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GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. Depending on the characteristic vector filed ζ, a generic lightlike submanifold M in an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection has various characterizations. In this paper, when the characteristic vector filed ζ belongs to the screen distribution $S(TM)$ of M, we provide some characterizations of (Lie-) recurrent generic lightlike submanifold M in an indefinite Kaehler manifold $\bar{M}$with a semi-symmetric metric connection. Moreover, we characterize various generic lightlike submanifolds in an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric connection.

1. Introduction

A lightlike submanifold M of an indefinite almost complex manifold $\bar{M}$, with an indefinite almost complex structure J, is called *generic* if there exists a screen distribution $S(TM)$ of M, which is a complementary non-degenerate distribution of $Rad(TM) = TM \cap TM^{\perp}$ in TM, such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.1}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$ such that $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [9] and later, studied by several authors [2, 5, 6, 10]. Moreover, Jin [8]

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studied generic lightlike submanifolds of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. Lightlike hypersurfaces of an indefinite almost complex manifold are important examples of the generic lightlike submanifold. Much of the theory of generic submanifolds will be immediately generalized in a formal way to general lightlike submanifolds.

In 1924, Friedmann-Schouten [4] introduced the idea of a semi-symmetric connection as follow: A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *semi-symmetric connection* if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}, \tag{1.2}$$

where θ is a 1-form associated with a smooth unit vector field ζ, which is called the *characteristic vector field* of $\bar{M}$, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. In the followings, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$. Moreover, if this connection is a metric one, *i.e.*, it satisfies $\bar{\nabla}\bar{g} = 0$, then $\bar{\nabla}$ is called a *semi-symmetric metric connection* on $\bar{M}$. The notion of a semi-symmetric metric connection on a Riemannian manifold was introduced by Yano [12].

Remark 1.1. *Denote $\widetilde{\nabla}$ by the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. It is well known that a linear connection $\bar{\nabla}$ on $\bar{M}$ is a semi-symmetric metric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta. \tag{1.3}$$

The object of this paper is to study generic lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection $\bar{\nabla}$ subject to the condition that the characteristic vector field ζ of $\bar{M}$ belongs to our screen distribution $S(TM)$ of M. In Section 3, we provide several new results on such a generic lightlike submanifold. In Section 4, we characterize generic lightlike submanifolds of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric connection subject such that ζ belongs to $S(TM)$.

2. Semi-symmetric metric connections

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be an indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure;

$$J^2\bar{X} = -\bar{X}, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = 0. \tag{2.1}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the semi-symmetric metric connection $\bar{\nabla}$, the third equation of three equations in (2.1) is reduced to

$$(2.2) \qquad (\bar{\nabla}_{\bar{X}} J)\bar{Y} = \theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X} - \bar{g}(\bar{X}, J\bar{Y})\zeta + \bar{g}(\bar{X}, \bar{Y})J\zeta.$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$ of dimension $(m + n)$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \leq r \leq \min\{m,\, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called the *screen distribution* and the *co-screen distribution* of M [1], such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \;\; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i,\, j,\, k,\, ... \;\; \in \{1,\, ...,\, r\}, \qquad a,\, b,\, c,\, ... \;\; \in \{r\, +\, 1,\, ...,\, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_1, \cdots, N_r\}$ be a null basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \cdots, \xi_r\}$ is a null basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

A lightlike submanifold $M = (M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is called an *r-lightlike submanifold* [1, 3] if $1 \leq r < \min\{m,\, n\}$. For an r-lightlike M, we see that $S(TM) \neq \{0\}$ and $S(TM^{\perp}) \neq \{0\}$. In the sequel, by saying that M is a lightlike submanifold we shall mean that it is an r-lightlike submanifold, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \cdots, \xi_r\,,\; N_1, \cdots, N_r\,,\; F_{r+1}, \cdots, F_m\,,\; E_{r+1}, \cdots, E_n\},$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{E_{r+1}, \cdots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{a=r+1}^{n} h_a^s(X,Y)E_a, \tag{2.3}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{2.4}$$

$$\bar{\nabla}_X E_a = -A_{E_a}X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \mu_{ab}(X)E_b; \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY)\xi_i, \tag{2.6}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.7}$$

where ∇ and ∇^* are induced linear connections induced from $\bar{\nabla}$ on M and $S(TM)$, respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are linear operators on M, which are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai} and μ_{ab} are 1-forms on M. Using (1.2), (1.3) and (2.3), we see that

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r} \{h_i^{\ell}(X,Y)\eta_i(Z) + h_i^{\ell}(X,Z)\eta_i(Y)\}, \tag{2.8}$$

$$T(X,Y) = \theta(Y)X - \theta(X)Y, \tag{2.9}$$

where η_i's are 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i).$$

From the facts that $h_i^{\ell}(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^{ℓ} and h_a^s are symmetric and independent of the choice of $S(TM)$. The above local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X,Y) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{2.10}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X,Y) - \sum_{k=1}^{r} \lambda_{ak}(X)\eta_k(Y), \tag{2.11}$$

$$h_i^*(X,PY) = g(A_{N_i}X,PY). \tag{2.12}$$

Applying $\bar\nabla_X$ to $\bar g(E_a,E_b) = \epsilon\delta_{ab}$, $g(\xi_i,\xi_j) = 0$, $\bar g(\xi_i,E_a) = 0$, $\bar g(N_i,N_j) = 0$ and $\bar g(N_i,E_a) = 0$ by turns, we obtain $\epsilon_b\mu_{ab} + \epsilon_a\mu_{ba} = 0$ and

$$\begin{aligned} &h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, && h_a^s(X,\xi_i) = -\epsilon_a\lambda_{ai}(X),\\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, && \bar g(A_{E_a}X,N_i) = \epsilon_a\rho_{ia}(X). \end{aligned} \tag{2.13}$$

Furthermore, using $(2.13)_1$, we see that

$$h_i^\ell(X,\xi_i) = 0, \qquad h_i^\ell(\xi_j,\xi_k) = 0, \qquad A^*_{\xi_i}\xi_i = 0. \tag{2.14}$$

Here, $(2.13)_i$ denotes the i-th equation of (2.13). We use the same notations for any others.

