Honam Mathematical J. **42** (2020), No. 3, pp. 601–620
https://doi.org/10.5831/HMJ.2020.42.3.601

η-RICCI SOLITONS ON TRANS-SASAKIAN MANIFOLDS WITH QUARTER-SYMMETRIC NON-METRIC CONNECTION

Oğuzhan Bahadir, Mohd Danish Siddiqi,
and Mehmet Akif Akyol*

Abstract. In this paper, firstly we discuss some basic axioms of trans Sasakian manifolds. Later, the trans-Sasakian manifold with quarter symmetric non-metric connection are studied and its curvature tensor and Ricci tensor are calculated. Also, we study the η-Ricci solitons on a Trans-Sasakian Manifolds with quarter-symmetric non-metric connection. Indeed, we investigated that the Ricci and η-Ricci solitons with quarter-symmetric non-metric connection satisfying the conditions $\widetilde{R}.\widetilde{S} = 0$. In a particular case, when the potential vector field ξ of the η-Ricci soliton is of gradient type $\xi = grad(\psi)$, we derive, from the η-Ricci soliton equation, a Laplacian equation satisfied by ψ. Finally, we furnish an example for trans-Sasakian manifolds with quarter-symmetric non-metric connection admitting the η-Ricci solitons.

1. Introduction

In 1985, Oubina[15] introduced a new class of almost contact Riemannian manifold known as trans-Sasakian manifold. Trans-Sasakian structures includes cosymplectic, $\alpha-$Sasakian, Sasakian, $\beta-$Kenmotsu, Kenmotsu structures.

In [11], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors ([1]- [5]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by S.Golab

Received April 14, 2020. Revised June 5, 2020. Accepted June 9, 2020.
2010 Mathematics Subject Classification. 53C15, 53C25, 53C40.
Key words and phrases. η-Ricci solitons, Trans-Sasakian manifolds, Quarter-symmetric non-metric connection.
*Corresponding author

[9]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\overline{T}$ is of the form

$$\overline{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y, \tag{1}$$

for any vector fields X, Y on a manifold, where u is a $1-$form and φ is a tensor of type $(1,1)$. If $\varphi = I$, then the quarter-symmetric connection is reduced to a semi-symmetric connection. Hence quarter-symmetric connection can be viewed as a generalization of semi-symmetric connection. The connection $\overline{\nabla}$ is said to be a metric connection if there is a Riemannian metric g in M such that $\overline{\nabla}g = 0$, otherwise it is non-metric. In [17], Sharfuddin and Hussian defined a semi-symmetric metric connection in an almost contact manifold, by setting

$$\overline{T}(X,Y) = \eta(Y)X - \eta(X)Y. \tag{2}$$

On the other hand the Ricci solitons are natural generalization of Einstein metrics on a Riemannian manifold being generalized fixed point of Hamilton's Ricci flow, $\frac{\partial g}{\partial t} = -2Ric$, is defined as [10]

$$\mathcal{L}_V g + 2Ric + 2\lambda g = 0, \tag{3}$$

where Ric is the Ricci tensor, L_V is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

If the vector field V is the gradient of a potential function -ψ , then g is called a gradient Ricci soliton and equation (3) assumes the form $\nabla\nabla\psi = S + \lambda g$.

Ricci flow as they correspond to self-similar solutions, and often arise as singularity models. They are also related to smooth metric measure spaces, since equation (3) is equivalent to ∞-Bakry-Emery Ricci tensor $Ric\psi = 0$. In physics, a smooth metric space $(M, g, e^{\psi}, dvol)$ with $Ric\psi = \lambda$g is called quasi-Einstein manifold. Therefore it is important to study geometry and topology of gradient Ricci solitons and their classifications.

Moreover, we focus on the case when the potential vector field ξ is of gradient type i.e., $\xi = grad(f)$, for f a nonconstant smooth function on M.

The evolution equation defining the Ricci flow is a type on nonlinear diffusion equation, counterpart of heat equation for metrics. It has become more important after G. Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. The more general notion of a Ricci soliton is that η-Ricci soliton introduced by J. T. Cho and M. Kimura [6]. In [22] K. Venu and G. Nagaraja study the η-Ricci solitos in trans-Sasakian manifolds. The geometrical axioms of the η-Ricci solitons have been studied for more details see ([3], [18], [19], [22]). It is natural and interesting to study η-Ricci solitons in trans-Sasakian manifolds with a quarter-symmetric non-metric connection not as real hypersurfaces of complex space forms but a special contact structures which is a generalization of Sasakian and Kenmotsu manifolds.

In this paper, Section 2 we give basic properties of trans Sasakian manifolds. In Section 3 the trans-Sasakian manifold with quarter symmetric non metric connection are studied and its curvature tensor and Ricci tensor are calculated. In Section 4 we study the η-Ricci solitons in a trans-Sasakian manifold with respect to a quarter-symmetric non-metric connection. Finally we give an example for trans-Sasakian manifolds with quarter-symmetric non-metric connection admitting the η-Ricci Soliton.

2. Preliminaries

Let M be an almost contact metric manifold of dimension $n(=2m+1)$ with an almost contact metric structure (φ, ξ, η, g), that is, φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that,

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta o \varphi = 0, \tag{4}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{5}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \; g(X, \xi) = \eta(X), \tag{6}$$

for all $X, Y \in TM$.

Definition 2.1. *An almost contact metric structure (φ, ξ, η, g) on M is called a trans-Sasakian structure if*

$$(\nabla_X \varphi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{7}$$

for some smooth functions α and β on M and we say that the trans-Sasakian structure is of type (α, β).

We note that the trans-Sasakian structures of type $(0,0)$ are cosymplectic, trans-Sasakian structures of type $(\alpha, 0)$ are α-Sasakian and trans-Sasakian structures of type $(0, \beta)$ are β-Kenmotsu.
Trans-Sasakian manifolds have been studied by authors[8] and they obtained the following results:

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi) \tag{8}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y) \tag{9}$$

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2 Y \\ &+ 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2 X \end{aligned} \tag{10}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X] \tag{11}$$

$$S(X,\xi) = [(n-1)(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (\varphi X)\alpha - (n-2)(X\beta) \tag{12}$$

when $\varphi(grad\alpha) = (n-2)grad\beta$, then from (12) the followings is provided.

$$S(X,\xi) = (n-1)(\alpha^2 - \beta^2)\eta(X), \tag{13}$$

$$S(\xi,\xi) = (n-1)(\alpha^2 - \beta^2). \tag{14}$$

3. Curvature Tensor and Ricci Tensor on a Trans- Sasakian Manifold with respect to Quarter-Symmetric Non-Metric Connection

Let $\widetilde{R}$ and R be the curvature tensors with respect to the quarter-symmetric non-metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ on a trans- Sasakian manifold M, respectively.

