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A NEW CONSTRUCTION OF TIMELIKE RULED SURFACES WITH CONSTANT DISTELI-AXIS

Rashad A. Abdel-Baky and Yasin Ünlütürk*

Abstract. In this study, we construct timelike ruled surfaces whose Disteli-axis is constant in Minkowski 3-space $\mathbb{E}_1^3$. Then we attain a general system characterizing these surfaces, and also give necessary and sufficient conditions for a timelike ruled surface to get a constant Disteli-axis.

1. Introduction

The motion of a straight line on a curve establishes ruled surface. These lines are said to be (rectilinear) generators, and every curve that intersects all the generators is called a directrix (base curve). Despite its long history, the ruled surface has been highlighted by researchers as well as mathematicians because it has numerous applications such as problems of designs in spatial mechanisms and physics, kinematics and computer aided design (CAD) [4, 12, 23].

Differential geometry of ruled surfaces has got fruitful conclusions in Minkowski 3-space $\mathbb{E}_1^3$. Since Lorentzian metric is not only positive definite metric, the function $\langle , \rangle$ can occur as positive, negative or zero, whereas the distance function is only positive in the Euclidean space. For instance, a continuously moving of a timelike line throughout a curve generates a timelike ruled surface. Turgut and Hacısalihoğlu have researched timelike ruled surfaces of Minkowski 3-space and presented some features of these surfaces [20]. Hassan and others have examined timelike ruled surfaces with timelike rulings [10]. Küçük has reached certain conclusions on the developable timelike ruled surfaces of Minkowski 3-space [13].

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*Corresponding author

Examining the motion of line space has established a concern between this space and dual numbers. All oriented lines in Euclidean 3-space $\mathbb{E}^3$ are straightly associated to points on the dual unit sphere in the dual 3-space $\mathbb{D}^3$ with respect to the E. Study map in screw and dual number algebra. More information on fundamental definitions of the dual elements and the bond between ruled surfaces and dual spherical motions of one parameter is able to be seen in [5, 11, 15, 18].

Some studies in the literature that may be included in the scope of our study are as follows: Disteli analyzed a pair of meshing skew gears whose tooth flanks are ruled surfaces in contact along lines, also used screw theory to examine the motion, and established a diagram giving the relation between the angular velocities of a pair of skew gears and the position and pitch of their relative screw motions [6, 7, 8]. Abdel-Baky and Al-Ghefari presented an exact formulation for the central problem of screw theory, namely, the determination of the principal screws of given relative screw motions [1]. Önder and Uğurlu obtained the Frenet frames and Frenet invariants of timelike ruled surfaces, and proved that a timelike ruled surface and its directing cone have the same base of Frenet frame [16]. Ekinci and Uğurlu introduced Disteli diagram for the hyperbolic motion in the Lorentzian space and found the relations between the angular velocities of pair skew gears and the position and pitch of their relative screw motion in the Lorentzian space via using this diagram [9]. Yapar and Sağıroğlu studied the relations among the dual integral invariants of the timelike ruled surfaces drawn by the timelike vectors under a motion of a line along two closed ruled surfaces in dual Lorentzian 3-space [21]. Bilici examined the ruled surfaces generated by a Frenet trihedron of closed dual involute for a given dual curve by a strongly connected dual angle between the dual binormal vector and dual Darboux vector of this dual base curve and presented some properties of these surfaces [3].

In this work, by using the E. Study map, we examine timelike ruled surfaces with constant Disteli-axis based upon the curvature theory of a dual hyperbolic (resp. Lorentzian) spherical curve which matches in a timelike ruled surface in Minkowski 3-space $\mathbb{E}^3_1$. In terms of this reference frame, we construct timelike ruled surfaces with constant Disteli-axis and give some examples of these surfaces with their figures.

2. Basic concepts

Let $\mathbb{R}^3$ mark the real vector space with its usual vector structure. Let (x_1, x_2, x_3) denote the coordinates of a vector via the standard basis of $\mathbb{R}^3$.

The Minkowski 3-space is the metric space $\mathbb{E}_1^3 = (\mathbb{R}^3, \langle , \rangle)$, where the Lorentzian metric $\langle , \rangle$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ [14]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\mathbb{E}_1^3$, the Lorentzian vector product is ([14]) defined by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} -i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \tag{2}$$

A vector $\mathbf{x} \in \mathbb{E}_1^3$ is said to be spacelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and lightlike or null if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. The zero vector $\mathbf{x} = \mathbf{0}$ is considered of spacelike one although it satisfies $\langle \mathbf{x}, \mathbf{x} \rangle = 0$. For $\mathbf{x} \in \mathbb{E}_1^3$ the norm is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$ and a timelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ becomes locally spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [17].

The hyperbolic and Lorentzian unit spheres are defined as

$$\mathbf{h}_+^2 = \{x \in \mathbb{E}_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = -1, \ x_1 > 0\}, \tag{3}$$

and

$$\mathbf{s}_1^2 = \{x \in \mathbb{E}_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = 1\} \ . \tag{4}$$

respectively [17].

2.1. E. Study map

The set of dual numbers is

$$\mathbb{D} = \{X = x + \varepsilon x^* \mid \ x, \ x^* \in \mathbb{R}\}, \tag{5}$$

where $\varepsilon \neq 0$ is said to be the dual operator with the algebraic feature of $\varepsilon^2 = 0$. The sums and product of dual numbers are well defined by utilizing the dual operator. In the same manner, for all pairs $(\mathbf{x}, \mathbf{x}^*) \in \mathbb{E}_1^3 \times \mathbb{E}_1^3$, the set

$$\mathbb{D}_1^3 = \{\mathbf{X} = \mathbf{x} + \varepsilon \mathbf{x}^*,\ \varepsilon \neq 0,\ \varepsilon^2 = 0\}, \tag{6}$$

together with the Lorentzian inner product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \varepsilon(\langle \mathbf{y}, \mathbf{x}^* \rangle + \langle \mathbf{y}^*, \mathbf{x} \rangle), \tag{7}$$

composes the dual Lorentzian 3-space $\mathbb{D}_1^3$ [22]. So that dual coordiantes of a point $\mathbf{X} = (X_1, X_2, X_3)^t$ are as in $X_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$.