Definition 2.1. *We say that a lightlike submanifold M of a semi-Riemannian manifold $(\bar M,\bar g)$ is irrotational*[11] *if $\bar\nabla_X\xi_i \in \Gamma(TM)$ for all $i \in \{1,\cdots,r\}$.*

Remark 2.2. *From (2.3) and $(2.13)_2$, the above definition is equivalent to*

$$h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = \lambda_{ai}(X) = 0. \tag{2.15}$$

3. Structure equations

Let M be a generic lightlike submanifold of $\bar M$. From (1.1) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$\begin{aligned} S(TM) =& \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o,\\ &H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o. \end{aligned}$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{3.1}$$

Consider r-th local null vector fields U_i and V_i, $(n-r)$-th local non-null unit vector fields W_a, and their 1-forms u_i, v_i and w_a defined by

$$(3.2)\quad U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a,$$
$$(3.3)\quad u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a).$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{3.4}$$

Applying J to (3.4) and using $(2.1)_1$, (3.2) and (3.4), we have

$$F^2X = -X + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{3.5}$$

By using $(2.3)_2$ and (3.4), we obtain

$$\begin{aligned} g(FX, FY) = g(X, Y) &- \sum_{i=1}^{r}\{u_i(X)v_i(Y) + u_i(Y)v_i(X)\} \\ &- \sum_{a=r+1}^{n} \epsilon_a w_a(X)w_a(Y). \end{aligned} \tag{3.6}$$

In the sequel, we say that F is the *structure tensor field* of M.

Now we shall assume that the characteristic vector field ζ belongs to the screen distribution $S(TM)$ of M. Applying $\bar{\nabla}_X$ to (3.2) and (3.4) by turns and using (2.2)$\sim$(2.7), (2.10)$\sim$(2.12) and (3.2)$\sim$(3.4), we get

$$\begin{cases} h_j^\ell(X, U_i) = h_i^*(X, V_j) - \theta(V_j)\eta_i(X), \\ \epsilon_a h_a^s(X, U_i) = h_i^*(X, W_a) - \theta(W_a)\eta_i(X), \\ h_j^\ell(X, V_i) = h_i^\ell(X, V_j), \qquad h_a^s(X, V_i) = \epsilon_a h_i^\ell(X, W_a), \\ \epsilon_b h_b^s(X, W_a) = \epsilon_a h_a^s(X, W_b), \end{cases} \tag{3.7}$$

$$
\begin{aligned}
(3.8) \qquad \nabla_X U_i \;=\; & F(A_{N_i}X) + \sum_{j=1}^{r} \tau_{ij}(X)U_j + \sum_{a=r+1}^{n} \rho_{ia}(X)W_a \\
& + \theta(U_i)X - v_i(X)\zeta - \eta_i(X)F\zeta,
\end{aligned}
$$

$$
\begin{aligned}
(3.9) \qquad \nabla_X V_i \;=\; & F(A^*_{\xi_i}X) - \sum_{j=1}^{r} \tau_{ji}(X)V_j + \sum_{j=1}^{r} h^\ell_j(X,\xi_i)U_j \\
& - \sum_{a=r+1}^{n} \epsilon_a \lambda_{ai}(X)W_a + \theta(V_i)X - u_i(X)\zeta,
\end{aligned}
$$

$$
\begin{aligned}
(3.10) \qquad \nabla_X W_a \;=\; & F(A_{E_a}X) + \sum_{i=1}^{r} \lambda_{ai}(X)U_i + \sum_{b=r+1}^{n} \mu_{ab}(X)W_b, \\
& + \theta(W_a)X - \epsilon_a w_a(X)\zeta,
\end{aligned}
$$

$$
\begin{aligned}
(3.11) \qquad (\nabla_X F)Y \;=\; & \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \\
& - \sum_{i=1}^{r} h^\ell_i(X,Y)U_i - \sum_{a=r+1}^{n} h^s_a(X,Y)W_a \\
& + \theta(FY)X - \theta(Y)FX - \bar{g}(X,JY)\zeta + g(X,Y)F\zeta.
\end{aligned}
$$

4. Recurrent and Lie recurrent generic submanifolds

Theorem 4.1. *There exist no generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$ and F is parallel with respect to the connection ∇.*

Proof. Assume that F is parallel with respect to the connection ∇. Replacing Y by ξ_j to (3.11) and using the fact that $F\xi_j = -V_j$, we obtain

$$
(4.1) \qquad \sum_{k=1}^{r} h^\ell_k(X,\xi_j)U_k + \sum_{a=r+1}^{n} h^s_a(X,\xi_j)W_a + \theta(V_j)X - u_j(X)\zeta = 0.
$$

Taking the scalar product with N_i to (4.1) and then, taking $X = \xi_j$, we get $\theta(V_i) = 0$. Also taking the scalar product with U_i to (4.1) and then, taking $X = U_j$ and using $\theta(V_j) = 0$, we get $\theta(U_i) = 0$. Therefore, we obtain

$$
\theta(V_i) = 0, \qquad\qquad \theta(U_i) = 0.
$$

Taking the scalar product with W_b to (4.1) and using $\theta(V_i) = 0$, we have

$$h^s_a(X, \xi_i) = \epsilon_a \theta(W_a) u_i(X). \tag{4.2}$$

Replacing Y by W_a to (3.11) such that $\nabla_X F = 0$, we have

$$\begin{aligned} A_{E_a} X = \sum_{i=1}^{r} h^\ell_i(X, W_a) U_i + \sum_{b=r+1}^{n} h^s_b(X, W_a) W_b \\ + \theta(W_a) FX - \epsilon_a w_a(X) F\zeta. \end{aligned}$$

Taking the scalar product with U_i to this equation, we obtain

$$h^s_a(X, U_i) = -\epsilon_a \theta(W_a) \eta_i(X). \tag{4.3}$$

Taking $X = U_i$ to (4.2) and also, taking $X = \xi_i$ to (4.3) and then, comparing these two resulting equations, we obtain $\theta(W_a) = 0$. Taking the scalar product with ζ to (4.1) and using the facts that $\theta(V_i) = \theta(U_i) = \theta(W_a) = 0$, we have $u_j(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $u_j(U_j) = 1$. Thus there exist no generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection subject such that ζ belongs to $S(TM)$ and F is parallel with respect to the connection ∇. □

Definition 4.2. *The structure tensor field F of M is said to be recurrent* [6] *if there exists a 1-form ϖ on TM such that*

$$(\nabla_X F)Y = \varpi(X) FY.$$

A generic lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is called recurrent if it admits a recurrent structure tensor field F.