In this section we shall find the relation between $\widetilde{R}$ and R. Also we shall find the relation between $\widetilde{S}$ and S; where $\widetilde{S}$ is the Ricci tensor with respect to quarter-symmetric non-metric connection $\widetilde{\nabla}$ on M, S is Ricci tensor with respect to Levi-Civita connection ∇ on M.
In [21] Tripathi introduced the following connection :

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi Y - \eta(X)Y - \eta(Y)X + g(X,Y)\xi, \tag{15}$$

where $\widetilde{\nabla}$ is quarter-symmetric non-metric connection. Using definition 2.1 and the equation (8), we obtain

$$(\widetilde{\nabla}_X \varphi)(Y) = \alpha(g(X,Y)\xi - \eta(Y)X) + (\beta - 1)(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{16}$$

$$\widetilde{\nabla}_X \xi = -\alpha\varphi X + (\beta - 1)X - \beta\eta(X)\xi. \tag{17}$$

Now, we will obtain curvature tensor and Ricci tesor by using this connection. The curvature tensor $\widetilde{R}$ is as follows;

Theorem 3.1. *Let M be a trans-Sasakian manifold with the quarter-symmetric non-metric connection $\widetilde{\nabla}$. Then, the following equality is provided, for any X, Y and Z vector fields on M.*

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + \alpha[(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi + (\eta(Y)\eta(Z) - g(\varphi Y,Z))X \\ &+ (-\eta(X)\eta(Z) + g(\varphi X,Z))Y - 2g(X,\varphi Y)Z - g(Y,Z)\varphi X + g(X,Z)\varphi Y \\ &- 2g(X,\varphi Y)\varphi Z] + (\beta-1)[(g(\varphi Y,Z)\eta(X) - g(Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y) \\ &+ g(X,Z)\eta(Y))\xi - (\eta(Y)\eta(Z) - g(Y,Z))X + (\eta(X)\eta(Z) - g(X,Z))Y \\ &+ \eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y] + \beta[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{18}$$

where R and $\widetilde{R}$ are the curvature tensors with respect to the Levi-Civita connection and quarter-symmetric non-metric connection, respectively.

Proof. The curvature tensor with respect to quarter-symmetric non-metric connection is given as follows:

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_Z X - \widetilde{\nabla}_{[X,Y]}Z \tag{19}$$

If we write (15) in (19) and using (4), (16) and (17), we can easily have the proof. □

Corollary 3.2. *If an n-dimensional trans-Sasakian manifold M, from (18) we get the following statements.*

1. *If M is an α-Sasakian manifold, then*

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + \alpha[(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi + (\eta(Y)\eta(Z) - g(\varphi Y,Z))X \\ &+ (-\eta(X)\eta(Z) + g(\varphi X,Z))Y - 2g(X,\varphi Y)Z - g(Y,Z)\varphi X + g(X,Z)\varphi Y \\ &- 2g(X,\varphi Y)\varphi Z] - [(g(\varphi Y,Z)\eta(X) - g(Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y) \\ &+ g(X,Z)\eta(Y))\xi - (\eta(Y)\eta(Z) - g(Y,Z))X + (\eta(X)\eta(Z) - g(X,Z))Y \\ &+ \eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y] \end{aligned} \tag{20}$$

2. *If M is a Sasakian manifold, then*

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + [(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi + (\eta(Y)\eta(Z) - g(\varphi Y,Z))X \\ &+ (-\eta(X)\eta(Z) + g(\varphi X,Z))Y - 2g(X,\varphi Y)Z - g(Y,Z)\varphi X + g(X,Z)\varphi Y \\ &- 2g(X,\varphi Y)\varphi Z] - [(g(\varphi Y,Z)\eta(X) - g(Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y) \\ &+ g(X,Z)\eta(Y))\xi - (\eta(Y)\eta(Z) - g(Y,Z))X + (\eta(X)\eta(Z) - g(X,Z))Y \\ &+ \eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y] \end{aligned} \tag{21}$$

3. *If M is a β-Kenmotsu manifold, then*

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + (\beta-1)[(g(\varphi Y,Z)\eta(X) - g(Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y) \\ &+ g(X,Z)\eta(Y))\xi - (\eta(Y)\eta(Z) - g(Y,Z))X + (\eta(X)\eta(Z) - g(X,Z))Y \\ &+ \eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y] + \beta[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{22}$$

4. *If M is a Kenmotsu manifold, then*

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y \tag{23}$$

5. *If M is a cosymplectic manifold, then*

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z - [(g(\varphi Y,Z)\eta(X) - g(Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y) \\ + g(X,Z)\eta(Y))\xi - (\eta(Y)\eta(Z) - g(Y,Z))X + (\eta(X)\eta(Z) - g(X,Z))Y \\ + \eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y]\end{aligned} \tag{24}$$

Proposition 3.3. *Let M be a trans-Sasakian manifold with the quarter-symmetric non-metric connection $\widetilde{\nabla}$. Then the following equality is provided.*

$$\begin{aligned}\widetilde{R}(X,Y)Z + \widetilde{R}(Y,Z)X + \widetilde{R}(Z,X)Y = \alpha[-2g(X,\varphi Y)\varphi Z - 2g(Y,\varphi Z)\varphi X - 2g(Z,\varphi X)\varphi Y] \\ + (\beta - 1)[2g(\varphi Y,Z)\eta(X)\xi + 2g(\varphi Z,X)\eta(Y)\xi \\ + 2g(\varphi X,Y)\eta(Z)\xi]\end{aligned} \tag{25}$$

Proof. Using first bianchi identity with respect to Levi-Civita connection and from Theorem (3.1) we get the result. □

From Proposition 3.3, we get the following corollary.

