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A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY USING MITTAG-LEFFLER FUNCTION

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Abstract. In this paper, new subclasses of analytic functions are proposed by using Mittag-Leffler function. Also some properties of these classes are studied in regard to coefficient inequality, distortion theorems, extreme points, radii of starlikeness and convexity and obtained numerous sharp results.

1. Introduction

Let $\mathbb{C}$ be the complex plane and let $\Omega = \{w : w \in \mathbb{C} \text{ and } |w| < 1\}$, the open unit disc. Further, by $\mathcal{A}$ we represent the class of functions analytic in Ω, satisfying the condition

$$l(0) = l'(0) - 1 = 0.$$

Thus each function l in $\mathcal{A}$ has a Taylor series representation

$$l(w) = w + o_2 w^2 + o_3 w^3 + \cdots = w + \sum_{n=2}^{\infty} o_n w^n \tag{1}$$

and let $\mathcal{S}, \mathcal{T}$ be the subclasses of $\mathcal{A}$ consisting of functions which are univalent in Ω [9], and with negative coefficients given by (see [27])

$$l(w) = w - \sum_{n=2}^{\infty} o_n w^n, \quad (o_n \geq 0)\,, \tag{2}$$

respectively.

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We now recall that the analytic function l is said to be *subordinate* to the analytic function m (indicated as $l \prec m$), if there exists a Schwarz function

$$\varpi(w) = \sum_{n=1}^{\infty} \mathfrak{c}_n w^n \quad (\varpi(0) = 0, \ |\varpi(w)| < 1),$$

analytic in Ω such that

$$l(w) = m(\varpi(w)) \qquad (w \in \Omega).$$

Furthermore, if the function m is univalent in Ω, then we have the following equivalence, see [9, 13].

$$l(w) \prec m(w) \Longleftrightarrow l(0) = m(0) \text{ and } l(\Omega) \subset m(\Omega) \qquad (w \in \Omega).$$

The convolution or Hadamard product of two functions l and m is denoted by $l * m$ is defined as

$$(l * m)(w) = \sum_{n=0}^{\infty} o_n b_n w^n,$$

where l is given by (1) and $m(w) = \sum_{n=2}^{\infty} b_n w^n \quad (w \in \Omega)$.

Recall that $D \subset \mathbb{C}$ is said to be a starlike with respect to the point $w_0 \in D$ if and only if the line segment joining w_0 to every other point $w \in D$ lies entirely in D, while the set D is said to be convex if and only if it is starlike with respect to each of its points. By $\mathcal{S}^*$ and $\mathcal{K}$ we mean the subclasses of $\mathcal{S}$ composed of starlike and convex functions, respectively. A function $l \in \mathcal{A}$ is said to be starlike of order $\overline{\alpha}$, $0 \leq \overline{\alpha} < 1$, if

$$\Re\left(\frac{wl'(w)}{l(w)}\right) > \overline{\alpha} \qquad (w \in \Omega),$$

and a function $l \in \mathcal{A}$ is said to be convex of order $\overline{\alpha}$, $0 \leq \overline{\alpha} < 1$, if

$$\Re\left(\frac{\left(wl'(w)\right)'}{l'(w)}\right) > \overline{\alpha} \qquad (w \in \Omega).$$

By K, we mean $l \in \mathcal{A}$ and the class of all close-to-convex functions of order $\overline{\alpha}$, $0 \leq \overline{\alpha} < 1$, if

$$\Re\left(\frac{wl'(w)}{g'(w)}\right) > \overline{\alpha}$$

where g is convex. In 1991, Goodman [10] introduced the class $\mathcal{UCV}$ of uniformly convex functions which was extensively studied by Ronning and independently by Ma and Minda [17, 23]. A more convenient characterization of class $\mathcal{UCV}$ was given by Ma and Minda as:

$$l \in \mathcal{UCV} \Longleftrightarrow l \in \mathcal{A} \text{ and } \Re\left\{1 + \frac{wl''(w)}{l'(w)}\right\} > \left|\frac{wl''(w)}{l'(w)}\right| \qquad (w \in \Omega).$$