Theorem 4.3. *There exist no recurrent generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that the characteristic vector field ζ of $\bar{M}$ belongs to $S(TM)$.*

Proof. From the above definition and (3.11), we obtain

$$\begin{aligned} \varpi(X) FY = \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \\ - \sum_{i=1}^{r} h^\ell_i(X, Y) U_i - \sum_{a=r+1}^{n} h^s_a(X, Y) W_a \\ + \theta(FY) X - \theta(Y) FX - \bar{g}(X, JY)\zeta + g(X, Y) F\zeta. \end{aligned} \tag{4.4}$$

Replacing Y by ξ_j to this and using the fact that $F\xi_j = -V_j$, we get (4.5)

$$\varpi(X)V_j = \sum_{k=1}^{r} h_k^\ell(X, \xi_j)U_k + \sum_{b=r+1}^{n} h_b^s(X, \xi_j)W_b + \theta(V_j)X - u_j(X)\zeta.$$

Taking the scalar product with N_i to this, we obtain $\theta(V_j)\eta_i(X) = 0$. Taking $X = \xi_j$ to this equation, we have $\theta(V_i) = 0$ for all i. Taking the scalar product with V_i and W_a to (4.5) and using $\theta(V_i) = 0$, we obtain

$$h_j^\ell(X, \xi_i) = 0, \qquad h_a^s(X, \xi_i) = \epsilon_a \theta(W_a)u_i(X). \tag{4.6}$$

Replacing Y by W_a to (4.4) and using the fact that $FW_a = 0$, we have

$$\begin{aligned} A_{E_a}X = \sum_{i=1}^{r} h_i^\ell(X, W_a)U_i &+ \sum_{b=r+1}^{n} h_b^s(X, W_a)W_b \\ &+ \theta(W_a)FX - \epsilon_a w_a(X)F\zeta. \end{aligned}$$

Taking the scalar product with U_i to this equation, we obtain

$$h_a^s(X, U_i) = -\epsilon_a \theta(W_a)\eta_i(X). \tag{4.7}$$

Taking $X = \xi_i$ to (4.7) and also, taking $X = U_i$ to $(4.6)_2$ and then, comparing two resulting equations, we get $\theta(W_a) = 0$. As $\theta(W_a) = 0$, we get

$$h_j^\ell(X, \xi_i) = 0, \qquad h_a^s(X, \xi_i) = 0.$$

Using these equations and the fact that $\theta(V_i) = 0$, Eq. (4.5) is reduced to

$$\varpi(X)V_j = -u_j(X)\zeta.$$

Taking the scalar product with ζ to this, we have $u_j(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $u_j(U_j) = 1$. Thus there exist no recurrent generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$. □

Definition 4.4. *The structure tensor field F of M is said to be Lie recurrent* [7] *if there exists a 1-form ϑ on M such that*

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

In case $\mathcal{L}_X F = 0$, we say that F is Lie parallel. A generic lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is called Lie recurrent if it admits a Lie recurrent structure tensor field F.

Theorem 4.5. *Let M be a Lie recurrent lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that the characteristic vector field ζ of $\bar{M}$ belongs to $S(TM)$. Then F is Lie parallel,*

Proof. Using the above definition, (2.9) and (3.11), we obtain

$$\begin{aligned}\vartheta(X)FY \;=\; & -\nabla_{FY}X + F\nabla_Y X - \bar{g}(X,JY)\zeta + g(X,Y)F\zeta \\ & + \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \\ & - \sum_{i=1}^{r} h_i^{\ell}(X,Y)U_i - \sum_{a=r+1}^{n} h_a^s(X,Y)W_a.\end{aligned} \tag{4.8}$$

Replacing Y by ξ_j and also, Y by V_j to (4.8), respectively, we have

$$\begin{aligned}-\vartheta(X)V_j \;=\; & \nabla_{V_j}X + F\nabla_{\xi_j}X + u_j(X)\zeta \\ & - \sum_{i=1}^{r} h_i^{\ell}(X,\xi_j)U_i - \sum_{a=r+1}^{n} h_a^s(X,\xi_j)W_a,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\vartheta(X)\xi_j = & -\nabla_{\xi_j}X + F\nabla_{V_j}X + u_j(X)F\zeta \\ & - \sum_{i=1}^{r} h_i^{\ell}(X,V_j)U_i - \sum_{a=r+1}^{n} h_a^s(X,V_j)W_a.\end{aligned} \tag{4.10}$$

Taking the scalar product with U_i to (4.9) and N_i to (4.10), we get

$$\begin{aligned}-\delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X,U_i) - \bar{g}(\nabla_{\xi_j}X,N_i) + \theta(U_i)u_j(X),\\ \delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X,U_i) - \bar{g}(\nabla_{\xi_j}X,N_i) + \theta(U_i)u_j(X),\end{aligned}$$

respectively. From these two equations, we get $\vartheta = 0$. Thus F is Lie parallel. □

Proposition 4.6. *Let M be a Lie recurrent lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that the characteristic vector field ζ of $\bar{M}$ belongs to $S(TM)$. Then τ_{ij} and ρ_{ia} satisfy $\tau_{ij}\circ F = 0$ and $\rho_{ia}\circ F = 0$. Moreover,*

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)g(A_{N_k}V_j, N_i).$$

Proof. Taking the scalar product with N_i to (4.9) such that $X = W_a$ and using (2.11), $(2.13)_4$ and (3.10), we get $h_a^s(U_i,V_j) = \rho_{ia}(\xi_j)$. Also, taking the scalar product with W_a to (4.10) such that $X = U_i$ and using

(3.8), we have $h^s_a(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h^s_a(U_i, V_j) = 0$.

Taking the scalar product with U_i to (4.9) such that $X = W_a$ and using (2.11), $(2.13)_{2,4}$ and (3.10), we get $\epsilon_a \rho_{ia}(V_j) = \lambda_{aj}(U_i)$. Also, taking the scalar product with W_a to (4.9) such that $X = U_i$ and using $(2.13)_2$ and (3.8), we get $\epsilon_a \rho_{ia}(V_j) = -\lambda_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = 0$ and $\lambda_{aj}(U_i) = 0$.

Taking the scalar product with V_i to (4.9) such that $X = W_a$ and using $(2.13)_2$, $(3.7)_4$ and (3.10), we obtain $\lambda_{ai}(V_j) = -\lambda_{aj}(V_i)$. Also, taking the scalar product with W_a to (4.9) such that $X = V_i$ and using $(2.13)_2$ and (3.9), we have $\lambda_{ai}(V_j) = \lambda_{aj}(V_i)$. Thus we obtain $\lambda_{ai}(V_j) = 0$.