Corollary 3.4. *If an n-dimensional trans-Sasakian manifold M is a Kenmotsu manifold, then first bianchi identity with respect to the quarter-symmetric non-metric connection $\widetilde{\nabla}$ is provided.*

Theorem 3.5. *Let M be a trans-Sasakian manifold with the quarter-symmetric non-metric connection $\widetilde{\nabla}$. Then the following equality is provided.*

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \alpha[(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))g(\xi,W) + (\eta(Y)\eta(Z) \\ - g(\varphi Y,Z))g(X,W) + (-\eta(X)\eta(Z) + g(\varphi X,Z))g(Y,W) - 2g(X,\varphi Y)g(Z,W) \\ - g(Y,Z)g(\varphi X,W) + g(X,Z)g(\varphi Y,W) - 2g(X,\varphi Y)g(\varphi Z,W)] \\ + (\beta - 1)[(g(\varphi Y,Z)\eta(X) - g(Y,Z)\eta(X) \\ - g(\varphi X,Z)\eta(Y) + g(X,Z)\eta(Y))g(\xi,W) \\ - (\eta(Y)\eta(Z) - g(Y,Z))g(X,W) + (\eta(X)\eta(Z) - g(X,Z))g(Y,W) \\ + \eta(Y)\eta(Z)g(\varphi X,W) - \eta(X)\eta(Z)g(\varphi Y,W)] \\ + \beta[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned}$$

Proof. Using the equality $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and Theorem (3.1) we get the result. □

From Theorem (3.5), we get the following results

Corollary 3.6. *If an n-dimensional trans-Sasakian manifold M, we obtain the following relations:*

$$\widetilde{R}(X,Y,Z,Z) = 2\alpha g(\varphi X,Y)g(Z,Z), \tag{26}$$

$$\widetilde{R}(X,Y,Z,W) + \widetilde{R}(Y,X,Z,W) = 0, \tag{27}$$

$$\widetilde{R}(X,Y,Z,W) + \widetilde{R}(X,Y,W,Z) = 4\alpha g(\varphi X,Y)g(Z,W), \tag{28}$$

$$\widetilde{R}(X,Y,Z,W) - \widetilde{R}(Y,X,W,Z) = -4\alpha g(\varphi X,Y)g(Z,W). \tag{29}$$

Corollary 3.7. *If an n-dimensional trans-Sasakian manifold M is Kenmotsu or β-Kenmotsu or cosymplectic manifold, we obtain the following equations:*

$$\widetilde{R}(X,Y,Z,Z) = 0, \tag{30}$$

$$\widetilde{R}(X,Y,Z,W) + \widetilde{R}(X,Y,W,Z) = 0, \tag{31}$$

$$\widetilde{R}(X,Y,Z,W) = \widetilde{R}(Y,X,W,Z). \tag{32}$$

Theorem 3.8. *In a trans-Sasakian manifold M, the Ricci tensor$\widetilde{S}$ and the scalar curvature $\widetilde{r}$ with respect to quarter-symmetric non-metric connection $\widetilde{\nabla}$ is obtained as follows:*

$$\begin{aligned}\widetilde{S}(X,Y) = S(X,Y) + \alpha[-g(X,Y) + n\eta(X)\eta(Y) - ng(\varphi X,Y)] + (\beta-1)[g(\varphi X,Y)\\ + (n-2)g(\varphi X,\varphi Y)] + \beta(n-1)g(X,Y),\end{aligned} \tag{33}$$

$$\widetilde{r} = r + 2\beta(n-1)^2 - (n-1)(n-2), \tag{34}$$

where S and r are Ricci tensor and scalar curvature with respect to Levi-Civita connection ∇ on M, respectively.

Proof.

$$
\begin{aligned}
\widetilde{S}(X,Y) &= \sum_{i=1}^{n} \widetilde{R}(e_i,X,Y,e_i) \\
&= \sum_{i=1}^{n} R(e_i,X,Y,e_i) + \alpha[\sum_{i=1}^{n}(g(X,Y)\eta(e_i) - g(e_i,Y)\eta(X))g(e_i,\xi) \\
&+ \sum_{i=1}^{n}(\eta(X)\eta(Y) - g(\varphi X,Y))g(e_i,e_i) + \sum_{i=1}^{n}(-\eta(e_i)\eta(Y) + g(\varphi e_i,Y))g(X,e_i) \\
&- \sum_{i=1}^{n} 2g(e_i,\varphi X)g(Y,e_i) - \sum_{i=1}^{n} g(X,Y)g(\varphi e_i,e_i) + \sum_{i=1}^{n} g(e_i,Y)g(\varphi X,e_i) \\
&- \sum_{i=1}^{n} 2g(e_i,\varphi X)g(\varphi Y,e_i)] + (\beta-1)[\sum_{i=1}^{n}(g(\varphi X,Y)\eta(e_i) - g(X,Y)\eta(e_i) \\
&- g(\varphi e_i,Y)\eta(X) + g(e_i,Y)\eta(X))g(\xi,e_i) - \sum_{i=1}^{n}(\eta(X)\eta(Y) - g(X,Y))g(e_i,e_i) \\
&+ \sum_{i=1}^{n}(\eta(e_i)\eta(Y) - g(e_i,Y))g(X,e_i) + \sum_{i=1}^{n}\eta(X)\eta(Y)g(\varphi e_i,e_i) \\
&- \sum_{i=1}^{n}\eta(e_i)\eta(Y)g(\varphi X,e_i)] + \beta(\sum_{i=1}^{n} g(X,Y)g(e_i,e_i) - \sum_{i=1}^{n} g(e_i,Y)g(X,e_i)) \\
&= S(X,Y) + \alpha[-g(X,Y) + n\eta(X)\eta(Y) - ng(\varphi X,Y)] + (\beta-1)[g(\varphi X,Y) \\
&+ (n-2)g(\varphi X,\varphi Y)] + \beta(n-1)g(X,Y)
\end{aligned}
$$

Using the equality

$$r = \sum_{i=1}^{n} S(e_i,e_i)$$

and from (33) we get the following result.
$\widetilde{r} = r + 2\beta(n-1)^2 - (n-1)(n-2)$ □

Corollary 3.9. *Let M be n-dimension a trans-Sasakian manifold with quarter-symmetric non-metric connection $\widetilde{\nabla}$. Ricci tensor with respect to quarter-symmetric non-metric connection $\widetilde{\nabla}$ is symmetric if and only if $n\alpha = \beta - 1$.*

