Honam Mathematical J. **42** (2020), No. 3, pp. 569–576
https://doi.org/10.5831/HMJ.2020.42.3.569

THE NORMING SET OF A POLYNOMIAL IN $\mathcal{P}({}^2 l_\infty^2)$

Sung Guen Kim

Abstract. An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}({}^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}({}^n E)$, we define

$$Norm(P) = \{x \in E : x \text{ is a norming point of } P\}.$$

$Norm(P)$ is called the *norming set* of P. We classify $Norm(P)$ for $P \in \mathcal{P}({}^2 l_\infty^2)$.

1. Introduction

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a real Banach space E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}({}^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [5].

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}({}^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}({}^n E)$, we define

$$Norm(P) = \{x \in E : x \text{ is a norming point of } P\}.$$

$Norm(P)$ is called the *norming set* of P. Notice that $x \in Norm(P)$ if and only if $-x \in Norm(P)$. Indeed, if $x \in Norm(P)$, then

$$|P(-x)| = |(-1)^n P(x)| = |P(x)| = \|P\|,$$

which shows that $-x \in Norm(P)$. If $-x \in Norm(P)$, then $x = -(-x) \in Norm(P)$. The following examples show that $Norm(P) = \emptyset$ or a finite set or an infinite set.

Received December 26, 2019. Revised May 7, 2020. Accepted June 25, 2020.
2010 Mathematics Subject Classification. 46A22.
Key words and phrases. Norming points, a polynomial of $\mathcal{P}({}^2 l_\infty^2)$.

Examples. (a) Let

$$P((x_i)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^2 \in \mathcal{P}(^2c_0).$$

Then, $Norm(P) = \emptyset$.

(b) Let

$$P((x_i)_{i\in\mathbb{N}}) = x_1^2 - \sum_{i=2}^{\infty} \frac{1}{2^i} x_i^2 \in \mathcal{P}(^2c_0).$$

Then,

$$Norm(P) = \{\pm e_1\}.$$

(c) Let

$$P((x_i)_{i\in\mathbb{N}}) = x_1^2 \in \mathcal{P}(^2c_0).$$

Then,

$$Norm(P) = \{((\pm 1, x_2, x_3, \ldots) \in c_0 : |x_j| \leq 1 \text{ for } j = 2, 3, \ldots\}.$$

If $Norm(P) \neq \emptyset$, $P \in \mathcal{P}(^nE)$ is called a *norm attaining* polynomial.(See [3])

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jimenez-Sevilla and Paya [6] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

It seems to be natural and interesting to study about $Norm(P)$ for $P \in \mathcal{P}(^nE)$. For $m \in \mathbb{N}$, let $l_\infty^m := \mathbb{R}^m$ with the supremum norm. Notice that for every $P \in \mathcal{P}(^n l_\infty^m)$, $Norm(P) \neq \emptyset$ since $S_{l_\infty^m}$ is compact.

In this paper, we classify $Norm(P)$ for every $P \in \mathcal{P}(^2 l_\infty^2)$.

2. Results

Lemma 2.1. *Let* $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2 l_\infty^2)$ *for some* $a,b,c \in \mathbb{R}$. *Then there exists (unique)* $P^{'}(x,y) = a^*x^2 + b^*y^2 + c^*xy \in \mathcal{P}(^2 l_\infty^2)$ *such that* $a^*, b^*, c^* \in \{\pm a, \pm b, \pm c\}$ *with* $a^* \geq |b^*|$, $c^* \geq 0$ *and* $\|P\| = \|P^{'}\|$.

Proof. If $a < 0$, taking $-P$, we assume that $a \geq 0$. If $|b| > a$,

$$\text{Let } P_1^{'}(x,y) \quad := \quad P(y,x) = |b|x^2 + ay^2 + cxy.$$

Then, $\|P_1^{'}\| = \|P\|$. If $c < 0$,

$$\text{Let } P_2^{'}((x,y) \quad := \quad P_1^{'}(x,-y) = |b|x^2 + ay^2 + |c|xy.$$

Then, $\|P_2^{'}\| = \|P\|$. Therefore, we can find a polynomial $P^{'} \in \mathcal{P}(^2 l_\infty^2)$ which satisfies the conditions of the lemma. $\square$

Theorem A([4]). *Let* $P(x,y) = ax^2+by^2+cxy \in \mathcal{P}(^2 l_\infty^2)$ *with* $a \geq |b|$ *and* $c \geq 0$. *Then, If* $b \geq 0$ *or* ($b < 0$ *and* $c > 2|b|$), *then* $\|P\| = a + b + c$.
If $b < 0$ *and* $c \leq 2|b|$, *then*

$$\|P\| = \frac{c^2}{4|b|} + a.$$

Notice that if $\|P\| = 1$, then $|a| \leq 1, |b| \leq 1, |c| \leq 1$.

Let $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2 l_\infty^2)$ for some $a,b,c \in \mathbb{R}$. Notice that $(x,y) \in Norm(P)$ if and only if $(-x,-y) \in Norm(P)$. By Lemma 2.1, we may assume that $a \geq |b|$ and $c \geq 0$. We are in position to prove the main result of this paper.

