GENERALIZED YANG’S CONJECTURE ON THE PERIODICITY OF ENTIRE FUNCTIONS

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ABSTRACT. On the periodicity of transcendental entire functions, Yang’s Conjecture is proposed in [6, 13]. In the paper, we mainly consider and obtain partial results on a general version of Yang’s Conjecture, namely, if \( f(z)^n f^{(k)}(z) \) is a periodic function, then \( f(z) \) is also a periodic function.

We also prove that if \( f(z)^n + f^{(k)}(z) \) is a periodic function with additional assumptions, then \( f(z) \) is also a periodic function, where \( n, k \) are positive integers.

1. Introduction and main results

Titchmarsh [12, p. 267] considered the real transcendental entire solutions of the differential equation

\[ f(z)f^{(k)}(z) = p(z)\sin^2 z, \]

where \( p(z) \) is a non-zero polynomial and obtained the following result. The real entire function \( f(z) \) means that \( f : \mathbb{R} \to \mathbb{R} \).

**Theorem A.** The differential equation \( f(z)f''(z) = -\sin^2 z \) has no real entire functions of finite order other than \( f(z) = \pm \sin z \).

Recently, Li, Lü and Yang [6, Theorem 1] considered Theorem A by removing the assumption that \( f(z) \) is real and of finite order as follows.

**Theorem B.** If \( f(z) \) is an entire function satisfying \( f(z)f''(z) = p(z)\sin^2 z \), where \( p(z) \) is a non-zero polynomial with real coefficients and real zeros, then \( p(z) \) must be a non-zero constant \( p \) and \( f(z) = a\sin z \), where \( a \) is a constant satisfying \( a^2 = -p \).

Obviously, Theorem B generalizes Theorem A, that is \( f(z)f''(z) = -\sin^2 z \) has entire solutions \( f(z) = \pm \sin z \) and no other solutions exist. Remark that \( -\sin^2 z \) is a periodic function and \( \pm \sin z \) are also periodic functions with the
different periods. An interesting conjecture below related to the periodicity of transcendental entire functions is proposed in [6], which is also mentioned in a former paper and called Yang’s Conjecture [13, Conjecture 1.1].

Yang’s Conjecture. Let \( f(z) \) be a transcendental entire function and \( k \) be a positive integer. If \( f(z)f^{(k)}(z) \) is a periodic function, then \( f(z) \) is also a periodic function.

Some results on the periodicity of transcendental meromorphic functions can be found in [1, 3–5, 11, 15]. Obviously, Yang’s Conjecture is also related to the properties of transcendental entire solutions of complex differential-difference equation
\[
f(z)f^{(k)}(z) = f(z+c)f^{(k)}(z+c),
\]
provided that \( f(z)f^{(k)}(z) \) is a periodic function with period \( c \), where \( c \) is non-zero constant. However, it is well known that the complex differential-difference equations are difficult to solve, the simplest complex differential-difference equation \( f'(z) = f(z+c) \) is not solved completely, where \( c \) is a non-zero constant, the partial results on the above equation can be found in [2, 7]. Until now, Yang’s Conjecture has also not been proved completely. We summarize the partial results in the below remark.

Remark 1.1. (1) Yang’s Conjecture is true for \( k = 1 \); it is proved by Wang and Hu [13, Theorem 1.1]. We recall an example given by Zhang and Yi [17, Corollary 1.7]. They obtained that all the entire solutions of
\[
f(z)f'(z) = \frac{1}{2}\sin 2z
\]
are \( f(z) = \pm i\cos z \) and \( f(z) = \pm \sin z \). Obviously, \( \pm i\cos z, \pm \sin z \) and \( \frac{1}{2}\sin 2z \) are periodic functions. Of course, \( f(z)f^{(k)}(z) = \frac{1}{2}\sin 2z \) also admits entire solutions \( f(z) = \pm i\cos z \) and \( f(z) = \pm \sin z \) when \( k = 4m + 1 \) and \( m \) is a positive integer.

(2) Yang’s Conjecture is true for the transcendental entire functions \( f(z) \) with a non-zero Picard exceptional value, see Liu and Yu [9, Theorem 1.1].