In 1999, Kanas and Wisniowska [12, 13] (see also [14, 15]) introduced the class $k-$uniformly convex functions, $k \geq 0$, denoted by $k-\mathcal{UCV}$ and a related class $k-\mathcal{ST}$ as:

$$l \in k-\mathcal{UCV} \Longleftrightarrow wl' \in k-\mathcal{ST} \Longleftrightarrow l \in \mathcal{A} \text{ and } \Re\left\{\frac{\left(wl'(w)\right)'}{l'(w)}\right\} > \left|\frac{wl''(w)}{l'(w)}\right| \qquad (w \in \Omega).$$

The class $k-\mathcal{UCV}$ was discussed earlier in [31], with same extra restriction and without geometrical interpretation by Bharati et al. [4].

Mittag-Leffler defined familiar Mittag-Leffler function [19, 20] $M_{\overline{\alpha}}(w)$ by

$$M_{\overline{\alpha}}(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\overline{\alpha}+1)},$$

and Wiman [33] generalized this function by

$$M_{\overline{\alpha},\mu}(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\overline{\alpha}n+\mu)} \quad (\overline{\alpha} \geq 0),$$

where $\Re(\overline{\alpha}) > 0$, $\Re(\mu) > 0$ and $\overline{\alpha}, \mu \in \mathbb{C}$. Many researchers explain the Mittag-Leffler function and its generalizations see [3, 8, 18, 24, 28, 29, 30].

An important theory that has contributed significantly in geometric function theory is differential operator theory. Numerous researchers have worked intensively in this way, for recent work see [1, 5, 7, 21]. Elhaddad [6] introduced the following differential operator for $l \in \mathcal{A}$

$$D_\chi^i(\overline{\alpha},\mu)l(w) = w + \sum_{n=2}^{\infty} [1+(n-1)\chi]^i \frac{\Gamma(\mu)}{\Gamma(\overline{\alpha}(n-1)+\mu)} o_n w^n, \tag{3}$$

and for $l \in \mathcal{T}$

$$D_\chi^i(\overline{\alpha},\mu)l(w) = w - \sum_{n=2}^{\infty} [1+(n-1)\chi]^i \frac{\Gamma(\mu)}{\Gamma(\overline{\alpha}(n-1)+\mu)} o_n w^n. \tag{4}$$

Definition 1.1. *A function $l \in \mathcal{A}$ is said to be in the class*

$$\mathrm{Q}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$$

$(-1 \leq I < H \leq 1, \overline{\alpha}, \mu, \gamma, \chi \geq 0, i \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, i > j, w \in \Omega)$ *if the following subordination relationship is satisfied:*

$$\frac{D_{\chi}^{i}(\overline{\alpha},\mu)l(w)}{D_{\chi}^{j}(\overline{\alpha},\mu)l(w)} - \gamma \left| \frac{D_{\chi}^{i}(\overline{\alpha},\mu)l(w)}{D_{\chi}^{j}(\overline{\alpha},\mu)l(w)} - 1 \right| \prec \frac{1+Hw}{1+Iw}.$$

For particular values of the parameters $\chi, \overline{\alpha}, \mu, H, I, i, j, \gamma$, we have the following subclasses studied by various authors:
(*i*) $Q_{1,i,j}^{0,1}(\gamma, 1-2\varepsilon, -1) = E_{i,j}(\gamma, \varepsilon)$ (see Sümer Eker and Owa [32]),
(*ii*) $Q_{1,1,0}^{0,1}(\gamma, 1-2\varepsilon, -1) = UE(\gamma, \varepsilon)$ (see Shams et al. [25]),
(*iii*) $Q_{1,2,0}^{0,1}(\gamma, 1-2\varepsilon, -1) = UE(\gamma, \varepsilon)$ (see Shams et al. [26]),
(*iii*) $Q_{1,1,0}^{0,1}(0, H, I) = S^*(H, I)$ (see Janowski [11]),
(*iv*) $Q_{1,2,0}^{0,1}(0, H, I) = K(H, I)$ (see Padmanabhan and Ganesan [22]).