Theorem 2.2. *Let* $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2 l_\infty^2)$ *be such that* $\|P\| = 1$ *with* $a \geq |b|$ *and* $c \geq 0$. *Then,*
Case 1: $b \geq 0$.
If $a = 1$, *then*

$$Norm(P) = \{\pm(1,t) : -1 \leq t \leq 1\}.$$

If ($0 < a < 1$ *and* $b = 0$) *or* ($b > 0$ *and* $c > 0$), *then*

$$Norm(P) = \{\pm(1,1)\}.$$

If ($a = b = 0$) *or* ($b > 0, c = 0$), *then*

$$Norm(P) = \{\pm(1,1), \pm(1,-1)\}.$$

Case 2: $b < 0$.
Subcase *1:* $c < 2|b|$.

If $|b| = a$, *then*

$$Norm(P) = \{\pm(1, -\frac{\sqrt{a(1-a)}}{a}), \pm(-\frac{\sqrt{a(1-a)}}{a}, 1)\}.$$

If $|b| < a$, *then*

$$Norm(P) = \{\pm(1, -\frac{c}{2b})\}.$$

Subcase *2:* $c = 2|b|$.

$$Norm(P) = \{\pm(1,1), \pm(1,-1)\}.$$

Subcase *3:* $c > 2|b|$.
If $|b| = a$, *then*

$$Norm(P) = \{\pm(1,1), \pm(1,-1)\}.$$

If $|b| < a$, *then*

$$Norm(P) = \{\pm(1,1)\}.$$

Proof. Case 1: $b \geq 0$.
By Theorem A,

$$\|P\| = 1 = a + b + c.$$

Suppose that $(x, y) \in Norm(P)$ for some $x, y \in [-1, 1]$. Notice that

$1 = |P(x, y)| = |ax^2+by^2+cxy| \leq a|x|^2+b|y|^2+c|x|\,|y| \leq a+b+c = 1$ $(*)$.

Suppose that $a = 1$. Then, $b = c = 0$. By $(*)$, $|x| = 1$. Hence,

$$Norm(P) = \{\pm(1, t) : -1 \leq t \leq 1\}.$$

Suppose that $(0 < a < 1$ and $b = 0)$ or $(b > 0$ and $c > 0)$. By $(*)$, $xy = 1$. Hence,

$$Norm(P) = \{\pm(1,1)\}.$$

Suppose that $(a = b = 0)$ or $(b > 0, c = 0)$. By $(*)$, $|xy| = 1$. Hence,

$$Norm(P) = \{\pm(1,1), \pm(1,-1)\}.$$

Case 2: $b < 0$.
Subcase 1: $c < 2|b|$.
Suppose that $|b| = a$. Then,

$$P = ax^2 - ay^2 + 2\sqrt{a(1-a)}xy \ (\frac{1}{2} < a \leq 1).$$

If $a = 1$, then

$$Norm(P) = \{\pm(1,0), \pm(0,1)\}.$$

Let $\frac{1}{2} < a < 1$. Suppose that $(1, y) \in Norm(P)$ for some $y \in [-1, 1]$. Then,

$$1 = |P(1, y)| = |a - ay^2 + 2\sqrt{a(1-a)}y|.$$

If $1 = P(1, y)$, then

$$y = -\frac{\sqrt{a(1-a)}}{a}$$

and $(1, -\frac{\sqrt{a(1-a)}}{a}) \in Norm(P)$. Notice that if $-1 = P(1, y)$, then there are no norming points of P.

Suppose that $(x, 1) \in Norm(P)$ for some $x \in [-1, 1]$. Notice that if $1 = P(x, 1)$, there are no norming points of P. Notice that if $-1 = P(x, 1)$, then

$$x = -\frac{\sqrt{a(1-a)}}{a}$$

and $(-\frac{\sqrt{a(1-a)}}{a}, 1) \in Norm(P)$. Hence,

$$Norm(P) = \{\pm(1, -\frac{\sqrt{a(1-a)}}{a}), \pm(-\frac{\sqrt{a(1-a)}}{a}, 1)\}.$$

Suppose that $|b| < a$. By Theorem A,

$$1 = \|P\| = \frac{c^2}{4|b|} + a,$$

so

$$c^2 = 4|b|(1-a).$$

Suppose that $(1, y) \in Norm(P)$ for some $y \in [-1, 1]$. Then,

$$1 = |P(1, y)| = |a + by^2 + cy|.$$

Notice that if $1 = P(1, y)$, then $y = -\frac{c}{2b}$. Hence, $\pm(1, -\frac{c}{2b}) \in Norm(P)$. Notice that if $-1 = P(1, y)$, then there are no norming points of P.

Suppose that $(x, 1) \in Norm(P)$ for some $x \in [-1, 1]$. Notice that there are no norming points of P in this case. Therefore,

$$Norm(P) = \{\pm(1, -\frac{c}{2b})\}.$$

Subcase 2: $c = 2|b|$.