(3) Yang’s Conjecture is true for the finite order transcendental entire functions \( f(z) \) with 0 as the Picard exceptional value, see Liu and Korhonen [8]. Actually, \( f(z) = e^{h(z)} \) in this case, where \( h(z) \) is a non-constant polynomial. The basic computations imply that \( h(z) \) is a linear polynomial, and thus \( f(z) \) is periodic.

(4) Recall that Rényi and Rényi [11] proved that if \( f(z) \) is a non-constant entire function and \( P(z) \) is a polynomial with \( \deg(P(z)) \geq 3 \), then \( f(P(z)) \) cannot be a periodic function. Hence, if \( e^{h(z)} \) is a periodic function, then \( \deg(h(z)) \leq 2 \) follows immediately. In addition,
\[
f(z) = e^{iz^3} + e^{z^3} = (e^{iz} + e^z) \circ (z^3).
\]
Thus \( e^{iz^3} + e^{z^3} \) is not a periodic function by Rényi and Rényi’s result. However, if \( f(z) = e^{H(z)} + Q(z) \) is a prime function, then Rényi and Rényi’s result cannot
be used, where $H(z)$ is an entire function and $Q(z)$ is a polynomial. Here, the prime function means that one of $h$ and $s$ must be linear for every factorization of $F(z) = h(s(z))$, $h, s$ are entire functions. Ozawa [10, Theorem 1] showed that for any $\rho \in [1, +\infty)$ there exists a prime periodic entire function $h$ of the order $\rho(h) = \rho$.

(5) Difference and differential-difference versions of Yang’s Conjecture are also considered in Liu and Korhonen [8].

In the paper, the generalized Yang’s Conjecture can be stated as follows. We give the first result based on the corresponding discussions in Remark 1.1.

**Generalized Yang’s Conjecture.** Let $f(z)$ be a transcendental entire function and $n, k$ be positive integers. If $f(z)^nf^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

**Theorem 1.2.** Let $f(z)$ be a transcendental entire function and $n, k$ be positive integers. If one of the following conditions is satisfied

(i) $k = 1$;

(ii) $f(z) = e^{h(z)}$, where $h(z)$ is a non-constant polynomial;

(iii) $f(z)$ has a non-zero Picard exceptional value and $f(z)$ is of finite order, and if $f(z)^nf^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

**Remark 1.3.** Remark that an assumption that $f(z)$ is of finite order is added in (iii). It remains open for us to remove this assumption.

**Theorem 1.4.** Let $f(z)$ be a transcendental entire function and $n, k$ be positive integers. If $f(z)^nf^{(k)}(z)$ and $f(z)^nf^{(k+1)}(z)$ are periodic functions with the same principal period $c$, then $f(z)$ is also a periodic function with period $c, 2c$ or $(n + 1)c$.

**Remark 1.5.** If $f^{(k)}(z)$ and $f^{(l)}(z)$ are periodic entire functions, then the principal periods are the same, where $k, l$ are positive integers. In fact, we assume that $f^{(k)}(z) = f^{(k)}(z + c)$ and $f^{(l)}(z) = f^{(l)}(z + b)$ for all $z \in \mathbb{C}$, where $c, b$ are the principal periods for $f^{(k)}(z)$ and $f^{(l)}(z)$. Without loss of generalization, we assume that $k < l$. In this case,

$$f^{(l)}(z) = f^{(l)}(z + c) = f^{(l)}(z + b)$$

for all $z \in \mathbb{C}$. So $c = b$. However, it remains open for us that $f(z)^nf^{(k)}(z)$ and $f(z)^nf^{(l)}(z)$ have the same principal periods or not if they are periodic functions.

Finally, we consider the periodicity of $f(z)$ if the differential polynomial $f(z)^n + f^{(k)}(z)$ is a periodic function and obtain the following result.

**Theorem 1.6.** Let $f(z)$ be a transcendental entire function and $n \geq 2$, $k$ be a positive integer. If $f(z)^n + f^{(k)}(z)$ is a periodic function with period $c$ and one of the following conditions is satisfied

(i) $k = 1$;
(iii) \( f(z + c) - f(z) \) has no zeros;

(iii) the zeros multiplicity of \( f(z + c) - f(z) \) is great than or equal to \( k \); then \( f(z) \) is also a periodic function with period \( c \) or \( 2c \).