The norm is defined by

$$\|\mathbf{X}\| = \|\mathbf{x}\| \, (1 + \varepsilon \frac{\langle \mathbf{x}, \mathbf{x}^* \rangle}{\|\mathbf{x}\|^2}). \tag{8}$$

The hyperbolic and Lorentzian dual unit spheres, respectively, are ([22]):

$$\mathbb{H}_+^2 = \{\mathbf{X} \in \mathbb{D}_1^3 \mid -X_1^2 + X_2^2 + X_3^2 = -1,\ X_1 > 0\}, \tag{9}$$

and

$$\mathbb{S}_1^2 = \{\mathbf{X} \in \mathbb{D}_1^3 \mid -X_1^2 + X_2^2 + X_3^2 = 1\}. \tag{10}$$

This yields

$$\mathbf{F}_1 \times \mathbf{F}_2 = \mathbf{F}_3, \quad \mathbf{F}_2 \times \mathbf{F}_3 = -\mathbf{F}_1, \quad \mathbf{F}_3 \times \mathbf{F}_1 = \mathbf{F}_2, \tag{11}$$

where the base vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, are at the origin point $\mathbf{0}\,(0,0,0)$ of the space $\mathbb{D}_1^3$.

In the Minkowski 3-space $\mathbb{E}_1^3$, the statement of the E. Study's map as follows: Dual unit vectors of dual unit hyperbolic and Lorentzian spheres $\mathbb{H}_+^2$ and $\mathbb{S}_1^2$ of the Minkowski 3-space $\mathbb{D}_1^3$ are injective relation with the directed lines of the space of Lorentzian lines $\mathbb{E}_1^3$, respectively [22]. Thus a differentiable curve of $\mathbb{H}_+^2$ matches in a timelike ruled surface at $\mathbb{E}_1^3$. Similarly a timelike (resp. spacelike) curve of $\mathbb{S}_1^2$ comes to any spacelike (resp. timelike) ruled surface at $\mathbb{E}_1^3$. Hence, the E. Study map is given as: the dual unit spheres are constructed as a conjugate hyperboloids' pair. The common asymptotic cone means null lines, the ring shaped hyperboloid performs spacelike lines, and the oval shaped hyperboloid produces timelike lines, opposite points of each hyperboloid represent the pair of opposite vectors on a line, see Fig. 1.

Definition 2.1. *Let us deal with two non-null dual vectors* $\mathbf{A} = (A_1, A_2, A_3)^t$ *and* $\mathbf{B} = (B_1, B_2, B_3)^t$ *in* $\mathbb{D}_1^3$*:*
i) Spacelike dual angle: Let us presume that $\mathbf{A}$ *and* $\mathbf{B}$ *are spacelike dual vectors, then:*

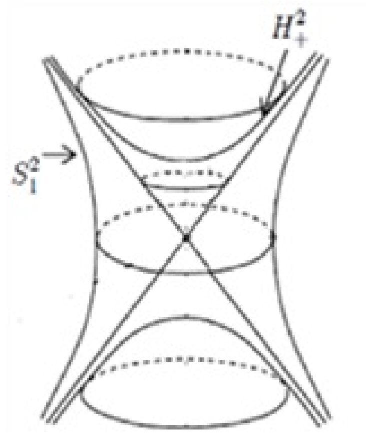

FIGURE 1. The dual Lorentzian S_1^2 and hyperbolic H_+^2 unit spheres

• *If they span a spacelike dual plane; there is a unique dual number $\Theta = \vartheta + \varepsilon\vartheta^*$; $0 \leq \vartheta \leq \pi$, and $\vartheta^* \in \mathbb{R}$ such that $\langle \mathbf{A}, \mathbf{B} \rangle = \|\mathbf{A}\| \|\mathbf{B}\| \cos\Theta$. This number is called the spacelike dual angle between the dual vectors* $\mathbf{A}$ *and* $\mathbf{B}$.

• *If they span a timelike dual plane; there exists a unique real number dual number $\Theta = \vartheta + \varepsilon\vartheta^* \geq 0$ such that $\langle \mathbf{A}, \mathbf{B} \rangle = \|\mathbf{A}\| \|\mathbf{B}\| \cosh\Theta$. This number is said to be the hyperbolic dual angle between the dual vectors* $\mathbf{A}$ *and* $\mathbf{B}$.

ii) Central dual angle: Let us consider that $\mathbf{A}$ *and* $\mathbf{B}$ *are timelike dual vectors, then there exists a unique dual number $\Theta = \vartheta + \varepsilon\vartheta^* \geq 0$ such that $\langle \mathbf{A}, \mathbf{B} \rangle = \|\mathbf{A}\| \|\mathbf{B}\| \cosh\Theta$. This number is called the central dual angle between the vectors* $\mathbf{A}$ *and* $\mathbf{B}$.

iii) Lorentzian timelike dual angle: Let us accept that $\mathbf{A}$ *is spacelike and* $\mathbf{B}$ *is timelike, then there exists a unique dual number $\Theta = \vartheta + \varepsilon\vartheta^* \geq 0$ such that $\langle \mathbf{A}, \mathbf{B} \rangle = \|\mathbf{A}\| \|\mathbf{B}\| \sinh\Theta$. This is said to be the Lorentzian timelike dual angle between the dual vectors* $\mathbf{A}$ *and* $\mathbf{B}$ [19].