Proof. Using (33) we get

$$
\begin{aligned}
\widetilde{S}(X,Y) - \widetilde{S}(Y,X) &= \alpha[-ng(\varphi X,Y) + ng(\varphi Y,X)] + (\beta-1)[g(\varphi X,Y) - g(\varphi Y,X)] \\
&= \alpha[-2ng(\varphi X,Y)] + (\beta-1)[2g(\varphi X,Y)] \\
&= [-2n\alpha + 2(\beta-1)]g(\varphi X,Y)
\end{aligned}
$$

$\widetilde{S}(X,Y)$ is symmetric if and only if $[-2n\alpha + 2(\beta-1)]g(\varphi X,Y) = 0$, i.e., $n\alpha = \beta - 1$ □

Using (13), (14) and theorem 2, we get the following proposition

Proposition 3.10. *The curvature tensor $\widetilde{R}$, the Ricci tensor $\widetilde{S}$ and the Ricci operator $\widetilde{Q}$ in a trans-Sasakian manifold with respect to the quarter symmetric non-metric connection for constants α and β, we have* [22]

$$\widetilde{R}(X,Y)\xi = (\alpha+\beta)(\alpha-\beta+1)[\eta(Y)X-\eta(X)Y] + (2\alpha\beta+\beta-\alpha-1)[\eta(Y)\varphi X-\eta(X)\varphi Y] \tag{35}$$

$$\widetilde{R}(\xi,X)\xi = (\alpha+\beta)(\alpha-\beta+1)(-X+\eta(X)\xi) + (\alpha-\beta+1)\varphi X \tag{36}$$

$$\widetilde{R}(X,\xi)\xi = (\alpha-\beta+1)(\alpha+\beta)(X-\eta(X)\xi) + (2\alpha\beta+\beta-\alpha-1)\varphi X \tag{37}$$

$$\widetilde{S}(X,\xi) = (n-1)(\alpha+\beta)(\alpha-\beta+1)\eta(X) \tag{38}$$

$$\widetilde{S}(\xi,\xi) = (n-1)(\alpha+\beta)(\alpha-\beta+1)$$

In a 3$-$ dimensional the curvature tensor $\widetilde{R}$ is defined as

$$\begin{aligned}\widetilde{R}(X,Y)Z &= \widetilde{S}(Y,Z)X - g(X,Z)\widetilde{Q}Y + g(Y,Z)\widetilde{Q}X \\ &- \widetilde{S}(X,Z)Y - \frac{\widetilde{r}}{2}(g(Y,Z)X - g(X,Z)Y),\end{aligned} \tag{39}$$

where $\widetilde{R}$, $\widetilde{Q}$, $\widetilde{S}$ and $\widetilde{r}$ are curvature tensor, Ricci operator, Ricci tensor and scalar curvature with respect to quarter symmetric non-metric connection, respectively.

Proposition 3.11. *In a 3- dimensional trans-Sasakian manifold with quarter-symmetric non-metric connection, Ricci tensor with respect to quarter-symmetric non-metric connection follows that*

$$\begin{aligned}\widetilde{S}(Y,Z) &= \Big\{3(\alpha-\beta+1)(\alpha+\beta) - \frac{\widetilde{r}}{2}\Big\}\eta(Y)\eta(Z) \\ &- \Big\{(\alpha-\beta+1)(\alpha+\beta) - \frac{\widetilde{r}}{2}\Big\}g(Y,Z) - (-2\alpha\beta+\alpha-\beta+1)g(\phi Y,Z)\end{aligned} \tag{40}$$

Proof. Using (36), we get

$$\widetilde{R}(Y,\xi,\xi,Z) = (\alpha-\beta+1)(\alpha+\beta)(g(Y,Z)-\eta(Y)\eta(Z)) + (2\alpha\beta+\beta-\alpha-1)g(\varphi Y,Z) \tag{41}$$

Using (37), putting $X=\xi$ in (40) and contracting with ξ, we obtain

$$\begin{aligned}\widetilde{R}(\xi,Y,Z,\xi) &= \widetilde{S}(Y,Z) - \{3(\alpha-\beta+1)(\alpha+\beta) - \frac{\widetilde{r}}{2}\}\eta(Y)\eta(Z) \\ &+ \{2(\alpha-\beta+1)(\alpha+\beta) - \frac{\widetilde{r}}{2}\}g(Y,Z)\end{aligned} \tag{42}$$

Comparing the last two equations with respect to Corollary3.6, we have the proof. □

Corollary 3.12. *In a 3-dimensional trans-Sasakian manifold M, we have the following statements.*

1. *If M is a Sasakian manifold, then*

$$\widetilde{S}(Y,Z) = \Big\{6 - \frac{\widetilde{r}}{2}\Big\}\eta(Y)\eta(Z) - \Big\{2 - \frac{\widetilde{r}}{2}\Big\}g(Y,Z) - 2g(\phi Y, Z).$$

2. *If M is a Kenmotsu manifold, then*

$$\widetilde{S}(Y,Z) = \frac{\widetilde{r}}{2}(g(Y,Z) - \eta(Y)\eta(Z)),$$

so, M is $\eta-$ Einstein manifold.

3. *If M is a cosymplectic manifold, then*

$$\widetilde{S}(Y,Z) = \frac{\widetilde{r}}{2}(g(Y,Z) - \eta(Y)\eta(Z)) - g(\phi Y, Z),$$

Also, we consider

Corollary 3.13. *In a 3- dimensional trans-Sasakian manifold with quarter-symmetric non-metric connection, Ricci tensor with respect to quarter-symmetric non-metric connection is symmetric if and only if either $\alpha = 0$ or $\alpha = \frac{2}{3}$.*

Proof. Using (41) we get

$$\widetilde{S}(X,Y) - \widetilde{S}(Y,X) = [-2\alpha\beta - \beta + \alpha + 1]g(\varphi X, Y). \tag{43}$$

From Corollary 3.9, we know that $\beta = 1 + 3\alpha$. Thus, using (43) we get

$$\widetilde{S}(X,Y) - \widetilde{S}(Y,X) = -2\alpha(2 + 3\alpha)g(\varphi X, Y). \tag{44}$$