Definition 1.2. *Let* $\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$ *denote the subclass of* $\mathcal{A}$ *consisting of functions* l *of the form* (2) *and we define the class* $\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$ *by*

$$\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I) = \mathrm{Q}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I) \cap \mathcal{T}.$$

For particular values of the parameters $\chi, \overline{\alpha}, \mu, H, I, i, j, \gamma$, we have the following subclasses studied by various authors:
(*i*) $TQ_{1,i+1,i}^{0,1}(\gamma, 1-2\varepsilon, -1) = TS(i, \gamma, \varepsilon)$ (see Aouf [2]),
(*ii*) $TQ_{1,1,0}^{0,1}(1, 1-2\varepsilon, -1) = S_pT(\varepsilon)$ (see Bharati et al. [4]),
(*iii*) $TQ_{1,1,0}^{0,1}(0, 1-2\varepsilon, -1) = T^*(\varepsilon)$ (see Silverman [27]).

2. Main Results

In this section, we will prove our main results.

Theorem 2.1. *A function* l *of the form* (1) *is in the class* $\mathrm{Q}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$ *if*

$$(6) \qquad \sum_{n=2}^{\infty} \upsilon \left[\{1+\gamma(1+|I|)\} \left(\phi^{i} - \phi^{j}\right) + \left|I\phi^{i} - H\phi^{j}\right| \right] |o_n| \leq H - I,$$

where $\phi = 1 + (n-1)\chi$ *and* $\upsilon = \frac{\Gamma(\mu)}{\Gamma(\overline{\alpha}(n-1)+\mu)}$.

Proof. We need to show that

$$\left|\frac{p(w)-1}{H-Ip(w)}\right|<1,$$

where

$$p(w)=\frac{D_{\chi}^{i}(\overline{\alpha},\mu)l(w)}{D_{\chi}^{j}(\overline{\alpha},\mu)l(w)}-\gamma\left|\frac{D_{\chi}^{i}(\overline{\alpha},\mu)l(w)}{D_{\chi}^{j}(\overline{\alpha},\mu)l(w)}-1\right|.$$

Hence, we obtain

$$\begin{aligned}
&\left|\frac{p(w)-1}{H-Ip(w)}\right|\\
=&\left|\frac{D_{\chi}^{i}(\overline{\alpha},\mu)l(w)-D_{\chi}^{j}(\overline{\alpha},\mu)l(w)-\gamma e^{i\theta}\left|D_{\chi}^{i}(\overline{\alpha},\mu)l(w)-D_{\chi}^{j}(\overline{\alpha},\mu)l(w)\right|}{HD_{\chi}^{j}(\overline{\alpha},\mu)l(w)-I\left[D_{\chi}^{i}(\overline{\alpha},\mu)l(w)-\gamma e^{i\theta}\left|D_{\chi}^{i}(\overline{\alpha},\mu)l(w)-D_{\chi}^{j}(\overline{\alpha},\mu)l(w)\right|\right]}\right|\\
=&\left|\frac{\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)o_{n}w^{n}-\gamma e^{i\theta}\left|\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)o_{n}w^{n}\right|}{(H-I)w-\left[\sum_{n=2}^{\infty}\upsilon\left(I\phi^{i}-H\phi^{j}\right)o_{n}w^{n}-\gamma Ie^{i\theta}\left|\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)o_{n}w^{n}\right|\right]}\right|\\
\leq&\frac{\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)|o_{n}|\,|w|^{n}+\gamma\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)|o_{n}|\,|w|^{n}}{(H-I)\,|w|-\left[\sum_{n=2}^{\infty}\upsilon\left|\left(I\phi^{i}-H\phi^{j}\right)\right||o_{n}|\,|w|^{n}+\gamma\,|I|\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)|o_{n}|\,|w|^{n}\right]}\\
\leq&\frac{\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)(1+\gamma)\,|o_{n}|}{(H-I)-\sum_{n=2}^{\infty}\upsilon\left|\left(I\phi^{i}-H\phi^{j}\right)\right||o_{n}|-\gamma\,|I|\sum_{n=2}^{\infty}\upsilon\left(\phi^{i}-\phi^{j}\right)|o_{n}|}.
\end{aligned}$$

This last expression is bounded previously by 1 if

$$\sum_{n=2}^{\infty}\upsilon\left[\left\{1+\gamma\left(1+|I|\right)\right\}\left(\phi^{i}-\phi^{j}\right)+\left|I\phi^{i}-H\phi^{j}\right|\right]|o_{n}|\leq H-I.$$

This completes the proof. □

In Theorem (2.2), it is shown that the condition in (6) is also necessary for functions l of the form (2) to be in the class $TQ_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$.