2.2. Timelike ruled surfaces in Minkowski 3-space

Let $\boldsymbol{\alpha}(u)$ be a curve in the Minkowski 3-space $\mathbb{E}_1^3$ defined on $I \subseteq \mathbb{R}$, and $\mathbf{x} = \mathbf{x}(u)$ be a unit ruling vector of an oriented line in $\mathbb{E}_1^3$. Then we acquire a timelike ruled surface's parametrization (X) as

$$(X) : \mathbf{y}(u, v) = \boldsymbol{\alpha}(u) + v\mathbf{x}(u), \ v \in \mathbb{R}. \tag{12}$$

Here $\boldsymbol{\alpha}(u)$ is a non-null space curve which is said to be the directrix, $\mathbf{x}(u)$ is a curve on the hyperbolic unit sphere h_+^2 (or s_1^2), and u is the motion parameter. Omitting $\mathbf{x}$ is constant or null or $\dfrac{d\mathbf{x}}{du}$ null, the distribution parameter for a ruled surface describes the winding speed of the tangent planes winding about the ruling. It can be determined by

$$\mu(u) = \frac{\det\left(\dfrac{d\boldsymbol{\alpha}}{du} \times \mathbf{x}, \dfrac{d\mathbf{x}}{du}\right)}{\left\|\dfrac{d\mathbf{x}}{du}\right\|^2} \tag{13}$$

which only depends on u. The directrix is not unique, since any curve of the pattern

$$\mathbf{C}(u) = \boldsymbol{\alpha}(u) - \eta(u)\mathbf{x}(u) \tag{14}$$

is used as its directrix, $\eta(s)$ is a differentiable function. If a common perpendicular to two neighboring rulings occurs on (X), then the common perpendicular's foot on the main ruling is said to be a central point. The locus of the central points is said to be the striction curve. In Eq. (14) if

$$\eta(u) = \frac{\left\langle \dfrac{d\mathbf{x}}{du}, \dfrac{d\boldsymbol{\alpha}}{du} \right\rangle}{\left\|\dfrac{d\mathbf{x}}{du}\right\|^2}, \tag{15}$$

then $\mathbf{C}(u)$ is said to be the striction curve on the ruled surface and it is unique. In the case of $\eta = 0$ the directrix is the striction curve [20].

2.3. Ruled surface as a curve on $\mathbb{H}_+^2$(or $\mathbb{S}_1^2$)

The dual version for equation (12) is rewritten by means of E. Study map as

$$\mathbf{X}(u) = \mathbf{x}(u) + \varepsilon\boldsymbol{\alpha} \times \mathbf{x}(u) = \mathbf{x}(u) + \varepsilon\mathbf{x}^*(u). \tag{16}$$

This means that

$$u \in \mathbb{R} \mapsto \mathbf{X}(u) \in \mathbb{H}_+^2(\text{or } \mathbb{S}_1^2), \tag{17}$$

performs a ruled surface (X) of Minkowski 3-space $\mathbb{E}_1^3$. Thus a differentiable curve on $\mathbb{H}_+^2$ matches in a timelike ruled surface (X) in $\mathbb{E}_1^3$. Similarly the dual curve on $\mathbb{S}_1^2$ means to a spacelike or timelike ruled surface (X) in $\mathbb{E}_1^3$ [2].

In order to study the geometrical features of $\mathbf{X}(u) \in \mathbb{H}^2_+$(or $\mathbb{S}^2_1$), we introduce a moving frame concominant with the point on $\mathbb{H}^2_+$(or $\mathbb{S}^2_1$). If we classify the point as the generator $\mathbf{X} = \mathbf{X}(u)$ on the sphere $\mathbb{H}^2_+$(or $\mathbb{S}^2_1$), then the Blaschke frame is built as

$$\left.\begin{array}{lll} \mathbf{X} = \mathbf{X}(u), & \mathbf{T}(u) = \left\| \dfrac{d\mathbf{X}}{du} \right\|^{-1} \dfrac{d\mathbf{X}}{du}, & \mathbf{G}(u) = \mathbf{X} \times \mathbf{T}, \\ \mathbf{X} \times \mathbf{T} = \mathbf{G}, & \mathbf{X} \times \mathbf{G} = \epsilon_1 \mathbf{T}, & \mathbf{T} \times \mathbf{G} = -\epsilon_1 \mathbf{X}, \\ \langle \mathbf{X}, \mathbf{X} \rangle = \epsilon_1(\pm 1), & \langle \mathbf{T}, \mathbf{T} \rangle = \epsilon_2(\pm 1), & \langle \mathbf{G}, \mathbf{G} \rangle = -\, \epsilon_1 \epsilon_2. \end{array}\right\} \tag{18}$$