This proves our assertion. □

Example 3.14. *We consider the 3-dimensional manifold $M = \big\{(x,y,z) \in \mathbb{R}^3, z \neq 0\big\}$, where (x,y,z) are standard coordinate of R^3.*
The vector fields

$$e_1 = z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}),\ e_2 = z\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M.
Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_1,e_2) = g(e_2,e_3) = 0,$$
$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η *be the* 1*-form defined by* $\eta(Z) = g(Z, e_3)$ *for any* $Z \in \chi(M)$ *. Let* ϕ *be the* $(1,1)$ *tensor field defined by*

$$\phi(e_1) = e_2,\ \phi(e_2) = -e_1,\ \phi(e_3) = 0.$$

Then using the linearity of ϕ *and* g*, we have*

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$*, the set of all smooth vector fields on* M*.*
Then for $e_3 = \xi$*, the structure* (ϕ, ξ, η, g) *defines an almost contact metric structure on* M*.*
Let ∇ *be the Levi-Civita connection with respect to metric* g*. Then we have*

$$[e_1, e_2] = ye_2 - z^2 e_3, \quad [e_1, e_3] = -\frac{1}{z}e_1, \quad [e_2, e_3] = -\frac{1}{z}e_2$$

Taking $e_3 = \xi$ *and using Koszul formula for the Riemannian metric* g*, we can easily calculate*

$$\nabla_{e_1} e_3 = -\frac{1}{z}e_1 + \frac{1}{2}z^2 e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2}z^2 e_3, \nabla_{e_1} e_1 = \frac{1}{z}e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{z}e_2 - \frac{1}{2}z^2 e_1,$$
$$\nabla_{e_2} e_2 = ye_1 + \frac{1}{z}e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}z^2 e_3 - ye_2, \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = -\frac{1}{2}z^2 e_1, \quad \nabla_{e_3} e_1 = \frac{1}{2}z^2 e_2.$$

(ϕ, ξ, η, g) *is a trans-Sasakian structure on* M*. Consequently manifold* M *is trans-Sasakian manifold with* $\alpha = -\frac{1}{2}z^2 \neq 0$, $\beta = -\frac{1}{z} \neq 0$ [7].
Hence,one can easily obtain by simple calculation that the curvature and Ricci tensor components are as follows:

$$R(e_1, e_2)e_1 = (\frac{3}{4}z^4 + \frac{1}{z^2} + y^2)e_2 + yz^2 e_3, R(e_1, e_2)e_2 = (-\frac{3}{4}z^4 - \frac{1}{z^2} - y^2)e_1 - \frac{y}{z}e_3,$$
$$R(e_2, e_3)e_1 = -\frac{y}{z}e_2, R(e_1, e_2)e_3 = -yz^2 e_1 + \frac{y}{z}e_2,\ R(e_1, e_3)e_1 = yz^2 e_2 + (\frac{2}{z^2} - \frac{z^4}{4})e_3,$$
$$R(e_1, e_3)e_2 = -yz^2 e_1, R(e_1, e_3)e_3 = (\frac{z^4}{4} - \frac{2}{z^2})e_1, R(e_2, e_3)e_2 = \frac{y}{z}e_1 + (-\frac{z^4}{4} + \frac{2}{z^2})e_3,$$
$$R(e_2, e_3)e_3 = (\frac{z^4}{4} - \frac{2}{z^2})e_2, S(e_1, e_1) = -\frac{z^4}{2} - \frac{3}{z^2} - y^2, S(e_1, e_2) = 0, S(e_1, e_3) = -\frac{y}{z},$$
$$S(e_2, e_2) = -\frac{z^4}{2} - \frac{3}{z^2} - y^2, S(e_2, e_3) = -yz^2, S(e_3, e_3) = \frac{z^4}{2} - \frac{4}{z^2}.$$

Now, we compute other relations with respect to quarter-symmetric non-metric connection. From (15) we obtain

$$\widetilde{\nabla}_{e_1}e_3 = (\frac{-1-z}{z})e_1 + \frac{1}{2}z^2e_2,\ \widetilde{\nabla}_{e_1}e_2 = -\frac{1}{2}z^2e_3,\ \widetilde{\nabla}_{e_1}e_1 = (\frac{1+z}{z})e_3,$$
$$\widetilde{\nabla}_{e_2}e_2 = ye_1 + (\frac{1+z}{z})e_3,\ \widetilde{\nabla}_{e_2}e_1 = \frac{1}{2}z^2e_3 - ye_2,\ \widetilde{\nabla}_{e_3}e_3 = -e_3,$$
$$\widetilde{\nabla}_{e_2}e_3 = (\frac{-1-z}{z})e_2 - \frac{1}{2}z^2e_1, \widetilde{\nabla}_{e_3}e_2 = (\frac{-z^2+2}{2})e_1 - e_2,\ \widetilde{\nabla}_{e_3}e_1 = -e_1 + (\frac{z^2-2}{2})e_2.$$

Using above the equations, the components of the curvature tensors with respect to quarter symmetric non-metric connection is obtained as follows:

$$\widetilde{R}(e_1,e_2)e_1 = -z^2e_1 + (\frac{3z^4-4z^2}{4} + \frac{(1+z)^2}{z^2} + y^2)e_2 + yz^2e_3,$$
$$\widetilde{R}(e_1,e_2)e_2 = (\frac{-3z^4+4z^2}{4} - \frac{(1+z)^2}{z^2} - y^2)e_1 - z^2e_2 - \frac{y}{z}e_3,$$
$$\widetilde{R}(e_1,e_2)e_3 = -yz^2e_1 + \frac{y}{z}e_2 - z^2e_3,\ \widetilde{R}(e_1,e_3)e_1 = yz^2e_2 + (\frac{-z^4+2z^2}{4} + \frac{2+z}{z^2})e_3,$$
$$\widetilde{R}(e_1,e_3)e_2 = -yz^2e_1 + (\frac{-z^2+2}{2} + \frac{1}{z})e_3,$$
$$\widetilde{R}(e_1,e_3)e_3 = (\frac{z^4-2z^2}{4} - \frac{2+z}{z^2})e_1 + (\frac{z^2-2}{2} - \frac{1}{z})e_2,$$
$$\widetilde{R}(e_2,e_3)e_1 = -\frac{y}{z}e_2 + (\frac{z^2-2}{2} - \frac{1}{z})e_3,\ \widetilde{R}(e_2,e_3)e_2 = \frac{y}{z}e_1 + (\frac{-z^4+2z^2}{4} + \frac{2+z}{z^2})e_3,$$
$$\widetilde{R}(e_2,e_3)e_3 = (-\frac{z^2-2}{2} + \frac{1}{z})e_1 + (\frac{z^4-2z^2}{4} - \frac{2+z}{z^2})e_2. \tag{45}$$