Theorem 2.2. *Let $l\in\mathcal{T}$. Then $l\in \mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$ if*

$$\sum_{n=2}^{\infty}\upsilon\left[\left\{1+\gamma\left(1+|I|\right)\right\}\left(\phi^{i}-\phi^{j}\right)+\left|I\phi^{i}-H\phi^{j}\right|\right]|o_{n}|\leq H-I,$$

where $\phi=1+(n-1)\chi$ and $\upsilon=\frac{\Gamma(\mu)}{\Gamma(\overline{\alpha}(n-1)+\mu)}$.

Proof. Since

$$TQ_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)\subseteq Q_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I),$$

by making use of the same technique given in the proof of Theorem (2.1), we immediately have Theorem (2.2). □

Corollary 2.3. *A function l defined by (2) is in the class $\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$, then*

$$o_n \leq \frac{H-I}{\upsilon\left[\{1+\gamma(1+|I|)\}(\phi^i-\phi^j)+|I\phi^i-H\phi^j|\right]} \quad (n \geq 2). \tag{7}$$

The result is sharp for the function

$$l(w) = w - \frac{H-I}{\upsilon\left[\{1+\gamma(1+|I|)\}(\phi^i-\phi^j)+|I\phi^i-H\phi^j|\right]} w^n \quad (n \geq 2). \tag{8}$$

The growth and distortion properties of the function l in the respective class $TQ_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$ are given as follows:

Theorem 2.4. *Let the function l defined by (2) belong to the class $\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$. Then*

$$|l(w)| \geq |w| - \frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}|w|^2$$

and

$$|l(w)| \leq |w| + \frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}|w|^2.$$

Proof. In view of Theorem (2.2), consider

$$\delta(n) = \upsilon\{1+\gamma(1+|I|)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|,$$

where $\phi = 1+(n-1)\chi$ and $\upsilon = \frac{\Gamma(\mu)}{\Gamma(\overline{\alpha}(n-1)+\mu)}$, $\delta(n)$ is an increasing function for n $(n \geq 2)$. This implies that

$$\delta(2)\sum_{n=2}^{\infty}|o_n| \leq \sum_{n=2}^{\infty}\delta(n)\,|o_n| \leq H-I,$$

that is

$$\sum_{n=2}^{\infty}|o_n| \leq \frac{H-I}{\delta(2)}.$$

Thus we have

$$|l(w)| \leq |w| + \sum_{n=2}^{\infty}|o_n|\,|w|^2,$$

$$|l(w)| \leq |w| + \frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}|w|^2.$$

Similarly, we get

$$\begin{aligned}|l(w)| &\geq |w| - \sum_{n=2}^{\infty} |o_n|\,|w|^n \geq |w| - \sum_{n=2}^{\infty} |o_n|\,|w|^2 \\ &\geq |w| - \frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}\,|w|^2 .\end{aligned}$$

Finally, the result is sharp for the function

(9)
$$l(w) = w - \frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}\,w^2$$

at $|w| = r$ and $w = re^{i(2k+1)\pi}$ $(k \in \mathbb{Z})$.This completes Theorem (2.4). □

Theorem 2.5. *Let the function l defined by* (2) *belong to the class* $\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma, H, I)$. *Then*

$$\left|l'(w)\right| \geq 1 - \frac{2\,(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}\,|w| ,$$

and

$$\left|l'(w)\right| \leq 1 + \frac{2\,(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}\,|w| .$$

The result is sharp.

