutative $\mathfrak{U}$-submodule of $(A \times_\alpha B)^*$.

Theorem 4.1. *The weak module amenability of $A \times_\alpha B$ implies weak module amenability of B. In addition if A is unital and symmetric(as B-bimodule) then A is also weak module amenable.*

Proof. Assume that X is a commutative $\mathfrak{U}$-submodule of A^* and Y is a commutative $\mathfrak{U}$-submodule of B^* then by the above observation $X \times Y$ is a commutative $\mathfrak{U}$-submodule of $(A \times_\alpha B)^*$. Also $\{0\} \times Y$ is commutative $\mathfrak{U}$-submodule of $(A \times_\alpha B)^*$. Let $D_1 \in Z_{\mathfrak{U}}(A, X)$ and $D_2 \in Z_{\mathfrak{U}}(B, Y)$, and define $D : A \times_\alpha B \longrightarrow \{0\} \times Y$; $D(a, b) = (0, D_2(b))$, it is easy to see that $D \in Z_{\mathfrak{U}}(A \times_\alpha B, \{0\} \times Y)$. Since $A \times_\alpha B$ is weak module amenable there is $(0, \psi) \in \{0\} \times Y$ such that $D = \delta_{(0,\psi)}$ and

$$\begin{aligned}(0, D_2(b)) = D(a, b) = \delta_{(0,\psi)}(a, b) &= (0, \psi) \cdot (a, b) - (a, b) \cdot (0, \psi) \\ &= (0, \psi \cdot b) - (0, b \cdot \psi) = (0, \delta_\psi(b)).\end{aligned}$$

Thus $D_2 = \delta_\psi$ and B is weak module amenable. Now assume that A is unital and symmetric B-module. Define $D_3 : B \longrightarrow X$; $D_3(b) = D_1(b \cdot 1)$ then $D_3 \in Z_{\mathfrak{U}}(B, X)$, D_1, D_2, D_3 satisfy in properties of proposition 2.4. Define $D : A \times_\alpha B \longrightarrow X \times Y$; $D(a, b) = (D_1(a) + D_3(b), D_2(b))$, then by an argument as in the proof of Proposition 2.4, $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X \times Y)$. Since $A \times_\alpha B$ is weak module amenable, there is $(\varphi, \psi) \in X \times Y$ such that $D = \delta_{(\varphi,\psi)}$. Thus by Proposition 2.5, $D_1 = \delta_\varphi$, $D_2 = \delta_\psi$, hence A, B are weak module amenable. □

Theorem 4.2. *If A is unital and symmetric B-bimodule then the weak module amenability of A, and B imply the weak module amenability of $A \times_\alpha B$.*

Proof. Suppose that $X \times Y$ is a commutative Banach $\mathfrak{U}$-submodule of $(A \times_\alpha B)^*$, and $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X \times Y)$. Then X is a commutative $\mathfrak{U}$-submodule of A^* and Y is a commutative $\mathfrak{U}$-submodule of B^*. Since $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X \times Y)$, by Proposition 2.4 there are $D_1 \in Z_{\mathfrak{U}}(A, X)$, and $D_2 \in Z_{\mathfrak{U}}(B, Y)$, and $D_3 \in Z_{\mathfrak{U}}(B, X)$ such that $D(a, b) = (D_1(a) + D_3(b), D_2(b))$. Since $D_1 \in Z_{\mathfrak{U}}(A, X)$, $D_2 \in Z_{\mathfrak{U}}(B, Y)$ and A, B are weak module amenable so there are $\varphi \in X$ and $\psi \in Y$ such that $D_1 = \delta_\varphi$, $D_2 = \delta_\psi$. Since $D_1(b \cdot 1) = b \circledcirc D_1(1) + D_3(b) \cdot 1$, we have $D_3 = \overline{\delta}_\varphi$. Thus $D = \delta_{(\varphi,\psi)}$, hence $A \times_\alpha B$ is module amenable. □

Example 4.3. Let $S = \mathbb{N}_\vee$ be as in Example 3.3, since $l^1(S)$ is $l^1(S)$-$l^1(S)$-module and $l^1(S)$ is weak module amenable, $l^1(S) \times_\alpha l^1(S)$ is weak module amenable.