The dual unit vectors $\mathbf{X}$,

$$\mathbf{T}(u) = = \mathbf{t}(u) + \varepsilon \mathbf{t}^*(u), \text{ and } \mathbf{G}(u) = \mathbf{g}(u) + \varepsilon \mathbf{g}^*(u)$$

correspond to three orthogonal lines in Minkowski 3-space $\mathbb{E}^3_1$. Their intersection point is central point $\mathbf{C}$ on the ruling $\mathbf{X}$. The dual vector $\mathbf{G}(u)$ called central tangent of the ruled surface at the central point is the limit position of common perpendicular to $\mathbf{X}(u)$ and $\mathbf{X}(u + du)$. Thus the Blaschke derivative formula is given as follows:

$$\frac{d}{du} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ -\epsilon_1 \epsilon_2 P & 0 & Q \\ 0 & \epsilon_1 Q & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}, \tag{19}$$

where

$$P(u) = p + \varepsilon p^* = \left\| \frac{d\mathbf{X}}{du} \right\|, \text{ and}$$

$$Q(u) = q + \varepsilon q^* = \left\| \frac{d\mathbf{X}}{du} \right\|^{-2} \left\langle \frac{d^2\mathbf{X}}{du^2}, \frac{d\mathbf{X}}{du} \times \mathbf{X} \right\rangle,$$

are called the Blaschke invariants. The dual arc length $d\widehat{s} = ds + \varepsilon ds^*$ of the dual curve $\mathbf{X}(u) \in \mathbb{H}^2_+$ or $\mathbb{S}^2_1$ is:

$$\widehat{s}(u) = \int_{u_0}^{u} \left\| \frac{d\mathbf{X}}{du} \right\| du = \int_{u_0}^{u} \left\| \frac{d\mathbf{x}}{du} \right\| du + \varepsilon \int_{u_0}^{u} \left\langle \mathbf{t}, \frac{d\mathbf{x}^*}{du} \right\rangle du. \tag{20}$$

In what follows, we will use the dual arc length parameter $\widehat{s}$ of the dual curve $\mathbf{X}$. If dash means to differentiation as per $\widehat{s}$, then from Eq. (19), we get

$$\begin{pmatrix} \mathbf{X}' \\ \mathbf{T}' \\ \mathbf{G}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\epsilon_1 \epsilon_2 & 0 & \Sigma \\ 0 & \epsilon_1 \Sigma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}, \tag{21}$$

where

$$\Sigma(\widehat{s}) = \frac{Q}{P} = \gamma + \varepsilon \left(\Gamma - \mu \gamma \right) = \langle \mathbf{X}'', \mathbf{X}' \times \mathbf{X} \rangle$$

is the dual geodesic curvature of $\mathbf{X}(\widehat{s}) \in \mathbb{H}^2_+$ (or $\mathbb{S}^2_1$). The Eq. (21) may also be written as

$$\begin{pmatrix} \mathbf{X}' \\ \mathbf{T}' \\ \mathbf{G}' \end{pmatrix} = \mathbf{\Omega} \times \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}, \tag{22}$$

where $\mathbf{\Omega}(\widehat{s}) = \boldsymbol{\omega} + \varepsilon \boldsymbol{\omega}^* = \Sigma \mathbf{X} - \epsilon_2 \mathbf{G}$ is said to be the dual Darboux vector of the Blaschke frame. When the Darboux vector $\mathbf{\Omega}$ is expressed as a multiple $\|\mathbf{\Omega}\|\, \mathbf{B}$ of a dual unit vector, then $\mathbf{B}$ is called as the Disteli-axis (curvature axis or evolute) of the ruled surface (X). Hence, we may write $\mathbf{B}$ of the form

$$\mathbf{B}(\widehat{s}) = \mathbf{b} + \varepsilon \mathbf{b}^* = \frac{\mathbf{\Omega}}{\|\mathbf{\Omega}\|} = \frac{\Sigma \mathbf{X} - \epsilon_2 \mathbf{G}}{\sqrt{|\Sigma^2 - \epsilon_2|}}. \tag{23}$$

For each generator of a cylindrical surface the Disteli-axis coincides with the curvature axis of the orthogonal sections. The tangent vector to the striction curve is found as

$$\frac{d\mathbf{C}}{ds} = \epsilon_1 \Gamma \mathbf{x} - \epsilon_1 \epsilon_2 \mu \mathbf{g}. \tag{24}$$

The functions $\gamma(s)$, $\Gamma(s)$ and $\mu(s)$ are said to be the curvature functions of the ruled surface. The geometrical interpretations of these invariants are described as: the function γ corresponds to the geodesic curvature of the spherical indicatrix curve $\mathbf{x} = \mathbf{x}(s)$; the function Γ clarifies the angle between the tangent vector to the striction curve and the surface's ruling; and finally the function μ conforms the ruled surface's distribution parameter in the ruling [2].

3. Timelike ruled surfaces with constant Disteli-axis

In this section, based on the different values of ϵ_1, and ϵ_2, we present the classifications of the timelike ruled surface (X):
(1) If $(\epsilon_1, \epsilon_2) = (-1, 1)$ then (X) is said to be timelike of type (X_-),
(2) If $(\epsilon_1, \epsilon_2) = (1, 1)$ then (X) is called as timelike of type (X_+).

In these classifications we adopt subscript "+" and "−" to display the casual character of the ruling vector.

3.1. Timelike ruled surfaces of type (X_-) with constant Disteli-axis

The ruled surface (X_-) is called timelike ruled surface with constant Disteli-axis if all rulings of (X_-) have a constant dual angle according to the Disteli-axis.