Proof. In view of Theorem (2.2), suppose that

$$\delta(n) = \upsilon\,\{1+\gamma(1+|I|)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right| ,$$

where $\phi = 1+(n-1)\,\chi$ and $\upsilon = \frac{\Gamma(\mu)}{\Gamma(\overline{\alpha}(n-1)+\mu)}$, $\frac{\Phi(n)}{n}$ is an increasing function for n $(n \geq 2)$. Similarly, we obtain

$$\frac{\delta(2)}{2}\sum_{n=2}^{\infty} n\,|o_n| \leq \sum_{n=2}^{\infty} \frac{\delta(n)}{n} n\,|o_n| = \sum_{n=2}^{\infty} \delta(n)\,|o_n| \leq (H-I) ,$$

that is

$$\sum_{n=2}^{\infty} n\,|o_n| \leq \frac{2\,(H-I)}{\delta(2)}$$

and consequently

$$\left|l'(w)\right| \leq 1 + \sum_{n=2}^{\infty} n\,|o_n|\,|w| ,$$

$$\left|l'(w)\right| \leq 1 + \frac{2\,(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}\,|w| .$$

Also, we get

$$\left|l^{'}(w)\right| \geq 1-\sum_{n=2}^{\infty} n\left|o_n\right||w|$$

$$\left|l^{'}(w)\right| \geq 1-\frac{2\,(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma\,(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I\,(1+\chi)^i-H\,(1+\chi)^j\right|\right]}\,|w|\,.$$

Finally, we can see that the assertions of Theorem (2.5) are sharp for the function l defined by (9). This completes the proof of Theorem (2.5). □

Now we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TQ_{\chi,i,j}^{\overline{\alpha},\mu}\,(\gamma,H,I)\,.$

Theorem 2.6. *Let the function l defined by* (2) *be in the class* $\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}\,(\gamma,H,I)$*. Then*
(i) l is starlike of order $\overline{\alpha}$ $(0\leq\overline{\alpha}<1)$ in $|w|<r_1$, where

$$r_1=\inf_{n\geq 2}\left\{\frac{\upsilon\left[\{1+\gamma\,(1+|I|)\}\,(\phi^i-\phi^j)+\left|I\phi^i-H\phi^j\right|\right]}{(H-I)}\times\left(\frac{1-\overline{\alpha}}{n-\overline{\alpha}}\right)\right\}^{\frac{1}{n-1}}, \tag{10}$$

(ii) l is convex of order $\overline{\alpha}$ $(0\leq\overline{\alpha}<1)$ in $|w|<r_2$, where

$$r_2=\inf_{n\geq 2}\left\{\frac{\upsilon\left[\{1+\gamma\,(1+|I|)\}\,(\phi^i-\phi^j)+\left|I\phi^i-H\phi^j\right|\right]}{(H-I)}\times\left(\frac{1-\overline{\alpha}}{n\,(n-\overline{\alpha})}\right)\right\}^{\frac{1}{n-1}}, \tag{11}$$

(ii) l is close to convex of order $\overline{\alpha}$ $(0\leq\overline{\alpha}<1)$ in $|w|<r_3$, where

$$r_3=\inf_{n\geq 2}\left\{\frac{\upsilon\left[\{1+\gamma\,(1+|I|)\}\,(\phi^i-\phi^j)+\left|I\phi^i-H\phi^j\right|\right]}{(H-I)}\times\left(\frac{1-\overline{\alpha}}{n}\right)\right\}^{\frac{1}{n-1}}. \tag{12}$$

Each of these results is sharp for the function l given by (8)*.*

Proof. It is sufficient to show that

$$\left|\frac{wl^{'}(w)}{l(w)}-1\right|\leq 1-\overline{\alpha}\quad\text{for }\;|w|<r_1,$$

where r_1 is given by (10). Indeed we find from (2) that is

$$\left|\frac{wl^{'}(w)}{l(w)}-1\right|\leq\frac{\sum_{n=2}^{\infty}(n-1)o_n\,|w|^{n-1}}{1-\sum_{n=2}^{\infty}o_n\,|w|^{n-1}}.$$