Taking the scalar product with W_a to (4.9) such that $X = \xi_i$ and using (2.7), (2.10) and $(2.13)_2$, we get $h^\ell_i(V_j, W_a) = \lambda_{ai}(\xi_j)$. Also, taking the scalar product with V_i to (4.10) such that $X = W_a$ and using (3.10), we have $h^\ell_i(V_j, W_a) = -\lambda_{ai}(\xi_j)$. Thus $\lambda_{ai}(\xi_j) = 0$ and $h^\ell_i(V_j, W_a) = 0$.

Summarizing the above results, we obtain

$$
\begin{aligned}
(4.11)\quad & \rho_{ia}(\xi_j) = 0,\ \rho_{ia}(V_j) = 0,\ \lambda_{ai}(U_j) = 0,\ \lambda_{ai}(V_j) = 0,\ \lambda_{ai}(\xi_j) = 0,\\
& h^s_a(U_i, V_j) = h^\ell_j(U_i, W_a) = 0, \qquad h^\ell_i(V_j, W_a) = h^s_a(V_j, V_i) = 0.
\end{aligned}
$$

Taking the scalar product with N_i to (4.8) and using $(2.13)_4$, we have

$$
\begin{aligned}
(4.12)\qquad & -\bar{g}(\nabla_{FY}X, N_i) + g(\nabla_Y X, U_i) + \theta(U_i)g(X, Y)\\
& + \sum_{k=1}^{r} u_k(Y)\bar{g}(A_{N_k}X, N_i) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) = 0.
\end{aligned}
$$

Taking $X = \xi_j$ and $Y = U_k$ to (4.12) and using (2.7) and (2.10), we have

$$h^\ell_j(U_k, U_i) = \eta_i(A_{N_k}\xi_j).$$

As h^ℓ_j are symmetric, from the last equation, we see that $\eta_i(A_{N_k}\xi_j)$ is symmetric with respect to i and k. From this result and $(2.13)_4$, we obtain

$$(4.13)\qquad g(A_{N_k}\xi_j, N_i) = 0, \qquad h^*_i(U_k, V_j) = 0.$$

Taking $X = \xi_j$ to (4.12) and using (2.7), (2.10), $(4.11)_1$ and $(4.13)_1$, we get

$$(4.14)\qquad h^\ell_j(X, U_i) = \tau_{ij}(FX).$$

Taking $X = U_i$ to (4.8) and using (2.12), (3.5), $(3.7)_{1,2}$ and (3.8), we get

$$
\begin{aligned}
&\sum_{k=1}^{r} u_k(Y) A_{N_k} U_i + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} U_i \\
&-A_{N_i} Y + \eta_i(Y)\zeta + v_i(Y) F\zeta - F(A_{N_i} FY) \\
&-\sum_{j=1}^{r} \tau_{ij}(FY) U_j - \sum_{a=r+1}^{n} \rho_{ia}(FY) W_a = 0.
\end{aligned} \tag{4.15}
$$

Taking the scalar product with V_j to (4.15) and using (2.11), (2.12), $(3.7)_1$, $(4.11)_6$ and $(4.13)_2$, we obtain

$$h_j^{\ell}(X, U_i) = -\tau_{ij}(FX).$$

Comparing this equation with (4.14), we obtain

$$\tau_{ij}(FX) = 0, \qquad h_j^{\ell}(X, U_i) = 0. \tag{4.16}$$

Taking $X = V_j$ to (4.12) and using (2.10), (3.9), $(4.11)_2$ and $(4.16)_2$, we get

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X) \bar{g}(A_{N_k} V_j, N_i).$$

Taking the scalar product with U_j to (4.15) and then, taking $Y = W_a$ and using (2.11), (2.12) and $(3.7)_2$, we have

$$h_i^*(W_a, U_j) = \epsilon_a h_a^s(U_i, U_j) = \epsilon_a h_a^s(U_j, U_i) = h_i^*(U_j, W_a). \tag{4.17}$$

Taking the scalar product with W_a to (4.15), we have

$$
\begin{aligned}
\epsilon_a \rho_{ia}(FY) = &-h_i^*(Y, W_a) + \theta(W_a)\eta_i(Y) \\
&+ \sum_{k=1}^{r} u_k(Y) h_k^*(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y) h_b^s(U_i, W_a).
\end{aligned}
$$

Taking the scalar product with U_i to (4.8) such that $X = W_a$ and using (2.11), (2.12), $(2.13)_4$, (3.5), $(3.7)_2$ and (4.17), we get

$$
\begin{aligned}
\epsilon_a \rho_{ia}(FY) = &\, h_i^*(Y, W_a) - \theta(W_a)\eta_i(Y) \\
&- \sum_{k=1}^{r} u_k(Y) h_k^*(U_i, W_a) - \sum_{b=r+1}^{n} \epsilon_b w_b(Y) h_b^s(U_i, W_a).
\end{aligned}
$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. □

Theorem 4.7. *There exist no generic lightlike submanifolds of an indefinite Kaehler manifold* $\bar{M}$ *with a semi-symmetric metric connection such that* ζ *belongs to* $S(TM)$, $V_i(i = 1, \cdots, r)$ *are parallel with respect to* ∇ *and* $h^s_a(X, \xi_i) = 0$ *for any vector field* X *on* M.

Proof. Assume that $V_i(i = 1, \cdots, r)$ are parallel with respect to the connection ∇ and $h^s_a(X, \xi_i) = 0$ for any vector field X on M. Taking the scalar product with W_a to (3.9) and using $\lambda_{ai}(X) = h^s_a(X, \xi_i) = 0$, we get

$$\epsilon_a \theta(V_i) w_a(X) = \theta(W_a) u_i(X).$$

Taking $X = W_a$ and $X = U_i$ to this equation by turns, we obtain

$$\theta(V_i) = 0, \qquad \theta(W_a) = 0.$$

Taking the scalar product with V_j to (3.9) and using $\theta(V_i) = 0$, we have

$$h^\ell_i(X, \xi_j) = 0.$$

Taking the scalar product with ζ and N_j to (3.9) by turns and using the last two equations, we obtain

$$h^\ell_i(X, F\zeta) = -u_i(X), \qquad h^\ell_i(X, U_j) = 0.$$

From these two equations, we have the following impossible result:

$$-\delta_{ij} = -u_i(U_j) = h^\ell_i(U_j, F\zeta) = h^\ell_i(F\zeta, U_j) = 0.$$