Remark 1.7. Theorem 1.6 is not true provided that \( n = 1 \). We see that \( f(z) = ze^{-z} \) is not a periodic function, but

\[
f(z) + f^{(k)}(z) = (-1)^{k-1}ke^{-z} + (-1)^kze^{-z} + ze^{-z}
\]

is a periodic function provided that \( k \) is odd. Here, it is open for us that Theorem 1.6 is true for any positive integers \( k \).

2. Lemmas

Lemma 2.1 ([14, Theorem 1]). Let \( m, n \) be positive integers satisfying \( \frac{1}{m} + \frac{1}{n} < 1 \). Then there are no non-constant entire solutions \( f(z) \) and \( g(z) \) that satisfy

\[
a(z)f(z)^n + b(z)g(z)^m = 1,
\]

where \( a(z), b(z) \) are small functions with respect to \( f(z) \).

Lemma 2.2 ([16, Theorem 1.62]). Suppose that \( f_j (j = 1, 2, \ldots, n) \) \( (n \geq 3) \) are meromorphic functions which are not constants except for \( f_n \). Furthermore, let

\[
\sum_{j=1}^{n} f_j = 1.
\]

If \( f_n \neq 0 \) and

\[
\sum_{j=1}^{n} N(r, \frac{1}{f_j}) + (n - 1) \sum_{j=1}^{n} N(r, f_j) < (\lambda + o(1))T(r, f_k),
\]

where \( r \in I, I \) is a set whose linear measure is infinite, \( k \in \{1, 2, \ldots, n - 1\} \) and \( \lambda < 1 \), then \( f_n \equiv 1 \).

3. Proofs of Theorems

Proof of Theorem 1.2. We assume that \( n \geq 2 \) in the following proof. The case of \( n = 1 \) has been showed in Remark 1.1.

Case (i). If \( k = 1 \) and \( f^n(z)f'(z) \) is a periodic function with period \( c \), then

\[
f^n(z)f'(z) = f^n(z + c)f'(z + c).
\]

Integrating the above equation, we have

\[
f(z + c)^{n+1} - f(z)^{n+1} = A,
\]

where \( A \) is a constant. If \( n \geq 2 \), then \( A \equiv 0 \) from Lemma 2.1. Hence, \( f(z + c) = tf(z) \) and \( T^{n+1} = 1 \), thus \( f(z) \) is a periodic function with period \( (n + 1)c \).

Case (ii). Since \( f(z)^{n}f^{(k)}(z) \) is a periodic function with period \( c \), we have

\[
f(z)^{n}f^{(k)}(z) = f(z + c)^{n}f^{(k)}(z + c).
\]
Substituting $f(z) = e^{h(z)}$ into (3), where $h(z)$ is a non-constant polynomial. We have
\[ e^{(n+1)[h(z+c)−h(z)]} = \frac{H(z)}{H(z + c)}, \]
where $H(z)$ is a differential polynomial of $h(z)$, hence $H(z)$ is also a polynomial in $z$. Since the rational function $\frac{H(z)}{H(z + c)}$ has neither zeros nor poles, we have $\frac{H(z)}{H(z + c)} \equiv 1$. Thus, we have $e^{(n+1)[h(z+c)−h(z)]} \equiv 1$, that is, $f(z)$ is a periodic function with period $c$ or $(n+1)c$.

Case (iii). Assume that $d$ is the non-zero Picard exceptional value of $f(z)$. Then $f(z) = e^{p(z)} + d$ follows by the Hadamard factorization theorem, where $p(z)$ is a non-constant polynomial. Substituting $f(z) = e^{p(z)} + d$ into (3), a basic computation implies that

\begin{equation}
P_1(z)(e^{p(z)} + d)^n e^{p(z)} = P_1(z + c)(e^{p(z+c)} + d)^n e^{p(z+c)},
\end{equation}

where $P_1(z)$ is a differential polynomial of $p(z)$. Furthermore, we obtain

\begin{equation}
e^{(n+1)p(z)} + C_n^1 d e^{np(z)} + \ldots + C_n^{n-1} d^{n-1} e^{2p(z)} + d^n e^{p(z)} = H_1(z) \left[ e^{(n+1)p(z+c)} + C_n^1 d e^{np(z+c)} + \ldots + d^n e^{p(z+c)} \right],
\end{equation}

where $H_1(z) = \frac{P_1(z+c)}{P_1(z)}$ and $T(r, H_1(z)) = S(r, f)$. We also obtain
\begin{equation}
\frac{H_1(z)}{d^n} \frac{e^{(n+1)p(z+c)} - p(z) + \ldots + H_1(z)e^{p(z+c)} - p(z)}{d^n - \ldots - ne^{p(z)}} = 1.
\end{equation}