Thus, we have

$$\left|\frac{wl^{'}(w)}{l(w)}-1\right|\leq 1-\overline{\alpha},$$

if and only if

$$\frac{\sum_{n=2}^{\infty}(n-\overline{\alpha})o_n\,|w|^{n-1}}{(1-\overline{\alpha})}\leq 1. \tag{13}$$

But, by Theorem (2.2), equation (13) will be true if

$$\left(\frac{n-\overline{\alpha}}{1-\overline{\alpha}}\right)|w|^{n-1}\leq\frac{\upsilon\left[\{1+\gamma\,(1+|I|)\}\left(\phi^{i}-\phi^{j}\right)+\left|I\phi^{i}-H\phi^{j}\right|\right]}{(H-I)},$$

that is, if

$$|w|\leq\left\{\frac{\upsilon\left[\{1+\gamma\,(1+|I|)\}\left(\phi^{i}-\phi^{j}\right)+\left|I\phi^{i}-H\phi^{j}\right|\right]}{(H-I)}\times\left(\frac{1-\overline{\alpha}}{n-\overline{\alpha}}\right)\right\}^{\frac{1}{n-1}}\qquad(n\geq 2)$$

this implies

$$r_1=\inf_{n\geq 2}\left\{\frac{\upsilon\left[\{1+\gamma\,(1+|I|)\}\left(\phi^{i}-\phi^{j}\right)+\left|I\phi^{i}-H\phi^{j}\right|\right]}{(H-I)}\times\left(\frac{1-\overline{\alpha}}{n-\overline{\alpha}}\right)\right\}^{\frac{1}{n-1}}\quad(n\geq 2)\,.$$

This completes the proof of equation (10).
To prove (11) and (12) it is sufficient to show that

$$\left|1+\frac{wl''(w)}{l'(w)}-1\right|\leq 1-\overline{\alpha}\qquad(|w|<r_2,\ 0\leq\overline{\alpha}<1)\,,$$

and

$$\left|l'(w)-1\right|\leq 1-\overline{\alpha}\qquad(|w|<r_3,\ 0\leq\overline{\alpha}<1)\,.$$

□

Next, we discussed extreme points for functions belonging to the class $TQ_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$.

Theorem 2.7. *Let*

$$l_1(w)=w,$$

$$l_n(w)=w-\frac{H-I}{\upsilon\left[\{1+\gamma\,(1+|I|)\}\left(\phi^{i}-\phi^{j}\right)+|I\phi^{i}-H\phi^{j}|\right]}w^{n}\qquad(n\geq 2)\,,$$

where $\phi=1+(n-1)\,\chi$ *and* $\upsilon=\frac{\Gamma(\mu)}{\Gamma(\overline{\alpha}(n-1)+\mu)}$, *then* $l(w)\in\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$ *if and only if it can be expressed in the following form*

$$l(w)=\sum_{n=1}^{\infty}\eta_n l_n(w),$$

where

$$\eta_n\geq 0,\qquad\sum_{n=1}^{\infty}\eta_n=1.$$

Proof. Suppose that

$$\begin{aligned} l(w) &= \sum_{n=1}^{\infty} \eta_n l_n(w) \\ &= w - \sum_{n=2}^{\infty} \eta_n \frac{H-I}{\upsilon\left[\{1+\gamma\left(1+|I|\right)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|\right]} w^n. \end{aligned}$$

Then, from Theorem (2.2), we have

$$\begin{aligned} &\sum_{n=2}^{\infty}\left[\frac{\upsilon\left[\{1+\gamma\left(1+|I|\right)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|\right](H-I)}{\upsilon\left[\{1+\gamma\left(1+|I|\right)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|\right]}\eta_n\right] \\ =& \quad (H-I)\sum_{n=2}^{\infty}\eta_n = (H-I)(1-\eta_1) \leq (H-I). \end{aligned}$$