Using (45), we get the components of the Ricci and scalar curvature with respect to quarter symmetric non-metric connection, as follows;

$$\widetilde{S}(e_1,e_1) = \frac{-z^4+z^2}{2} - \frac{3}{z^2} - \frac{3}{z} - y^2 - 1,\ \widetilde{S}(e_1,e_2) = \frac{3}{2}z^2 - \frac{1}{z} - 1,$$
$$\widetilde{S}(e_1,e_3) = -\frac{y}{z},\ \widetilde{S}(e_2,e_1) = -\frac{3}{2}z^2 + \frac{1}{z} + 1,\ \widetilde{S}(e_2,e_2) = \frac{-z^4+z^2}{2} - \frac{3}{z^2} - \frac{3}{z} - y^2 - 1,$$
$$\widetilde{S}(e_2,e_3) = -yz^2,\ \widetilde{S}(e_3,e_1) = -\frac{y}{z},\ \widetilde{S}(e_3,e_2) = -yz^2,\ \widetilde{S}(e_3,e_3) = \frac{z^4-2z^2}{2} - \frac{4}{z^2} - \frac{2}{z},$$
$$r = -\frac{z^4}{2} - \frac{10}{z^2} - 2y^2,\ \widetilde{r} = -\frac{z^4}{2} - \frac{10}{z^2} - \frac{8}{z} - 2y^2 - 2.$$

The above equations prove some results in this paper.

4. η-Ricci solitons on $(M, \phi, \xi, \eta, g,)$ admitting quarter symmetric non-metric connection

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. Consider the equation

$$\mathcal{L}_\xi g + 2\widetilde{S} + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{46}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, $\widetilde{S}$ is the Ricci curvature tensor field with respect to the quarter symmetric non-metric connection of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi$ in terms of the quarter-symmetric metric connection $\widetilde{\nabla}$, we obtain:

$$2\widetilde{S}(X, Y) = -g(\widetilde{\nabla}_X \xi, Y) - g(X, \widetilde{\nabla}_Y \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y), \tag{47}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (46) is said to be an *η-Ricci soliton* on M [6]. In particular if $\mu = 0$ then (g, ξ, λ) is called *Ricci soliton* [10] and it is called *shrinking*, *steady* or *expanding*, according as λ is negative, zero or positive respectively [6].

On a trans-Sasakian manifold with quarter-symmetric non-metric connection using equation (17) and $\mathcal{L}_\xi g = 2[\beta g - \beta\eta \otimes \eta]$, the equation (47) becomes:

$$\widetilde{S}(X, Y) = -(\lambda + \beta)g(X, Y) + (\beta - \mu)\eta(X)\eta(Y). \tag{48}$$

In particular, $X = \xi$, we obtain

$$\widetilde{S}(X, \xi) = -(\lambda + \mu)\eta(X). \tag{49}$$

In this case, the Ricci operator $\widetilde{Q}$ is defined by $g(\widetilde{Q}X, Y) = \widetilde{S}(X, Y)$ has the expression

$$\widetilde{Q}X = -(\lambda + \beta)X + (\beta - \mu)\eta(X)\xi. \tag{50}$$

From the above discussion we can conclude the results in the form of following proposition.

Proposition 4.1. *In a trans-Sasakian manifold with quarter- symmetric non-metric connection, the existence of an η-Ricci soliton implies that the characteristic vector field ξ is an eigenvector of Ricci operator corresponding to the eigenvalue $-(\lambda + \mu + \beta - 1)$.*

Recall that the manifold is called *quasi-Einstein* if the Ricci curvature tensor field S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero 1-from η

satisfying $\eta = g(X, \xi)$, for ξ a unit vector field and respectively, *Einstein* if S is collinear with g.

Theorem 4.2. *If (ϕ, ξ, η, g) is a trans-Sasakian manifold with quarter-symmetric non-metric connection on M and (46) defines an η-Ricci soliton on M, then*

1. $\widetilde{Q} \circ \phi = \phi \circ \widetilde{Q}$
2. *$\widetilde{Q}$ and $\widetilde{S}$ are parallel along ξ.*

Proof. The first statement follows from a direct computation and for the second one, note that

$$(\widetilde{\nabla}_\xi \widetilde{Q})X = \widetilde{\nabla}_\xi \widetilde{Q}X - \widetilde{Q}(\widetilde{\nabla}_\xi X) \tag{51}$$

and

$$(\widetilde{\nabla}_\xi \widetilde{S})(X, Y) = \xi(\widetilde{S}(X, Y)) - \widetilde{S}(\widetilde{\nabla}_\xi X, Y) - \widetilde{S}(X, \widetilde{\nabla}_\xi Y). \tag{52}$$

Replacing $\widetilde{Q}$ and $\widetilde{S}$ from (50) and (49) we get the conclusion. □

A particular case arises when the manifold is ϕ-Ricci symmetric, which means that $\phi^2 \circ \nabla\widetilde{Q} = 0$, that fact is stated in the next theorem.

Theorem 4.3. *Let (M, ϕ, ξ, η, g) be a trans-Sasakian manifold with quarter-symmetric non-metric connection. If M is ϕ-Ricci symmetric and (46) defines an η-Ricci soliton on M, then $\mu = 1$ and (M, g) is Einstein manifold.*

Proof. Replacing $\widetilde{Q}$ from (49) in (50) and applying ϕ^2 we obtain

$$(\beta - \mu)\eta(Y)[X - \eta(X)\xi] = 0, \tag{53}$$

for any $X, Y \in \chi(M)$. Follows $\mu = \beta$ and $S = -(\lambda + \beta - 1)g$. □

In particular, the existence of an η-Ricci soliton on a trans-Sasakian manifold with quarter-symmetric non-metric connection which is *Ricci symmetric* (i.e. $\widetilde{\nabla}S = 0$) implies that M is an *Einstein* manifold. The class of Ricci symmetric manifold represents an extension of class of Einstein manifold to which the locally symmetric manifold (*i.e. satisfying* $\widetilde{\nabla}R = 0$). The condition $\widetilde{\nabla}S = 0$ implies $\widetilde{R}.\widetilde{S} = 0$ and the manifolds satisfying this condition are called *Ricci semi-symmetric*.