Since $p(z)$ is a non-constant polynomial, then $mp(z+c) - p(z)$ $(m = 2, \ldots, n+1)$ cannot be constants other than $p(z + c) - p(z)$. From Lemma 2.2 and (6), we have
\[ \frac{P_1(z + c)e^{p(z+c)} - p(z)}{P_1(z)} \equiv 1. \]

Thus, we have $p(z)$ is a linear polynomial. Furthermore, we have $P_1(z)$ and $P_1(z + c)$ are the same constants. Hence, we have $e^{p(z+c)} = e^{p(z)}$, so $f(z) = f(z + c)$. Thus, $f(z)$ is a periodic function with period $c$. \(\square\)

**Proof of Theorem 1.4.** We assume that $f(z)^n f^{(k)}(z)$ and $f(z)^n f^{(k+1)}(z)$ are periodic functions with period $c$. Then

\begin{equation}
\begin{cases}
 f(z)^n f^{(k)}(z) = f(z + c)^n f^{(k)}(z + c), \\
 f(z)^n f^{(k+1)}(z) = f(z + c)^n f^{(k+1)}(z + c).
\end{cases}
\end{equation}

Let $F(z) = f^{(k)}(z)$. Thus $F'(z) = f^{(k+1)}(z)$, $F(z + c) = f^{(k)}(z + c)$, $F'(z + c) = f^{(k+1)}(z + c)$. By (7), we have
\[ \frac{F'(z + c)}{F(z + c)} = \frac{F'(z)}{F(z)} \]
Integrating the above equation, we have

\( F(z + c) = e^A F(z), \)

that is

\( f^{(k)}(z + c) = e^A f^{(k)}(z), \)

where \( A \) is a constant. From the first equation of (7) and (9), we have

\( f(z + c)^n e^A = f(z)^n. \)

Case 1. If \( n = 1 \), then (10) means

\( f(z + c) e^A = f(z). \)

Differentiating the equation (11) \( k \) times, we have

\( f^{(k)}(z + c) e^A = f^{(k)}(z), \)

that is

\( F(z + c) e^A = F(z). \)

Thus \( e^{2A} = 1 \) follows by (8) and (12). The equation (11) implies that \( f(z) \) is a periodic function with period \( c \) or \( 2c \).

Case 2. In \( n \geq 2 \), by integrating (9) \( k \) times, then

\( f(z + c) = e^A (f(z) + P(z)), \)

where \( P(z) \) is a polynomial with degree less than \( k \). Thus,

\( f(z + c)^n = e^{nA} (f(z) + P(z))^n. \)

We have \( P(z) \equiv 0 \) and \( e^{(n+1)A} = 1 \) by comparing (10) and (13). So \( f(z) \) must be a periodic function with period \((n + 1)c\) from (11).

**Proof of Theorem 1.6.** We assume that \( f(z)^n + f^{(k)}(z) \) is a periodic function with period \( c \), then

\( f(z + c)^n + f^{(k)}(z + c) = f(z)^n + f^{(k)}(z). \)

Thus,

\( f(z + c)^n - f(z)^n = f^{(k)}(z) - f^{(k)}(z + c). \)

The above equation implies that either \( f(z + c) - f(z) \equiv 0 \), i.e., \( f(z) \) is a periodic function with period \( c \) or \( f(z + c) - f(z) \) has no zeros under one of the conditions (i), (ii), (iii) in Theorem 1.6. We assume that \( f(z + c) - f(z) = e^{h(z)} \) from the Hadamard factorization theorem, where \( h(z) \) is an entire function.