Thus, in view of Theorem (2.2), we find that $l \in TQ_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$. Conversely, let us suppose that $l \in TQ_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$, then, since

$$o_n \leq \frac{H-I}{\upsilon\left[\{1+\gamma\left(1+|I|\right)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|\right]}$$

by setting

$$\eta_n = \frac{\upsilon\left[\{1+\gamma\left(1+|I|\right)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|\right]}{(H-I)}o \qquad (n \geq 2)$$

and

$$\eta_1 = 1 - \sum_{n=2}^{\infty}\eta_n,$$

we have

$$l(w) = \sum_{n=1}^{\infty}\eta_n l_n(w).$$

This completes the proof of Theorem. □

Corollary 2.8. *The extreme points of the class* $\mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$ *are given by*

$$l_1(w) = w,$$

$$l_n(w) = w - \frac{H-I}{\upsilon\left[\{1+\gamma\left(1+|I|\right)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|\right]}w^n \qquad (n \geq 2).$$

In order to state the integral means inequality for $l \in TQ_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$ we need the following subordination result due to Littlewood [16].

Lemma 2.9. *If the functions l and m are analytic in Ω with*

$$l \prec m$$

then for $p>0$ and $w=re^{i\theta}(0<r<1)$,

$$\int_0^{2\pi} |l(w)|^p \, d\theta \leq \int_0^{2\pi} |m(w)|^p \, d\theta. \tag{14}$$

Theorem 2.10. *Suppose that $l \in \mathrm{TQ}_{\chi,i,j}^{\overline{\alpha},\mu}(\gamma,H,I)$ and $l_2(w)$ is defined by*

$$l_2(w) = w - \frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma\mu\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}w^2$$

then for $w=re^{i\theta}(0<r<1)$, we have

$$\int_0^{2\pi} |l(w)|^p \, d\theta \leq \int_0^{2\pi} |l_2(w)|^p \, d\theta.$$

Proof. Let $l(w) = w - \sum\limits_{n=2}^{\infty} o_n w^n$ ($o_n \geq 0$) then we must show that

$$\begin{aligned} &\int_0^{2\pi} \left|1-\sum_{n=2}^{\infty} o_n w^{n-1}\right|^p d\theta \\ \leq \; &\int_0^{2\pi} \left|1-\frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}w\right|^p d\theta. \end{aligned}$$

By Lemma (2.9), it is enough to show that

$$1-\sum_{n=2}^{\infty} o_n w^{n-1} \prec 1-\frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}w.$$

By setting

$$1-\sum_{n=2}^{\infty} o_n w^{n-1} = 1-\frac{(H-I)\,\Gamma(\overline{\alpha}+\mu)}{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}\varpi(w)$$

and using (6), we obtain

$$\begin{aligned} |\varpi(w)| &= \left|\sum_{n=2}^{\infty} \frac{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}{(H-I)\,\Gamma(\overline{\alpha}+\mu)} o_n w^{n-1}\right| \\ &\leq |w| \sum_{n=2}^{\infty} \frac{\Gamma(\mu)\left[\{1+\gamma(1+|I|)\}\left((1+\chi)^i-(1+\chi)^j\right)+\left|I(1+\chi)^i-H(1+\chi)^j\right|\right]}{(H-I)\,\Gamma(\overline{\alpha}+\mu)} o_n \\ &\leq |w| \sum_{n=2}^{\infty} \frac{\upsilon\left[\{1+\gamma(1+|I|)\}\left(\phi^i-\phi^j\right)+\left|I\phi^i-H\phi^j\right|\right]}{H-I} o_n \\ &\leq |w| < 1. \end{aligned}$$

This completes the proof of Theorem (2.10). □

3. Consequences and observations

In our present investigation, we have introduced and studied the properties of the analytic function classes

$$Q_{\chi,i,j}^{\overline{\alpha},\mu}\left(\gamma,H,I\right),\ \ TQ_{\chi,i,j}^{\overline{\alpha},\mu}\left(\gamma,H,I\right)$$
$$\left(-1\leq I<H\leq 1,\overline{\alpha},\mu,\gamma,\chi\geq 0,i\in\mathbb{N},\ j\in\mathbb{N}_0,\ i>j,w\in\Omega\right),$$

involving the Mittag-Leffler function. For functions belonging to these classes, we have derived coefficient conditions, extreme points, convolution conditions. The results obtained here are sharp. Our investigation involving the Mittag-Leffler function is potentially useful in motivating further researches on this subject.

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