By Theorem A, $b = a - 1$. Hence,

$$P = ax^2 - (1-a)y^2 + 2(1-a)xy \ (\frac{1}{2} \leq a < 1).$$

Suppose that $(1, y) \in Norm(P)$ for some $y \in [-1, 1]$. Then,

$$1 = |P(1, y)| = |a - (1-a)y^2 + 2(1-a)y|.$$

Notice that if $1 = P(1, y)$, then $y = -1$, so $(1, -1) \in Norm(P)$. Notice that if $-1 = P(1, y)$, then $y = -1$, so $(1, -1) \in Norm(P)$.

Suppose that $(x, 1) \in Norm(P)$ for some $x \in [-1, 1]$. Notice that if $1 = P(x, 1)$, then $x = 1$, so $(1, 1) \in Norm(P)$. Notice that if $-1 = P(x, 1)$, then $x = -1$, so $(-1, 1) \in Norm(P)$. Therefore,

$$Norm(P) = \{\pm(1, 1), \pm(1, -1)\}.$$

Subcase 3: $c > 2|b|$.

Suppose that $|b| = a$. Then

$$P = ax^2 - ay^2 + xy \ (0 < a < \frac{1}{2}).$$

Obviously, $(1, 1), (1, -1) \in Norm(P)$. Suppose that $(1, y) \in Norm(P)$ for some $y \in [-1, 1]$. Notice that if $1 = P(1, y)$, then $y = 1$, so $(1, 1) \in Norm(P)$. Notice that if $-1 = P(1, y)$, then $y = -1$, so $(1, -1) \in Norm(P)$.

Suppose that $(x, 1) \in Norm(P)$ for some $x \in [-1, 1]$. Notice that if $1 = P(x, 1)$, then $x = 1$, so $(1, 1) \in Norm(P)$. Notice that if $-1 = P(x, 1)$, then $x = -1$, so $(-1, 1) \in Norm(P)$. Therefore,

$$Norm(P) = \{\pm(1, 1), \pm(1, -1)\}.$$

Suppose that $|b| < a$. Then, $2|b| < c < 2a$. Note that

$$2|b| < c < 2a \Leftrightarrow \frac{1 + |b|}{3} < a < 1 - |b|$$

and

$$3 + b - 2\sqrt{2b + 2} < \frac{1 + |b|}{3}.$$

Suppose that $(1, y) \in Norm(P)$ for some $y \in [-1, 1]$. Then,

$$1 = |P(1, y)| = |by^2 + cy + a|.$$

Notice that if $1 = P(1, y)$, then

$$y = \frac{c \pm \sqrt{c^2 - 4b(a - 1)}}{2|b|} = 1,$$

so $(1, 1) \in Norm(P)$. Suppose that $-1 = P(1, y)$. Then

$$y = \frac{c \pm \sqrt{c^2 - 4b(a + 1)}}{2|b|}.$$

Notice that

$$|\frac{c \pm \sqrt{c^2 - 4b(1 + a)}}{2|b|}| > 1,$$

a contradiction. Hence, if $-1 = P(1, y)$, then there are no norming points of P.

Suppose that $(x, 1) \in Norm(P)$ for some $x \in [-1, 1]$. Then,

$$1 = |P(x, 1)| = |ax^2 + cx + b|.$$

Notice that if $1 = P(x, 1)$, then

$$y = \frac{-c \pm \sqrt{c^2 - 4a(b-1)}}{2a} = 1,$$

so $(1, 1) \in Norm(P)$. Note that

$$c^2 - 4a(b+1) \geq 0 \Leftrightarrow 0 \leq a \leq 3 + b - 2\sqrt{2b+2}.$$

Notice that if $-1 = P(x, 1)$, then

$$y = \frac{-c \pm \sqrt{c^2 - 4a(b+1)}}{2a} \notin \mathbb{R}$$

since $c^2 - 4a(b+1) < 0$. Hence, if $-1 = P(1, y)$, then there are no norming points of P. Hence,

$$Norm(P) = \{\pm(1, 1)\}.$$

Therefore, we complete the proof. $\square$

Acknowledgements

The author is thankful to the referees for the careful reading and considered suggestions leading to a better presented paper.

References

[1] R.M. Aron, C. Finet and E. Werner, *Some remarks on norm-attaining n-linear forms*, Function spaces (Edwardsville, IL, 1994), 19–28, Lecture Notes in Pure and Appl. Math., **172**, Dekker, New York, 1995.

[2] E. Bishop and R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97-98.

[3] Y.S. Choi and S.G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. (2) **54** (1996), 135-147.

[4] Y.S. Choi and S.G. Kim, *The unit ball of* $\mathcal{P}(^2 l_2^2)$, Arch. Math. (Basel) **71** (1998), 472-480.

[5] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London (1999).

[6] M. Jimenez Sevilla and R. Paya, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Math. **127** (1998), 99-112.

Sung Guen Kim
Department of Mathematics, Kyungpook National University,
Daegu 702-701, Korea.
E-mail: sgk317@knu.ac.kr