5. Module approximate amenability

Let A be as above, then A is module approximately amenable (as an $\mathfrak{U}$-bimodule), if for any commutative Banach A-$\mathfrak{U}$-bimodule X, each module derivation $D : A \longrightarrow X^*$ is approximately inner.
A derivation $D : A \longrightarrow X$ is said to be approximately inner if there exists a net $(x_i)_i \subseteq X$ such that $D(a) = \lim_i(a \cdot x_i - x_i \cdot a)$, $a \in A$.([15]).

Lemma 5.1. *If A is unital and symmetric B-bimodule, and D_1, D_2, D_3 are such as in the Proposition 2.4, and $D(a, b) = (D_1(a) + D_3(b), D_2(b))$ then D is approximately inner if and only if D_1, D_2 are approximately inner.*

Proof. Assume that X is a commutative A-$\mathfrak{U}$-bimodule and also Y be commutative B-$\mathfrak{U}$-bimodule then $X \times Y$ is a commutative $A \times_\alpha B$-$\mathfrak{U}$-bimodule. Let D be approximately inner there is $(x_i, y_i)_i \subseteq X^* \times Y^*$ such that

$$\begin{aligned} D(a,b) &= \lim_i \Big((a,b) \cdot (x_i, y_i) - (x_i, y_i) \cdot (a,b) \Big) \\ &= \lim_i \Big((a \cdot x_i + (b \cdot 1) \cdot x_i, b \bullet y_i) - (x_i \circ a + x_i \circ (b \cdot 1), y_i \centerdot b) \Big) \\ &= \lim_i (a \cdot x_i - x_i \circ a + (b \cdot 1) \circ x_i - x_i \circ (b \bullet 1), b \bullet y_i - y_i \centerdot b) \end{aligned}$$

Since $D(a, b) = (D_1(a)+D_3(b), D_2(b))$. Take $a = 0$, to get $(D_3(b), D_2(b))$ $= \Big(\lim_i((b \cdot 1) \cdot x_i - x_i \cdot (b \cdot 1)), \lim_i(b \cdot y_i - y_i \cdot b) \Big)$ hence D_3, D_2 are approximately inner. Take $b = 0$, to get $D_1(a) = \lim_i(a \cdot x_i - x_i \cdot a)$ so D_1 is approximately inner. Let D_1, D_2 are approximately inner, there are $(x_i)_{i \in I} \subseteq X^*, (y_j)_{j \in J} \subseteq Y^*$ such that $D_1(a) = \lim_i(a \cdot x_i - x_i \circ a)$, $D_2(b) = \lim_j(b \bullet y_j - y_j \centerdot b)$. Also Since $D_1(1 \circ b) = D_1(1) \circledcirc b + 1 \cdot D_3(b)$, thus

$$\begin{aligned} D_3(b) &= \lim_i \Big((1 \circ b) \cdot x_i - x_i \circ (1 \circ b) \Big) \\ &= \lim_i (b \circledcirc x_i - x_i \circledcirc b). \end{aligned}$$

Since the index sets $(I, \leq)$,$(J, \leq)$ are ordered sets, so the set $\Lambda = I \times J = \{(i, j) : i \in I, j \in J\}$ is ordered as follows

$$(i, j) \leq (i', j') \Longleftrightarrow (i \leq i', j \leq j').$$

For $\lambda = (i, j) \in \Lambda$ set $t_\lambda = (x_i, y_j)$. Let $\epsilon > 0$ be given. Since $D_1(a) = \lim_i(a \cdot x_i - x_i \circ a)$, $D_2(b) = \lim_j(b \bullet y_j - y_j \centerdot b)$ and $D_3(b) = \lim_i(b \circledcirc$

$x_i - x_i \circledcirc b)$, there are $i_0 \in I$, $j_0 \in J$ such that

1) For all $i \geq i_0$, $\|D_1(a) - (a \cdot x_i - x_i \circ a)\| \leq \frac{\varepsilon}{3}$ and $\|D_3(b) - (b \circledcirc x_i - x_i \circledcirc b\| \leq \frac{\varepsilon}{3}$.