The timelike ruled surface of type (X_{-}) with constant Disteli-axis can be described as follows: Since $\mathbf{X}(\widehat{s}) \in \mathbb{H}_{+}^{2}$, then from Eq. (21) we have

$$\widehat{\kappa}(\widehat{s}) = \kappa + \varepsilon\kappa^{*} = \|\mathbf{X}' \times \mathbf{X}''\| = \sqrt{1-\Sigma^{2}}, \text{ with } |\Sigma| < 1 \ , \tag{25}$$

and

$$\widehat{\tau}(\widehat{s}) = \tau + \varepsilon\tau^{*} = \frac{\det(\mathbf{X}', \mathbf{X}'', \mathbf{X}''')}{\|\mathbf{X}' \times \mathbf{X}''\|^{2}} = \pm\frac{\Sigma'}{1-\Sigma^{2}}, \tag{26}$$

where $\widehat{\kappa}(\widehat{s})$ and $\widehat{\tau}(\widehat{s})$, are, respectively curvature and torsion functions of dual spherical curve in the dual Hyperbolic sphere $\mathbb{H}_{+}^{2}$ which represents timelike ruled surfaces with constant Disteli-axis in Minkowski space $\mathbb{E}_{1}^{3}$. Thus, the Disteli-axis is obtained as

$$\mathbf{B}(\widehat{s}) = \frac{\Sigma\mathbf{X} - \mathbf{G}}{\sqrt{1-\Sigma^{2}}} = \sinh\Psi\mathbf{X} - \cosh\Psi\mathbf{G}, \tag{27}$$

where $\Psi = \psi + \varepsilon\psi^{*}$ is the central dual angle (radius of curvature) between $\mathbf{X}$ and $\mathbf{B}$. Thus, we may write the following relationships:

$$\left.\begin{array}{ll} \widehat{\kappa}(\widehat{s}) = \sqrt{1-\Sigma^{2}} = \dfrac{1}{\cosh\Psi}, & \widehat{\tau}(\widehat{s}) = \pm\Psi', \\ \Sigma(\widehat{s}) = \gamma + \varepsilon(\Gamma - \gamma\mu) = \tanh\Psi. & \end{array}\right\} \tag{28}$$

Obviously Eq. (28) also describes in more detail that the dual curve $\mathbf{X}(\widehat{s}) \in \mathbb{H}_{+}^{2}$ is a circle ($|\Sigma| < 1$).

Corollary 3.1. *If $\Sigma(\widehat{s})$ is a dual constant, then the dual spherical curve $\mathbf{X}(\widehat{s}) \in \mathbb{H}_{+}^{2}$ is a circle.*

Proof. From Eqs. (28) we can figure out that $\Sigma(\widehat{s})$ =constant yields that $\widehat{\tau}(\widehat{s}) = 0$, and $\widehat{\kappa}(\widehat{s})$ is constant, which implies $\mathbf{X}(\widehat{s})$ is a circle. □

Theorem 3.2. *A non-developable timelike ruled surface of type (X_{-}) is a constant Disteli-axis if and only if (a) γ =constant, and (b) $\Gamma - \gamma\mu$ =constant.*

Proof. It is direct result of Eq. (28). □

Example 3.3. *In what follows, we will construct a constant Disteli-axis timelike ruled surface of type (X_{-}). If Σ is a constant, from Eq. (21) we have*

$$\mathbf{X}'''(\widehat{s}) - \widehat{\kappa}^{2}(\widehat{s})\mathbf{X}'(\widehat{s}) = 0. \tag{29}$$

Without loss of generality, we may assume $\mathbf{X}'(0) = (0, 1, 0)$. Thus,

$$\mathbf{X}'(\widehat{s}) = (A_{1}\sinh\widehat{\kappa}\widehat{s}, \cosh\widehat{\kappa}\widehat{s} + A_{2}\sinh\widehat{\kappa}\widehat{s}, A_{3}\sinh\widehat{\kappa}\widehat{s}),$$

for some dual constants A_1, A_2, and A_3. Since $\|\mathbf{X}'\|^2 = 1$, we have $A_1^2 - A_3^2 = 1$, and $A_2 = 0$. From this, we obtain

$$\mathbf{X}(\widehat{s})=\left(\frac{A_1}{\widehat{\kappa}}\cosh\widehat{\kappa}\widehat{s}+B_1, \frac{1}{\widehat{\kappa}}\sinh\widehat{\kappa}\widehat{s}, \frac{A_3}{\widehat{\kappa}}\cosh\widehat{\kappa}\widehat{s}+B_3\right), \tag{30}$$

for some dual constants B_1, and B_3; satisfying $B_3^2 - B_1^2 = \frac{1}{\widehat{\kappa}^2} - 1$, $A_1B_1 - A_3B_3 = 0$, and $A_1^2 - A_3^2 = 1$. We now change the dual coordinates (X_1, X_2, X_3) by $(\overline{X}_1, \overline{X}_2, \overline{X}_3)$ as:

$$\begin{pmatrix} \overline{X_1} \\ \overline{X_2} \\ \overline{X_3} \end{pmatrix} = \begin{pmatrix} A_1 & 0 & -A_3 \\ 0 & 1 & 0 \\ -A_3 & 0 & A_1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

With respect to the coordinates $(\overline{X}_1, \overline{X}_2, \overline{X}_3)$, $\mathbf{X}(\widehat{s})$ turns into

$$\mathbf{X}(\Phi) = (\cosh\Psi\cosh\Phi, \cosh\Psi\sinh\Phi, \sinh\Psi), \tag{31}$$

where $\Phi = \varphi + \varepsilon\varphi^* = \widehat{\kappa}\widehat{s}$. This means that the spacelike lines $\mathbf{B}$ and $\mathbf{F}_3$ are coincident, and

$$\psi = c_1 \text{ (real const.)},\ \psi^* = c_2 \text{ (real const.)}.$$

Thus the Eq. (31) has only two parameters φ, and φ^*, in addition it performs a timelike line congruence. A relation such as $f(\varphi, \varphi^*) = 0$, between the parameters restrains it to the set of timelike straight lines, i.e. a timelike ruled surface in the congruence. Then, let us deal with $\mathbf{X} = \mathbf{X}(t)$, $t \in \mathbb{R}$ meaning to a timelike ruled surface of type (X_-). Hence we get the following relation:

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \cosh\Psi\cosh\Phi & \cosh\Psi\sinh\Phi & \sinh\Psi \\ \sinh\Phi & \cosh\Phi & 0 \\ \sinh\Psi\cosh\Phi & \sinh\Psi\sinh\Phi & \cosh\Psi \end{pmatrix} \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{pmatrix}. \tag{32}$$

Thus we get

$$\frac{d}{dt}\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} 0 & \frac{d\Phi}{dt}\cosh\Psi & 0 \\ \frac{d\Phi}{dt}\cosh\Psi & 0 & \frac{d\Phi}{dt}\sinh\Psi \\ 0 & -\frac{d\Phi}{dt}\sinh\Psi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}. \tag{33}$$

By Eq. (33), we obtain

$$d\widehat{s} = \cosh\Psi d\Phi, \text{ and } \Sigma(t) = \tanh\Psi. \tag{34}$$

Thus we reach, by means of the real and dual parts of Eq. (34), to the result

$$\mu = \psi^* \tanh \psi + \frac{d\varphi^*}{d\varphi},\ \Gamma = \psi^* + \left(\frac{d\varphi^*}{d\varphi}\right) \tanh \psi,\ \gamma = \tanh \psi. \tag{35}$$

Now we determine the equation of this timelike ruled surface of type (X_-) in terms of the Plücker coordinates. If we compute the real and dual parts of Eq. (31), we have

$$\mathbf{x}(\varphi) = (\cosh \psi \cosh \varphi, \cosh \psi \sinh \varphi, \sinh \psi)\ , \tag{36}$$

and

$$\mathbf{x}^*(\varphi, \varphi^*) = \begin{pmatrix} \psi^* \cosh \varphi \sinh \psi + \varphi^* \sinh \varphi \cosh \psi \\ \psi^* \sinh \varphi \sinh \psi + \varphi^* \cosh \varphi \cosh \psi \\ \psi^* \cosh \psi \end{pmatrix}. \tag{37}$$

Let $\alpha(\alpha_1, \alpha_2, \alpha_3)$ denote a point in the timelike line $\mathbf{X}$. Since $\mathbf{x} \times \mathbf{x}^* = \alpha$, we get the following system of linear equations in terms of α_1, α_2, and α_3:

(38)

$$\left.\begin{array}{l} \alpha_2 \sinh \psi - \alpha_3 \cosh \psi \sinh \varphi = \psi^* \cosh \varphi \sinh \psi + \varphi^* \sinh \varphi \cosh \psi, \\ \alpha_1 \sinh \psi - \alpha_3 \cosh \psi \cosh \varphi = \psi^* \sinh \varphi \sinh \psi + \varphi^* \cosh \varphi \cosh \psi, \\ \alpha_2 \cosh \psi \cosh \varphi - \alpha_1 \cosh \psi \sinh \varphi = \psi^* \cosh \psi. \end{array}\right\}$$

The matrix of coefficients of unknowns α_1, α_2, and α_3 is as

$$\begin{pmatrix} 0 & \sinh \psi & -\cosh \psi \sinh \varphi \\ \sinh \psi & 0 & -\cosh \psi \cosh \varphi \\ \cosh \psi \sinh \varphi & \cosh \psi \cosh \varphi & 0 \end{pmatrix},$$

and therefore its rank is 2 with $\varphi(t)$ or $\psi \neq 0$. In addition the following matrix's rank

$$\begin{pmatrix} 0 & \sinh \psi & -\cosh \psi \sinh \varphi & \begin{array}{c} \psi^* \cosh \varphi \sinh \psi \\ +\varphi^* \sinh \varphi \cosh \psi \end{array} \\ \sinh \psi & 0 & -\cosh \psi \cosh \varphi & \begin{array}{c} \psi^* \sinh \varphi \sinh \psi \\ +\varphi^* \cosh \varphi \cosh \psi \end{array} \\ \cosh \psi \sinh \varphi & \cosh \psi \cosh \varphi & 0 & \psi^* \cosh \psi \end{pmatrix},$$

is 2. Hence this system has infinite solutions presented with

$$\left.\begin{array}{c} \alpha_1 = (\alpha_3 + \varphi^*) \cosh \psi \cosh \varphi + \psi^* \sinh \varphi, \\ \alpha_2 = (\alpha_3 + \varphi^*) \cosh \psi \sinh \varphi + \psi^* \cosh \varphi, \\ \alpha_3 = \alpha_3(t). \end{array}\right\} \tag{39}$$

Since $\alpha_3(t)$ is taken at random, then we may take $\alpha_3 = -\varphi^*$. In this case, Eq. (39) reduces to

$$\left.\begin{array}{l}\alpha_1(t) = \psi^* \sinh\varphi, \\ \alpha_2(t) = \psi^* \cosh\varphi, \\ \alpha_3(t) = -\varphi^*.\end{array}\right\} \tag{40}$$

Thus if we take $\varphi^*(t) = \cosh\varphi$, and $\varphi(t) = \varphi$ as the motion parameter, then we have the base curve as

$$\boldsymbol{\alpha}(\varphi) = (\psi^* \sinh\varphi, \psi^* \cosh\varphi, -\cosh\varphi). \tag{41}$$

We can also show that $\left\langle \frac{d\boldsymbol{\alpha}}{d\varphi}, \frac{d\mathbf{x}}{d\varphi} \right\rangle = 0$, so the base curve $\boldsymbol{\alpha}= \boldsymbol{\alpha}(\varphi)$ of (X_-) is its striction curve. Furthermore, we get:

$$(X_-) : \mathbf{y}(\varphi, v) = \begin{pmatrix} \psi^* \sinh\varphi + v\cosh\psi\cosh\varphi \\ \psi^* \cosh\varphi + v\cosh\psi\sinh\varphi \\ -\cosh\varphi + v\sinh\psi \end{pmatrix}. \tag{42}$$