Case 1. If \( n = 2 \), then \( f(z + c) + f(z) = H(z) \) follows (14), where \( H(z) \) is a differential polynomial of \( h(z) \) and \( T(r, H(z)) = S(r, e^{h(z)}). \) Hence, we have

\[
\begin{align*}
  f(z) &= \frac{H(z) - e^{h(z)}}{2}, \\
  f(z + c) &= \frac{H(z) + e^{h(z)}}{2} = \frac{H(z + c) - e^{h(z + c)}}{2}.
\end{align*}
\]
Thus, we have
\[ T(r, f(z)) = T(r, f(z + c)) + S(r, e^{h(z)}) \]
from the second main theorem of Nevanlinna theory. Hence,
\[ T(r, e^{h(z)}) = T(r, e^{h(z+c)}) + S(r, e^{h(z+c)}). \]

Thus, we have \( T(r, e^{h(z)}) = O(T(r, e^{h(z+c)})) \). Hence,
\[ T(r, H(z + c)) = S(r, e^{h(z)}). \]

Furthermore, (15) also implies that \( H(z + c) - H(z) = e^{h(z+c)} + e^{h(z)} \) and
\[ T(r, e^{h(z+c)}) = T(r, e^{h(z)}) + S(r, e^{h(z)}). \]

If \( H(z + c) - H(z) \neq 0 \), then
\[
T(r, e^{h(z)}) \leq N\left(r, \frac{1}{e^{h(z)}}\right) + N\left(r, \frac{1}{e^{h(z+c)} - H(z + c) + H(z)}\right) + S(r, e^{h(z)})
\]
\[
\leq N\left(r, \frac{1}{e^{h(z+c)}}\right) + S(r, e^{h(z)})
\]
\[
\leq S(r, e^{h(z)}),
\]
from the second main theorem of Nevanlinna theory. Hence, \( H(z + c) \equiv H(z) \) and \( e^{h(z+c)} = -e^{h(z)} \) follows. Thus, we have \( f(z) \) is a periodic function with period \( 2c \) from (15).

Case 2. If \( n \geq 3 \), from (14), then we also obtain
\[
\begin{align*}
&f(z + c) - f(z) = e^{h(z)}, \\
&f(z + c)^{n-1} + f(z + c)^{n-2}f(z) + \cdots + f(z)^{n-1} = H(z),
\end{align*}
\]
where \( H(z) \) is a differential polynomial of \( h(z) \) and \( T(r, H(z)) = S(r, e^{h(z)}) \). The equation (16) means
\[
\begin{align*}
&\begin{cases}
\left( f(z) \right)^n f(z + c) - f(z) = e^{h(z)} , \\
f(z)^{n-1} \left[ f(z + c)^n - f(z)^n - 1 \right] + f(z + c)^{n-2} + \cdots + f(z + c) + 1 = H(z). \end{cases}
\end{align*}
\]

Let \( M(z) = \frac{f(z + c)}{f(z)} \). Obviously, if \( M(z) \equiv 1 \), then \( f(z) \) is a periodic function with period \( c \). Suppose that \( M(z) \neq 1 \) and \( M(z) \) is not a constant. The equation (17) implies that
\[
\frac{(M(z) - 1)^{n-1}}{M(z)^{n-1} + M(z)^{n-2} + \cdots + M(z) + 1} = \frac{e^{(n-1)h(z)}}{H(z)}. \]

From (18), we obtain
\[
(n - 1)T(r, M(z)) = (n - 1)T(r, e^{h(z)}) + S(r, e^{h(z)}),
\]
\[
N\left(r, \frac{1}{M(z)^{n-1} + M(z)^{n-2} + \cdots + M(z) + 1}\right) = S(r, e^{h(z)}),
\]
and $M - 1$ has no zeros. Using the second main theorem of Nevanlinna theory, we have

$$(n - 2)T(r, M) \leq N\left(r, \frac{1}{M - 1}\right) + N\left(r, \frac{1}{M^{n-1} + M^{n-2} + \ldots + M + 1}\right)$$

$$+ S(r, M)$$

$$\leq S(r, e^{h(z)}) + S(r, M) = S(r, M),$$

which is impossible for $n \geq 3$. Hence, we have $M(z)$ must be a constant and $M(z) \neq 1$. The first equation of (17) means

$$T(r, f) = T(r, e^{h(z)}),$$

the second equation of (17) means

$$(n - 1)T(r, f) = T(r, H(z)) = S(r, e^{h(z)}) = S(r, f),$$

which leads to a contradiction. \qed

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