2) For all $j \geq j_0$, $\|D_2(b) - (b \bullet y_j - y_j \centerdot b)\| \leq \frac{\varepsilon}{3}$.

Now set $\lambda_0 = (i_0, j_0)$, then for all $\lambda \geq \lambda_0$ we have

$$\begin{aligned}
&\|D(a,b) - ((a,b) \cdot t_\lambda - t_\lambda \cdot (a,b))\| \\
&= \|D(a,b) - ((a,b) \cdot (x_i, y_j) - (x_i, y_j) \cdot (a,b))\| \\
&= \|D(a,b) - \big(a \cdot x_i - x_i \circ a + b \circledcirc x_i - x_i \circledcirc b, b \bullet y_j - y_j \centerdot b\big) \\
&= \|(D_1(a) + D_3(b), D_2(b)) - (a \cdot x_i - x_i \circ a + b \circledcirc x_i - x_i \circledcirc b, b \bullet y_j - y_j \centerdot b) \\
&= \|\big((D_1(a) - (a \cdot x_i - x_i \circ a)) \\
&+ (D_3(b) - (b \circledcirc x_i - x_i \circledcirc b)), (D_2(b) - (b \bullet y_j - y_j \centerdot b))\big) \\
&\leq \|D_1(a) - (a \cdot x_i - x_i \circ a)\| + \|D_3(b) - (b \circledcirc x_i - x_i \circledcirc b)\| \\
&+ \|D_2(b) - (b \bullet y_j - y_j \centerdot b)\| \\
&< \varepsilon.
\end{aligned}$$

Hence $D(a,b) = \lim_\lambda((a,b) \cdot t_\lambda - t_\lambda \cdot (a,b))$, i.e. D is approximately inner. □

Theorem 5.2. *If $A \times_\alpha B$ is module approximately amenable then B is module approximately amenable. In addition if A is unital and symmetric B-bimodule also A is module approximately amenable.*

Proof. In an argument as in the proof of Theorem 3.1 and use of above lemma. □

Theorem 5.3. *If A is unital and symmetric(as B-module) then the module approximate amenability of A, B implies the module approximate amenability of $A \times_\alpha B$.*

Proof. Let $X \times Y$ be a commutative $A \times_\alpha B$-$\mathfrak{U}$-bimodule and $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X^* \times Y^*)$. Since $X \times Y$ is commutative $A \times_\alpha B$-$\mathfrak{U}$-bimodule, X is a commutative A-$\mathfrak{U}$-bimodule and Y is a commutative B-$\mathfrak{U}$-module. Since $D \in Z_{\mathfrak{U}}(A \times_\alpha B, X^* \times Y^*)$, by Proposition 2.4, there are $D_1 \in Z_{\mathfrak{U}}(A, X^*)$, $D_2 \in Z_{\mathfrak{U}}(B, Y^*)$, $D_3 \in Z_{\mathfrak{U}}(B, X^*)$ such that $D(a,b) = (D_1(a) + D_3(b), D_2(b))$. Also since A, B are module approximate amenable, D_1, D_2 are approximatly inner, thus by the above lemma, D is approximately inner. □

Example 5.4. Let S be an amenable inverse semigroup such that $l^1(S)$ be unital, let the set of idempotents E_S be equal to S and $l^1(S)$ be $l^1(E_S)$ symmetric bimodule. Since S is amenable, $l^1(S)$ is module approximately amenable, [15]. Also $l^1(S)$ is $l^1(S)$-$l^1(S)$-bimodule, thus $l^1(S) \times_\alpha l^1(S)$ is module approximately amenable.

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