Thus we have our theorem □

5. Indefinite complex space forms

Denote by $\bar{R}, R$ and R^* the curvature tensor of the semi-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$ and the induced linear connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively:

$$
\begin{aligned}
(5.1)\qquad \bar{R}(X,Y)Z = R(X,Y)Z
&+ \sum_{i=1}^{r}\{h_i^{\ell}(X,Z)A_{N_i}Y - h_i^{\ell}(Y,Z)A_{N_i}X\} \\
&+ \sum_{a=r+1}^{n}\{h_a^{s}(X,Z)A_{E_a}Y - h_a^{s}(Y,Z)A_{E_a}X\} \\
+ \sum_{i=1}^{r}\{(\nabla_X h_i^{\ell})(Y,Z) - (\nabla_Y h_i^{\ell})(X,Z) \\
&+ \sum_{j=1}^{r}[\tau_{ji}(X)h_j^{\ell}(Y,Z) - \tau_{ji}(Y)h_j^{\ell}(X,Z)] \\
&+ \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^{s}(Y,Z) - \lambda_{ai}(Y)h_a^{s}(X,Z)] \\
&- \theta(X)h_i^{\ell}(Y,Z) + \theta(Y)h_i^{\ell}(X,Z)\}N_i \\
+ \sum_{a=r+1}^{n}\{(\nabla_X h_a^{s})(Y,Z) - (\nabla_Y h_a^{s})(X,Z) \\
&+ \sum_{i=1}^{r}[\rho_{ia}(X)h_i^{\ell}(Y,Z) - \rho_{ia}(Y)h_i^{\ell}(X,Z)] \\
&+ \sum_{b=r+1}^{n}[\mu_{ba}(X)h_b^{s}(Y,Z) - \mu_{ba}(Y)h_b^{s}(X,Z)] \\
&- \theta(X)h_a^{s}(Y,Z) + \theta(Y)h_a^{s}(X,Z)\}E_a,
\end{aligned}
$$

$$
\begin{aligned}
(5.2)\qquad R(X,Y)PZ = R^*(X,Y)PZ
&+ \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\} \\
+ \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\
&+ \sum_{k=1}^{r}[\tau_{ik}(Y)h_k^*(X,PZ) - \tau_{ik}(X)h_k^*(Y,PZ)] \\
&- \theta(X)h_i^*(Y,PZ) + \theta(Y)h_i^*(X,PZ)\}\xi_i.
\end{aligned}
$$

Definition. An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c;

$$(5.3)\ \widetilde{R}(\bar{X},\bar{Y})\bar{Z} = \frac{c}{4}\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y} + \bar{g}(J\bar{Y},\bar{Z})J\bar{X} - \bar{g}(J\bar{X},\bar{Z})J\bar{Y} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\},$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

By directed calculations from (1.2) and (1.3), we see that

$$(5.4)\ \bar{R}(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(\bar{X},\bar{Y})\bar{Z} + \bar{g}(\bar{X},\bar{Z})\bar{\nabla}_{\bar{Y}}\zeta - \bar{g}(\bar{Y},\bar{Z})\bar{\nabla}_{\bar{X}}\zeta + \{(\bar{\nabla}_{\bar{X}}\theta)(\bar{Z}) - \bar{g}(\bar{X},\bar{Z})\}\bar{Y} - \{(\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z}) - \bar{g}(\bar{Y},\bar{Z})\}\bar{X}.$$

Taking the scalar product with ξ_i and N_i to (5.4) by turns and then, substituting (5.1) and (5.3) into the resulting equation and using (5.2) and the facts that $g(\zeta,\xi_i) = \bar{g}(\zeta,N_i) = \bar{g}(\zeta,E_a) = 0$ and $\bar{\nabla}$ is metric, we obtain

$$\begin{aligned}(5.5)\quad & (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ & + \sum_{k=1}^{r}\{\tau_{ki}(X)h_k^\ell(Y,Z) - \tau_{ki}(Y)h_k^\ell(X,Z)\} \\ & + \sum_{a=r+1}^{n}\{\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)\} \\ & - \theta(X)h_i^\ell(Y,Z) + \theta(Y)h_i^\ell(X,Z) \\ & - g(X,Z)h_i^\ell(Y,\zeta) + g(Y,Z)h_i^\ell(X,\zeta) \\ & = \frac{c}{4}\{u_i(X)\bar{g}(JY,Z) - u_i(Y)\bar{g}(JX,Z) + 2u_i(Z)\bar{g}(X,JY)\},\end{aligned}$$

$$\begin{aligned}(5.6)\quad & (\nabla_X h_i^*)(Y,\, PZ) - (\nabla_Y h_i^*)(X,\, PZ) \\ & - \sum_{k=1}^{r}\{\tau_{ik}(X)h_k^*(Y,PZ) - \tau_{ik}(Y)h_k^*(X,PZ)\} \\ & - \sum_{k=1}^{r}\{h_k^\ell(Y,PZ)\eta_i(A_{N_k}X) - h_k^\ell(X,PZ)\eta_i(A_{N_k}Y)\} \\ & - \sum_{a=r+1}^{n}\{h_a^s(Y,\, PZ)\eta_i(A_{E_a}X) - h_a^s(X,\, PZ)\eta_i(A_{E_a}Y)\}\end{aligned}$$

$$\begin{aligned}
&- \theta(X)h_i^*(Y,PZ) + \theta(Y)h_i^*(X,PZ)\\
&- g(X,PZ)h_i^*(Y,\zeta) + g(Y,PZ)h_i^*(X,\zeta)\\
&- (\bar\nabla_X\theta)(PZ)\eta_i(Y) + (\bar\nabla_Y\theta)(PZ)\eta_i(X)\\
&= (\frac{c}{4}+1)\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\}\\
&+ \frac{c}{4}\{v_i(X)\bar g(JY,PZ) - v_i(Y)\bar g(JX,PZ) + 2v_i(PZ)\bar g(X,JY)\}.
\end{aligned}$$

Theorem 5.1. *Let M be a Lie recurrent generic lightlike submanifold of an indefinite complex space form $\bar M(c)$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$. Then $c = 0$, i.e., $\bar M(c)$ is flat.*

Proof. In case M is Lie recurrent. As $\tau_{ij}(FX) = 0$, from (4.14) we get

$$h_i^\ell(Y,U_j) = 0. \tag{5.7}$$

Applying ∇_X to this equation and using (3.8) and (5.7), we have

$$\begin{aligned}
(\nabla_X h_i^\ell)(Y,U_j) &= -h_i^\ell(Y,F(A_{N_j}X)) - \sum_{a=r+1}^{n}\rho_{ja}(X)h_i^\ell(Y,W_a)\\
&\quad - \theta(U_j)h_i^\ell(Y,X) + v_j(X)h_i^\ell(Y,\zeta) + \eta_i(X)h_i^\ell(Y,F\zeta).
\end{aligned}$$