In what follows, we shall consider η-Ricci solitons requiring for the curvature to satisfy $\widetilde{R}(\xi, X).\widetilde{S} = 0$.

5. η-Ricci solitions on a trans-Sasakian manifold with quarter-symmetric non-metric connection satisfying $\bar{R}(\xi, X).\bar{S} = 0$

Now we consider a trans-Sasakian manifold with quarter-symmetric non-metric connection $\widetilde{\nabla}$ satisfying the condition

$$\widetilde{S}(\widetilde{R}(\xi, X)Y, Z) + \widetilde{S}(Y, \widetilde{R}(\xi, X)Z) = 0, \tag{54}$$

for any $X, Y \in \chi(M)$.

Replacing the expression of $\bar{S}$ from (48) and from the symmetries of $\widetilde{R}$ we get

$$(\alpha+\beta)(\alpha-\beta+1)(\beta-\mu)[\eta(Y)g(X,Z)+\eta(Z)g(X,Y)-2\eta(X)\eta(Y)\eta(Z)]=0, \tag{55}$$

for any $X, Y \in \chi(M)$.
For $Z = \xi$ we have

$$(\alpha + \beta)(\alpha - \beta + 1)(\beta - \mu)g(\phi X, \phi Y) = 0, \tag{56}$$

for any $X, Y \in \chi(M)$.
Hence we can state the following theorem:

Theorem 5.1. *If a trans-Sasakian manifold with quarter-symmetric non-metric connection $\widetilde{\nabla}$, (g, ξ, λ, μ) is an η-Ricci soliton on M and satisfies $\widetilde{R}(\xi, X).\widetilde{S} = 0$, then the manifold is an η-Einstein manifold.*

For $\mu = 0$, we deduce:

Corollary 5.2. *On a trans-Sasakian manifold with quarter- symmetric non-metric connection satisfying $\widetilde{R}(\xi, X).\widetilde{S} = 0$, there is no η-Ricci soliton with the potential vector field ξ.*

6. η-Ricci solitions on a trans-Sasakian manifold with quarter-symmetric non-metric connection satisfying $\xi = grad(\psi)$

In this section, we consider the case when the potential vector filed ξ of the η-Ricci soliton is of the gradient type, $\xi = grad(\psi)$.

Again, consider the equation (46)

$$\mathcal{L}_\xi g + 2\widetilde{S} + 2\lambda g(X, Y) + 2\mu\eta \otimes \eta = 0, \tag{57}$$

where g is the Riemannian metric and $\widetilde{S}$ is the Ricci curvature tensor with respect to the quarter-symmetric non-metric connection, ξ is a vector filed, λ and μ are real constant.

Writing explicitly the Lie derivative $\mathcal{L}_\xi g$ and using equations (57) and (17) we obtain

(58)
$$\widetilde{S}(X,Y)=-(\lambda+\beta)g(X,Y)+(\beta-\mu)\eta(X)\eta(Y)-\frac{1}{2}[g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)],$$

Contracting (58) we get

$$\widetilde{r}=-(\lambda+\beta)n+(\beta-\mu)-div(\xi). \tag{59}$$

Let $(M,\phi,\xi,\eta,g,)$ be a trans-Sasakian manifold with a quarter-symmetric non-metric connection, (g,ξ,λ,μ) is an η-Ricci soliton on M. From (41) and (58) we obtain

$$\begin{aligned}[3(\alpha+\beta)(\alpha-\beta+1)-\frac{\widetilde{r}}{2}+(\lambda+\beta)]g(X,Y)\\ +[-(\alpha+\beta)(\alpha-\beta+1)-\frac{\widetilde{r}}{2}-(\beta-\mu)]\eta(X)\eta(Y)\\ -(-2\alpha\beta+\alpha-\beta+1)g(\phi Y,Z)+\frac{1}{2}[g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)]=0.\end{aligned} \tag{60}$$

Consider $\{e_i\}_{1\leq i\leq n}$ an orthonormal frame field on trans-Sasakian manifold, then $\{\phi e_1,\phi e_2,...\phi e_n,\xi\}$ is also orthonormal basis of vectors. Putting $X=Y=e_i$, we get

$$n(\lambda+\beta)-(\beta-\mu)=-(3n-1)(\alpha+\beta)(\alpha-\beta+1)-\frac{(n+1)}{2}\widetilde{r}-div(\xi). \tag{61}$$

Now, writing (60) for $X=Y=\xi$, we obtain

$$(\lambda+\beta)-(\beta-\mu)=-2(\alpha+\beta)(\alpha-\beta+1)+\widetilde{r}. \tag{62}$$

$$\begin{cases}\lambda=3(\alpha+\beta)(\alpha-\beta+1)-\frac{(n+3)}{2(n-1)}\widetilde{r}-\frac{div(\xi)}{(n-1)}-\beta\\ \mu=-(\alpha+\beta)(\alpha-\beta+1)+\frac{3n+1}{2(n-1)}\widetilde{r}+\frac{div(\xi)}{(n-1)}+\beta\end{cases} \tag{63}$$

Therefore, we can state the following theorem:

Theorem 6.1. *Let (M,ϕ,ξ,η,g) be a trans-Sasakian manifold with a quarter-symmetric non-metric connection and η be the g dual 1-form of the gradient vector field $\xi=grad(\psi)$. If (46) defines an η-Ricci soliton on M with respect to the quarter-symmetric non-metric connection, then the Laplacian equation satisfied by ψ becomes*

$$\Delta(\psi)=-(n-1)[\mu+(\alpha+\beta)(\alpha-\beta+1)+\beta]+\frac{(3n+1)}{2}\widetilde{r}. \tag{64}$$

Remark 6.2. *From equation (63) and (8) if $div(\xi) = -(n-1)\beta$. Then $\widetilde{r}$ is a constant. In addition, if M is compact by the divergence theorem then $\beta = 0$.*

Significance of Laplacian equation in Physics. The general theory of solution of Laplace equation is known as *potential theory* and the solution of Laplace equation are harmonic functions, which are important in branches of physics, electrostatics, gravitation and fluid dynamics. In modern physics there are two fundamental forces of the nature known at the time, namely gravity and the electrostatics forces, could be modeled using functions called the gravitational potential and electrostatics potential both of which satisfy Laplace equation. For example consider the case if ψ be the gravitational filed, ρ the mass density and G the gravitational constant. The Gauss's law of gravitational in differential form is

$$\nabla\psi = -4\pi G\rho \tag{65}$$

in case of gravitational field ψ is conservative and can be expressed as the negative gradient of gravitational potential i.e $\psi = -gradf$ then by the Gauss's law of gravitational , we have

$$\nabla^2 f = 4\pi G\rho. \tag{66}$$

This physical phenomena is directly identical to the theorem (6.1)and equation (64), which is Laplacian equation with potential vector filed of gradient type.