Also the curvatures of the timelike ruled surface of type (X_-) are found

$$\mu = \psi^* \tanh\psi - \sinh\varphi, \quad \Gamma = \psi^* - \sinh\varphi\tanh\psi, \quad \gamma = \tanh\psi, \tag{43}$$

where ψ^*, and ψ are constants. According to Eq. (42), we have the following cases:

(1) In the case of $\psi = 0$, and $\psi^* \neq 0$, then

$$(X_-) : \mathbf{y}(\varphi, v) = \begin{pmatrix} \psi^* \sinh\varphi + v\cosh\varphi \\ \psi^* \cosh\varphi + v\sinh\varphi \\ -\cosh\varphi \end{pmatrix}.$$

The timelike ruled surface's curvature is calculated as $\mu = -\sinh\varphi$, $\Gamma = \psi^*$, and $\gamma = 0$. For $\psi^* = 0.5$, and special values φ and v, the graph of the surface is shown in Fig. 2.

(2) In the case of $\psi \neq 0$, and $\psi^* = 0$, we have

$$(X_-) : \mathbf{y}(\varphi, v) = \begin{pmatrix} v\cosh\psi\cosh\varphi \\ v\cosh\psi\sinh\varphi \\ -\cosh\varphi + v\sinh\psi \end{pmatrix}.$$

The curvature functions are $\mu = -\sinh\varphi$, $\Gamma = -\sinh\varphi\tanh\psi$, and $\gamma = \tanh\psi$. For $\psi = 0.5$, and special values φ and v, the graph of the surface is exhibited in Fig. 3.

(3) In the case of $\psi = 0$, and $\psi^* = 0$, then we have

$$(X_-) : \mathbf{y}(\varphi, v) = \begin{pmatrix} v\cosh\varphi \\ v\sinh\varphi \\ -\cosh\varphi \end{pmatrix}.$$

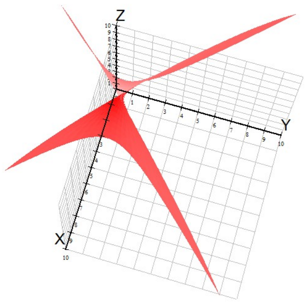

FIGURE 2

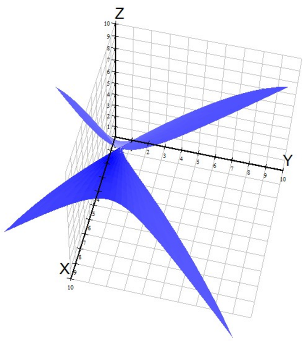

FIGURE 3

The curvature functions are $\mu = -\sinh\varphi$, and $\Gamma = \gamma = 0$. For special values φ and v, the graph of the surface is displayed in Fig. 4.

3.2. Timelike ruled surfaces of type (X_+) with constant Disteli-axis

As stated in the above case, since $\mathbf{X}(\widehat{s}) \in \mathbb{S}_1^2$, then from Eq. (21) we have

$$\widehat{\kappa}(\widehat{s}) = \sqrt{\Sigma^2 - 1}, \text{ and } \widehat{\tau}(\widehat{s}) = \pm\frac{\Sigma'}{\Sigma^2 - 1}, \tag{44}$$

where $\widehat{\kappa}(\widehat{s})$ and $\widehat{\tau}(\widehat{s})$, are, respectively curvature and torsion functions of dual spherical curve in the dual Hyperbolic sphere $\mathbb{S}_1^2$ which represents timelike ruled surface with constant Disteli-axis in Minkowski space $\mathbb{E}_1^3$.

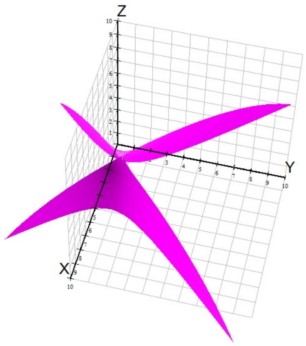

FIGURE 4

Therefore, the Disteli-axis is found as

$$\mathbf{B}(\widehat{s}) = \frac{\Sigma\mathbf{X} - \mathbf{G}}{\sqrt{\Sigma^2 - 1}} = \cosh\Psi\mathbf{X} - \sinh\Psi\mathbf{G}, \text{ with } |\Sigma| > 1, \tag{45}$$

where $\Psi = \psi + \varepsilon\psi^*$ is the hyperbolic dual angle (curvature radius) between $\mathbf{X}$ and $\mathbf{B}$. And, we write the following relationships:

$$\left.\begin{array}{ll} \widehat{\kappa}(\widehat{s}) = \sqrt{\Sigma^2 - 1} = \dfrac{1}{\sinh\Psi}, & \widehat{\tau}(\widehat{s}) = \pm\Psi' \\ \Sigma(\widehat{s}) = \gamma + \varepsilon(\Gamma - \gamma\mu) = \coth\Psi. & \end{array}\right\} \tag{46}$$

Example 3.4. *As in the above case, an analogous arguments show that:*

$$\mathbf{X}'''(\widehat{s}) + \widehat{\kappa}^2(\widehat{s})\mathbf{X}'(\widehat{s}) = 0.$$