Substituting the last two equations into (5.5) such that $Z = U_j$, we have

$$\begin{aligned}
&h_i^\ell(X,F(A_{N_j}Y)) - h_i^\ell(Y,F(A_{N_j}X))\\
&- \sum_{a=r+1}^{n}\{\rho_{ja}(X)h_i^\ell(Y,W_a) - \rho_{ja}(Y)h_i^\ell(X,W_a)\}\\
&+ \sum_{a=r+1}^{n}\{\lambda_{ai}(X)h_a^s(Y,U_j) - \lambda_{ai}(Y)h_a^s(X,U_j)\}\\
&+ \eta_j(X)h_i^\ell(Y,F\zeta) - \eta_j(Y)h_i^\ell(X,F\zeta)\\
&= \frac{c}{4}\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar g(X,JY)\}.
\end{aligned}$$

Taking $X = \xi_j$ and $Y = U_i$ to this and using $(4.11)_{3,5}$ and (5.7), we get

$$h_i^\ell(\xi_j,F(A_{N_j}U_i)) + \sum_{a=r+1}^{n}\rho_{ja}(U_i)h_i^\ell(\xi_j,W_a) = \frac{3}{4}c. \tag{5.8}$$

Replacing X by ξ_j to (2.10) and using $(2.14)_2$ and the fact that h_i^ℓ is symmetric, we get $h_i^\ell(X,\xi_j) = g(A^*_{\xi_i}\xi_j,X)$. From this result and

$(2.13)_1$, we obtain $g(A^*_{\xi_i}\xi_j + A^*_{\xi_j}\xi_i,\, X) = 0$ for all X. As $S(TM)$ is non-degenerate, we get $A^*_{\xi_i}\xi_j = -A^*_{\xi_j}\xi_i$. Thus $A^*_{\xi_i}\xi_j$ is skew-symmetric with respect to i and j.

On the other hand, taking $Y = U_j$ to (4.15), we have

$$A_{N_j}U_i = A_{N_i}U_j.$$

Applying F to this equation, we have $F(A_{N_j}U_i) = F(A_{N_i}U_j)$. Thus $F(A_{N_i}U_j)$ is symmetric with respect to i and j. Therefore, we obtain

$$h^\ell_i(\xi_j,\, F(A_{N_j}U_i)) = g(A^*_{\xi_i}\xi_j, F(A_{N_j}U_i)) = 0. \tag{5.9}$$

Also, from $(2.13)_2$, $(3.7)_4$, $(4.11)_4$ and the fact that h^s_a is symmetric, we get

$$h^\ell_i(\xi_j, W_a) = \epsilon_a h^s_a(\xi_j, V_i) = \epsilon_a h^s_a(V_i, \xi_j) = -\lambda_{aj}(V_i) = 0. \tag{5.10}$$

From (5.8)$\sim$ (5.10), we obtain $c = 0$. □

Definition 5.2. *A lightlike submanifold M is said to be screen conformal* [5] *if there exist non-vanishing smooth functions φ_i on $\mathcal{U}$ such that*

$$h^*_i(X, PY) = \varphi_i h^\ell_i(X, PY), \qquad \forall\, i. \tag{5.11}$$

Theorem 5.3. *Let M be a screen conformal irrotational generic lightlike submanifold of an indefinite complex space form $\bar M(c)$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$. Then $c = 0$, i.e., $\bar M(c)$ is flat.*

Proof. Using $(3.7)_{1,3}$ and (5.11), we get

$$h^\ell_j(X, U_i - \varphi_i V_i) = -\,\theta(V_j)\eta_i(X).$$

Replacing X by ξ_j to this equation and using $(2.14)_1$, we have

$$\theta(V_i) = 0, \qquad\qquad h^\ell_j(X, U_i - \varphi_i V_i) = 0. \tag{5.12}$$

If M is irrotational, then we have (2.15). Using $(3.7)_{2,4}$ and (5.11), we get

$$h^s_a(X, U_i - \varphi_i V_i) = -\,\epsilon_a\theta(W_a)\eta_i(X).$$

Replacing X by ξ_i to this equation and using $(2.15)_2$, we obtain

$$\theta(W_a) = 0, \qquad\qquad h^s_a(X, U_i - \varphi_i V_i) = 0. \tag{5.13}$$

Applying $\bar\nabla_X$ to $\theta(V_i) = 0$ and using $(2.15)_{1,2}$, (3.9) and $(5.12)_1$, we obtain

$$(\bar\nabla_X\theta)(V_i) = h^\ell_i(X, F\zeta) + u_i(X). \tag{5.14}$$

Applying ∇_X to $h_i^*(Y, PZ) = \varphi_i h_i^\ell(Y, PZ)$, we have

$$(\nabla_X h_i^*)(Y, PZ) = (X\varphi_i)h_i^\ell(Y, PZ) + \varphi_i(\nabla_X h_i^\ell)(Y, PZ).$$

Substituting this equation into (4.6) and using (4.5), we have

$$\begin{aligned}
&(X\varphi_i)h_i^\ell(Y, PZ) - (Y\varphi_i)h_i^\ell(X, PZ)\\
&- \sum_{j=1}^{r}\{\varphi_i\tau_{ji}(X) + \varphi_j\tau_{ij}(X) + \eta_i(A_{N_j}X)\}h_j^\ell(Y, PZ)\\
&+ \sum_{j=1}^{r}\{\varphi_i\tau_{ji}(Y) + \varphi_j\tau_{ij}(Y) + \eta_i(A_{N_j}Y)\}h_j^\ell(X, PZ)\\
&- \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h_a^s(Y, PZ) - \rho_{ia}(Y)h_a^s(X, PZ)\}\\
&- (\bar{\nabla}_X\theta)(PZ)\eta_i(Y) + (\bar{\nabla}_Y\theta)(PZ)\eta_i(X)\\
&= (\frac{c}{4} + 1)\{\eta_i(X)g(Y, PZ) - \eta_i(Y)g(X, PZ)\}\\
&+ \frac{c}{4}\{[v_i(X) - \varphi_i u_i(X)]g(FY, PZ) - [v_i(Y) - \varphi_i u_i(Y)]g(FX, PZ)\\
&\qquad + 2[v_i(PZ) - \varphi_i u_i(PZ)]\bar{g}(X, JY)\}.
\end{aligned}$$