7. Example of η-Ricci solitons on $(M, \phi, \xi, \eta, g,)$ with a quarter-symmetric non-metric connection

Example 7.1. *Let $M(\phi, \xi, \eta, g)$ be the a trans-Sasakian manifold with a quarter-symmetric non-metric connection considered in example 15 (page 9-10).*

From equation (45) the components of Ricci curvature tensor with respect to quarter-symmetric non-metric connection are given by:

$$\widetilde{S}(e_1,e_1)=\frac{-z^4+z^2}{2}-\frac{3}{z^2}-\frac{3}{z}-y^2-1,$$
$$;\widetilde{S}(e_2,e_2)=\frac{-z^4+z^2}{2}-\frac{3}{z^2}-\frac{3}{z}-y^2-1,$$
$$\widetilde{S}(e_3,e_3)=\frac{z^4-2z^2}{2}-\frac{4}{z^2}-\frac{2}{z},$$

Now using the equation (48) and $\beta=-\frac{1}{z}$, we obtain $\widetilde{S}(e_1,e_1)=-(\lambda+\beta)$ and $\widetilde{S}(e_3,e_3)=(\beta-\mu)$ therefore $\lambda=\frac{z^4-z^2}{2}+\frac{3+4z}{z^2}+y^2$ and $\mu=\frac{4+z}{z^2}-\frac{z^4-z^2}{2}$.

Acknowledgement. The author is thankful to the referee's for their valuable criticisms, comments and suggestions towards the improvement of the paper.

References

[1] N. S. Agashe and M. R. Chafle, *A semi symmetric non-metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math. **23** (1992), 399-409.

[2] M. Ahmad, J. B. Jun, and M. D. Siddiqi, *On some properties of semi-invariant submanifolds of a nearly trans-Sasakian manifolds admitting a quarter-symmetric non-metric connection*, Journal of the Chungcheong Math. Soc., **25(1)** (2012), 73-90.

[3] A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat, **30(2)** (2016), 489-496.

[4] D. E. Blair, *Contact manifold in Riemannian geometry*, Lecture Notes in Math., Springer Verlag, 1976.

[5] A. Haseeb, M. A. Khan, M. D. Siddiqi, *Some more results on an epsilon-Kenmotsu manifold with a semi-symmetric semi-metric connection*, Acta Mathematica Universitatis Comenianae, **85(1)** (2016), 9-20.

[6] J. T. Cho, M. Kimura, *Ricci solitons and Real hypersurfaces in a complex space form*, Tohoku Math.J., **61** (2009), 205-212.

[7] U. C. De, K. De, *On a class of three-dimensional trans-Sasakian manifolds*, Commun. Korean Math. Soc., **27(4)** (2012), 795-808.

[8] U. C. De, M. M. Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J., **2** (2003), 247-255.

[9] S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor **29** (1975), 249-254.

[10] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity*, (Santa Cruz. CA, 1986), Contemp. Math., AMS, **71** (1988), 237-262.

[11] H. A. Hayden, *Subspaces of a space with torsion*, Proc. London Math. Soc. **34** (1932), 27-50.

[12] D. Janssens, L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J., **4** (1981), 1-27.

[13] J. S. Kim, R. Prasad and M. M. Tripathi, *On generalized Ricci-recurrent trans-Sasakian manifolds*, J. Korean Math. Soc., **39(6)** (2012), 953-961.

[14] J. C. Marrero, , *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl., **162(4)** (1992), 77-86.

[15] J. A. Oubina, *New class of almost contact metric structures, Publ. Math. Debrecen*, **32** (1985), 187-193.

[16] C. Patra and A. Bhattacharyya, *Trans-Sasakian manifold admitting quarter-symmetric non-metric connection*, Acta Universitatis Apulensis, **36** (2013), 39-49.

[17] S. Sharfuddin and S. I Husain, *Semi-symmetric metric connexions in almost contact manifolds*, Tensor., **30**(1976), 133-139.

[18] M. D. Siddiqi, *η-Ricci Solitons in δ-Lorentzian Trans Sasakian Manifolds with a Semi-symmetric metric Connection*, Kyungpook Mathematical Journal, **59(3)** (2019), 537-562.

[19] M. D. Siddiqi, *η-Ricci Solitons in* 3*-Dimensinonal Normal Almost contact metric manifolds*, Bull. of the Transilvania Univ. of Brasov. Math., Informatics, Physics Series III **11(2)** (2018), 215-233.

[20] M. D. Siddiqi, M. Ahmad, and J. P. Ojha, *CR-Submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection*, African Diaspora Journal of Mathamatics, New Series **17(1)** (2014), 93-105.

[21] M. M. Tripathi, *A new connection in a Riemannian manifold*, International electronic journal of geometry **1(1)**(2008) 15-24.

[22] K. Vinu and H.G. Nagaraja, *η-Ricci solitons in trans-Sasakian manifolds*, Commun. Fac. sci. Univ. Ank. Series A1 **66(2)** (2017), 218-224.

[23] K. Yano and M. Kon *Structures on Manifolds*, Singapore:World Scientific Publishing co. pte. ltd., (1984).

Oğuzhan Bahadır
Department of Mathematics, Faculty of Science and Letters,
Kahramanmaras Sutcu Imam University, Kahramanmaras, Turkey.
E-mail: oguzbaha@gmail.com

Mohd. Danish Siddiqi
Department of Mathematics Faculty of Science, Jazan University,
Jazan, Kingdom of Saudi Arabia.
E-mail: anallintegral@gmail.com, msiddiqi@jazanu.edu.sa

Mehmet Akif Akyol
Department of Mathematics, Faculty of Arts and Sciences,
Bingol University, 12000 Bingol, Turkey.
E-mail: mehmetakifakyol@bingol.edu.tr