We regard the initial condition $\mathbf{X}'(0) = (0, 1, 0)$. *Thus,*

$$\mathbf{X}'(\widehat{s}) = (A_1\sinh(\widehat{s\kappa}), \cosh(\widehat{s\kappa}) + A_2\sinh(\widehat{s\kappa}), A_3\sinh(\widehat{s\kappa}))$$

for some dual constants A_1, A_2 *and* A_3. *Since* $\epsilon_2 = 1$ *we havs* $A_1^2 - A_3^2 = 1$, *and* $A_2 = 0$. *From this, we can obtain*

$$\mathbf{X}(\widehat{s}) = \left(\frac{A_1}{\widehat{\kappa}}\cosh(\widehat{s\kappa}) + B_1, \frac{1}{\widehat{\kappa}}\sinh(\widehat{s\kappa}), \frac{A_3}{\widehat{\kappa}}\cosh(\widehat{s\kappa}) + B_3\right)$$

for some dual constants B_1, *and* B_3 *satisfying* $A_1^2 - A_3^2 = 1$, $A_1B_1 - A_3B_3 = 0$, *and* $B_3^2 - B_1^2 = \dfrac{1}{\widehat{\kappa}^2} + 1$. *If we take another coordinate system*

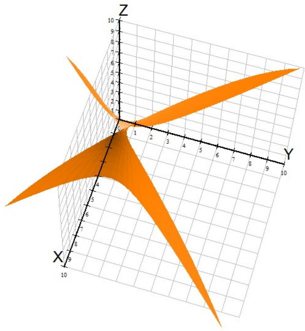

FIGURE 5

$(\overline{X}_1, \overline{X}_2, \overline{X}_3)$ *such that:*

$$\begin{pmatrix} \overline{X_1} \\ \overline{X_2} \\ \overline{X_3} \end{pmatrix} = \begin{pmatrix} -A_3 & 0 & A_1 \\ 0 & 1 & 0 \\ A_1 & 0 & -A_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

then $\mathbf{X}$ *takes the form*

$$\mathbf{X}(\Phi) = (\sinh\Psi\cosh\Phi, \sinh\Psi\sinh\Phi, \cosh\Psi)\,, \tag{47}$$

where $\Phi = \varphi + \varepsilon\varphi^* = \widehat{\kappa}\widehat{s}$, *with* $\psi = c_1$ *(real const.),,* $\psi^* = c_2$ *(real const.). If repeat the discussion of the above case for* (X_+) *instead of* (X_-) *we have:*

$$(X_+) : \mathbf{y}(\varphi, v) = \begin{pmatrix} \psi^*\sinh\varphi + v\sinh\psi\cosh\varphi \\ \psi^*\cosh\varphi + v\sinh\psi\sinh\varphi \\ -\cosh\varphi + v\cosh\psi \end{pmatrix}, \tag{48}$$

and

$$\mu = \psi^*\coth\psi - \sinh\varphi,\ \Gamma = \psi^* - \sinh\varphi\coth\psi,\ \gamma = \coth\psi, \tag{49}$$

where ψ^**, and* ψ *are constants. We also have the following cases:*
(1) In the case of $\psi = 0$*, and* $\psi^* \neq 0$*, Eq. (48) reduce to*

$$(X_+) : \mathbf{y}(\varphi, v) = \begin{pmatrix} \psi^*\sinh\varphi \\ \psi^*\cosh\varphi \\ -\cosh\varphi + v \end{pmatrix}.$$

For $\psi^* = 0.5$*, and special values* φ *and* v*, the graph of the surface is presented in Fig. 5.*

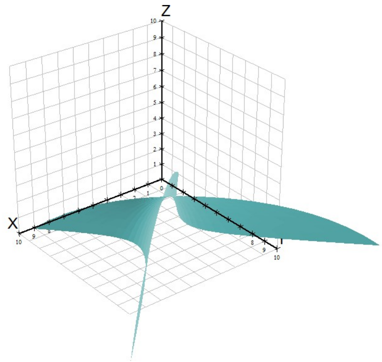

Figure 6

(2) In the case of $\psi \neq 0$, *and* $\psi^{*} = 0$, *Eq. (48) reduce to*

$$(X_{+}) : \mathbf{y}(\varphi, v) = \begin{pmatrix} v \sinh\psi \cosh\varphi \\ v \sinh\psi \sinh\varphi \\ -\cos\varphi + v\cosh\psi \end{pmatrix}.$$

The curvature functions are $\mu = -\sinh\varphi$, $\Gamma = -\sinh\varphi \coth\psi$, *and* $\gamma = \coth\psi$. *For* $\psi = 0.5$, *and special values* φ *and* v, *the graph of the surface is exhibited in Fig. 6.*

4. Conclusion

The survey of ruled surfaces has had rich results in dual Lorentzian 3-space $\mathbb{D}_1^3$. In this research, a necessary and sufficient condition for a timelike ruled surface to have a constant Disteli-axis has been presented in $\mathbb{D}_1^3$. Timelike ruled surfaces with constant Disteli-axis have been applied to present the instantaneous features of a point and line trajectories in Lorentzian spatial kinematics. Blaschke frames and Blaschke invariants of timelike ruled surfaces with constant Disteli-axis have been obtained by using E. Study map. It has been exhibited that a timelike ruled surface and its Disteli-axis have the same Blaschke frame. Finally, dual instantaneous rotation vectors of the Blaschke frames of timelike ruled surfaces with constant Disteli-axis have been defined in $\mathbb{D}_1^3$.

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Rashad A. Abdel-Baky
Department of Mathematics, Sciences Faculty for Girls,
University of Jeddah, P.O. Box 126300, Jeddah 21352, Saudi Arabia.
Department of Mathematics, University of Assiut,
Assiut 71516, Egypt.
E-mail: rbaky@live.com

Yasin Ünlütürk
Department of Mathematics, Kırklareli University,
39100, Kırklareli, Turkey.
E-mail: yasinunluturk@klu.edu.tr