Taking $Y = \xi_i$ and $PZ = V_j$ to this and using (2.15) and (5.14), we have

$$\begin{aligned}
&-(\xi_i\varphi_i)h_i^\ell(X, V_j) - h_j^\ell(X, F\zeta)\\
&+ \sum_{j=1}^{r}\{\varphi_i\tau_{ji}(\xi_i) + \varphi_j\tau_{ij}(\xi_i) + \eta_i(A_{N_j}\xi_i)\}h_j^\ell(X, V_j)\\
&+ \sum_{a=r+1}^{n}\epsilon_a\rho_{ia}(\xi_i)h_a^s(X, V_j) = -\frac{3}{4}cu_j(X).
\end{aligned}$$

Taking $X = U_j + \varphi_j V_j$ to this and using $(5.12)_2$ and $(5.13)_2$, we get $c = 0$ □

Definition 5.4. [1] *We say that $S(TM)$ is totally umbilical in M if there exist smooth functions γ_i on a coordinate neighborhood $\mathcal{U}$ such that*

$$h_i^*(X, PY) = \gamma_i g(X, PY), \qquad \forall\, i. \tag{5.15}$$

In case $\gamma_i = 0$ on $\mathcal{U}$, we say that $S(TM)$ is totally geodesic in M.

Theorem 5.5. *Let M be an irrotational generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric* *connection such that ζ belongs to $S(TM)$. If $S(TM)$ is totally umbilical in M, then $c = 0$ and $\gamma_i = 0$, i.e., $S(TM)$ is totally geodesic in M.*

Proof. If $S(TM)$ is totally umbilical, then, from $(3.7)_1$ and (5.15), we have

$$h_j^\ell(X, U_i) = \gamma_i u_j(X) - \theta(V_j)\eta_i(X).$$

Replacing X by ξ_j, V_k, U_k and ζ to this by turns and using $(2.14)_1$, we get

$$\theta(V_i) = 0, \quad h_j^\ell(V_k, U_i) = 0, \; h_j^\ell(U_k, U_i) = \gamma_i\delta_{kj}, \; h_j^\ell(U_i, \zeta) = 0, \tag{5.16}$$

$$h_j^\ell(X, U_i) = \gamma_i u_j(X). \tag{5.17}$$

If M is irrotational, then we have (2.15). From $(3.7)_2$ and (5.15), we get

$$h_a^s(X, U_i) = \gamma_i w_a(X) - \theta(W_a)\eta_i(X).$$

Replacing X by ξ_i, V_k, U_k and ζ to this by turns and using $(2.15)_2$, we have

$$\theta(W_a) = 0, \quad h_a^s(V_k, U_i) = 0, \quad h_a^s(U_k, U_i) = 0, \quad h_a^s(U_i, \zeta) = 0. \tag{5.18}$$

Applying $\bar\nabla_X$ to $\theta(V_i) = 0$ and using (2.10), (2.15), (3.4) and (3.9), we obtain

$$(\bar\nabla_X\theta)(V_i) = h_i^\ell(X, F\zeta) + u_i(X).$$

Taking $X = F\zeta$ to (5.17), we get $h_j^\ell(U_i, F\zeta) = 0$. Replacing X by U_j to the last equation and using the fact that $h_j^\ell(U_i, F\zeta) = 0$, we obtain

$$(\bar\nabla_{U_j}\theta)(V_i) = \delta_{ij}. \tag{5.19}$$

Applying ∇_X to $h_i^*(Y, PZ) = \gamma_i g(Y, PZ)$ and using (2.7), we obtain

$$(\nabla_X h_i^*)(Y, PZ) = (X\gamma_i)g(Y, PZ) + \gamma_i \sum_{j=1}^r h_j^\ell(X, PZ)\eta_j(Y).$$

Substituting this equation and (5.15) into (5.6), we have

$$\begin{aligned}
&\{X\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(X) - [\frac{c}{4}+1]\eta_i(X)\}g(Y,PZ)\\
&-\{Y\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(Y) - [\frac{c}{4}+1]\eta_i(Y)\}g(X,PZ)\\
&+\sum_{j=1}^{r}\{\gamma_i\eta_j(Y)+\eta_i(A_{N_j}Y)\}h_j^{\ell}(X,PZ)\\
&-\sum_{j=1}^{r}\{\gamma_i\eta_j(X)+\eta_i(A_{N_j}X)\}h_j^{\ell}(Y,PZ)\\
&-\sum_{a=r+1}^{n}\{h_a^s(Y,\,PZ)\eta_i(A_{E_a}X) - h_a^s(X,\,PZ)\eta_i(A_{E_a}Y)\}\\
&-(\bar\nabla_X\theta)(PZ)\eta_i(Y)+(\bar\nabla_Y\theta)(PZ)\eta_i(X)\\
&=\frac{c}{4}\{v_i(X)g(FY,PZ)-v_i(Y)g(FX,PZ)+2v_i(PZ)\bar g(X,JY)\}.
\end{aligned}$$

Replacing Y by ξ_k to this and using (2.15), (3.2) and (3.3), we have

$$\begin{aligned}
&\{\xi_k\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(\xi_k) - [\frac{c}{4}+1]\delta_{ik}\}g(X,PZ)\\
&-\sum_{j=1}^{r}\{\gamma_i\delta_{jk}+\eta_i(A_{N_j}\xi_k)\}h_j^{\ell}(X,PZ)\\
&-\sum_{a=r+1}^{n}\eta_i(A_{E_a}\xi_k)h_a^s(X,PZ)\\
&+(\bar\nabla_X\theta)(PZ)\delta_{ik}-(\bar\nabla_{\xi_k}\theta)(PZ)\eta_i(X)\\
&=\frac{c}{4}\{v_i(X)u_k(PZ)+2v_i(PZ)u_k(X)\}.
\end{aligned} \tag{5.20}$$

Taking $X = U_h$ and $PZ = V_h$ and using $(5.16)_2$, $(5.18)_2$ and (5.19), we have

$$\xi_k\gamma_i - \sum_{j=1}^{r}\gamma_j\tau_{ij}(\xi_k) = \frac{3}{4}c\,\delta_{ik}. \tag{5.21}$$

Applying $\bar{\nabla}_X$ to $g(\zeta,\zeta) = 1$ and using the fact that $\bar{\nabla}$ is metric, we obtain

$$(\bar{\nabla}_X\theta)(\zeta) = 0. \tag{5.22}$$

Taking $X = U_k$ and $Z = \zeta$ to (5.20) and using $(5.16)_4$, (5.21) and (5.22), we get $\theta(U_i) = 0$. As $\bar{g}(J\zeta,\zeta) = 0$, we see that $g(F\zeta,\zeta) = 0$. Thus

$$\theta(U_i) = 0, \qquad g(F\zeta,\zeta) = 0. \tag{5.23}$$

As $\theta(V_i) = \theta(U_i) = \theta(W_a) = 0$, we get $J\zeta = F\zeta \in \Gamma(S(TM))$. Applying $\bar{\nabla}_X$ to $\theta(U_i) = 0$ and using (3.8), $(5.18)_1$ and (5.23), we obtain

$$(\bar{\nabla}_X\theta)(U_i) = \gamma_i g(X, F\zeta) + v_i(X).$$

Taking $X = V_j$ and $X = U_j$ to this equation by turns, we obtain

$$(\bar{\nabla}_{V_j}\theta)(U_i) = \delta_{ij}, \qquad (\bar{\nabla}_{U_j}\theta)(U_i) = 0. \tag{5.24}$$

Taking $X = V_h$ and $PZ = U_h$ to (5.20) and using $(5.16)_2$, $(5.18)_2$, (5.21) and $(5.24)_1$, we have $c = 0$. Thus $\bar{M}(c)$ is flat.

As $\eta_i(A_{N_j}\xi_k)$ is skew-symmetric with respect to i and j by $(2.13)_3$ and $h^\ell_j(U_i, U_k)$ is symmetric with respect to i and j by $(5.16)_3$, we see that

$$\eta_i(A_{N_j}\xi_k)h^\ell_j(U_i, U_k) = 0. \tag{5.25}$$

As $c = 0$, Eq. (5.20) reduces

$$\sum_{j=1}^{r}\{\gamma_i\delta_{jk} + \eta_i(A_{N_j}\xi_k)\}h^\ell_j(X, PZ) + \sum_{a=r+1}^{n} \eta_i(A_{E_a}\xi_k)h^s_a(X, PZ)$$
$$= \{(\bar{\nabla}_X\theta)(PZ) - g(X, PZ)\}\delta_{ik} - (\bar{\nabla}_{\xi_k}\theta)(PZ)\eta_i(X).$$

Taking $X = U_i$ and $Z = U_k$ to this and using $(5.16)_3$, $(5.18)_3$, $(5.24)_2$ and (5.25), we have $\gamma_i = 0$. Thus $S(TM)$ is totally geodesic in M. □

Theorem 5.6. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}(c)$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$ and U_is are parallel with respect to the connection ∇. If either $\rho_{ia} = 0$ or $\tau_{ij} = 0$, then $c = 0$, i.e., $\bar{M}(c)$ is flat.*

Proof. (1) In case $\rho_{ia} = 0$. Taking the scalar product with W_a to (3.8), we get $\epsilon_a\theta(U_i)w_a(X) - \theta(W_a)v_i(X) = 0$. Taking $X = W_a$ and $X = V_i$ to this result by turns, we have

$$\theta(U_i) = 0, \qquad \theta(W_a) = 0. \tag{5.26}$$

Taking the scalar product with U_j, N_j, ζ and $F\zeta$ to (3.8) by turns and using (3.6), (5.26) and the fact that $g(F\zeta,\zeta)=0$, we obtain

$$\begin{aligned}(5.27)\qquad &\bar{g}(A_{N_i}X,N_j)=0, &&h_i^*(X,U_j)=0,\\ &g(F(A_{N_i}X),\zeta)=v_i(X), &&h_i^*(X,\zeta)=\eta_i(X).\end{aligned}$$

Applying $\bar{\nabla}_X$ to $\theta(U_i)=0$ and using (3.8) and $(5.27)_3$, we have

$$(5.28)\qquad (\bar{\nabla}_X\theta)(U_i)=0.$$

Applying ∇_Y to $(5.27)_2$ and using the fact that $\nabla_Y U_j=0$, we have

$$(\nabla_X h_i^*)(Y,U_j)=0.$$

Substituting this equation and $(5.27)_2$ into (5.6) such that $PZ=U_j$ and using $(2.13)_4$, $(5.27)_{1,2,4}$, (5.28) and the fact that $\rho_{ia}=0$, we have

$$\frac{c}{4}\{v_j(Y)\eta_i(X)-v_j(X)\eta_i(Y)+v_i(Y)\eta_j(X)-v_i(X)\eta_j(Y)\}=0.$$

Taking $X=\xi_i$ and $Y=V_j$ to this equation, we obtain $c=0$.

(2) In case $\tau_{ij}=0$. Taking the scalar product with V_j to (3.8), we get $\theta(U_i)u_j(X)-\theta(V_j)v_i(X)=0$. Taking $X=U_j$ and $X=V_j$ to this equation by turns, we have

$$(5.29)\qquad \theta(U_i)=0, \qquad \theta(V_i)=0.$$

Taking the scalar product with U_j, N_j, $F\zeta$ and ζ to (3.8) by turns and using (3.6), (5.29) and the fact that $g(F\zeta,\zeta)=0$, we obtain

$$\begin{aligned}(5.30)\qquad &\bar{g}(A_{N_i}X,N_j)=0, \quad h_i^*(X,U_j)=0, \quad h_i^*(X,\zeta)=\eta_i(X),\\ &g(F(A_{N_i}X),\,\zeta)+\sum_{a=r+1}^{n}\theta(W_a)\rho_{ia}(X)=v_i(X).\end{aligned}$$

Applying $\bar{\nabla}_X$ to $\theta(U_i)=0$ and using (3.8) and $(5.30)_4$, we have

$$(5.31)\qquad (\bar{\nabla}_X\theta)(U_i)=0.$$

Applying ∇_Y to $(5.30)_2$ and using the fact that $\nabla_Y U_i=0$, we have

$$(\nabla_X h_i^*)(Y,U_j)=0.$$

Substituting this equation and $(5.30)_2$ into (5.6) with $PZ=U_j$ and using $(5.30)_{1,3}$ and (5.31), we have

$$\frac{c}{4}\{v_j(Y)\eta_i(X)-v_j(X)\eta_i(Y)+v_i(Y)\eta_j(X)-v_i(X)\eta_j(Y)\}=0.$$

Taking $X=\xi_i$ and $Y=V_j$ to this equation, we obtain $c=0$. □

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