QUASI CONTACT METRIC MANIFOLDS WITH KILLING CHARACTERISTIC VECTOR FIELDS

JIHONG BAЕ, YEONGJAE JANG, JEONGHYEONG PARK, AND KOUEI SEKIGAWA

Abstract. An almost contact metric manifold is called a quasi contact metric manifold if the corresponding almost Hermitian cone is a quasi Kähler manifold, which was introduced by Y. Tashiro [9] as a contact O*-manifold. In this paper, we show that a quasi contact metric manifold with Killing characteristic vector field is a K-contact manifold. This provides an extension of the definition of K-contact manifold.

1. Introduction

A (2n+1)-dimensional smooth manifold is called an almost contact manifold if it admits a triple \((\phi, \xi, \eta)\) of a \((1,1)\)-tensor field \(\phi\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying the following condition [1]:

\[
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \\
\phi \xi &= 0, \quad \eta \circ \phi = 0.
\end{align*}
\]

(1)

From (1), we may deduce equality \(\eta(\xi) = 1\). Further, an almost contact manifold \(M = (M, \phi, \xi, \eta)\) equipped with a Riemannian metric \(g\) such that

\[
\begin{align*}
g(\phi U, \phi V) &= g(U, V) - \eta(U)\eta(V), \\
\eta(U) &= g(\xi, U),
\end{align*}
\]

(2)

for any smooth vector fields on \(U, V\) on \(M\), is called an almost contact metric manifold. On the other hand, \(M\) admitting a 1-form \(\eta\) satisfying \(\eta \wedge (d\eta)^n \neq 0\) everywhere on \(M\) is called a contact manifold with the contact 1-form \(\eta\). Now, an almost contact metric manifold \(M = (M, \phi, \xi, \eta, g)\) satisfying the condition

\[
d\eta(U, V) = g(U, \phi V),
\]

(3)

for any smooth vector fields \(U, V\) on \(M\), which is called a contact metric manifold with contact 1-form \(\eta\). Then from (3), we can check that \(\eta \wedge (d\eta)^n \neq 0\) on \(M\), and hence, \(M = (M, \eta)\) is a contact manifold with the contact 1-form \(\eta\).
Now, let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $\tilde{M} = M \times \mathbb{R}$ be a product manifold of $M$ and a real line $\mathbb{R}$ equipped with almost Hermitian structure $(\tilde{J}, \tilde{g})$ defined by

\begin{align}
\tilde{J}U &= \phi U - \eta(U) \frac{\partial}{\partial t}, \quad \tilde{J} \frac{\partial}{\partial t} = \xi, \\
\tilde{g}(U, V) &= e^{-2t} g(U, V), \quad \tilde{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = e^{-2t}, \quad \tilde{g}(U, \frac{\partial}{\partial t}) = 0,
\end{align}

for any $U, V \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on $M$, and $t \in \mathbb{R}$. The almost Hermitian manifold $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ is called an almost Hermitian cone corresponding to the almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ ([2], Remark 1.1). Gray and Hervella [4] classified 16 classes of almost Hermitian manifolds. Among their classes, the class of Kähler manifolds, almost Kähler manifolds and quasi Kähler manifolds (also known as O*-manifolds) have been examined extensively by many researchers. We remark that Kähler manifolds are included in almost Kähler manifolds and also almost Kähler manifolds are included in quasi Kähler manifolds [4].

Now, let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ be the corresponding almost Hermitian cone. Then, we may check that $M = (M, \phi, \xi, \eta, g)$ is a Sasakian manifold if and only if $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ is a Kähler manifold, $M = (M, \phi, \xi, \eta, g)$ is a contact metric manifold if and only if $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ is a contact metric manifold if and only if $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ is an almost Kähler manifold. Thus, a contact metric manifold is necessary a quasi contact metric manifold.

**Remark 1.** Any 4-dimensional quasi Kähler manifold is necessarily an almost Kähler manifold.

Thus, taking account of [2] and Remark 1, the following question was raised by the third and fourth authors in the previous paper in [5]:

**Question.** Does there exist a $(2n + 1)$ $(\geq 5)$-dimensional quasi contact metric manifold which is not a contact metric manifold?

Concerning the above question, several related results have been obtained [2, 5, 7].

Now, a contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called a K-contact manifold if the characteristic vector field $\xi$ is a Killing vector field. The main purpose of the paper is to prove the following.

**Theorem A.** A quasi contact metric manifold $M = (M, \phi, \xi, \eta, g)$ with Killing characteristic vector field $\xi$ is a K-contact manifold.

We remark that the main theorem of the paper [6] follows immediately from Theorem A, taking account of the result by Tanno [8].
2. Preliminaries

In this section, we prepare some basic preliminaries and fundamental formulas on quasi contact metric manifolds. Unless otherwise stated, all manifolds are assumed to be \((2n + 1)\)-dimensional smooth Riemannian manifolds. Let \(M = (M, \phi, \xi, \eta, g)\) be an almost contact metric manifold and \(h\) be the \((1, 1)\)-tensor field by

\[
h = \frac{1}{2} \mathcal{L}_\xi \phi,
\]

where \(\mathcal{L}_\xi\) denotes the Lie derivative with respect to \(\xi\). The tensor field \(h\) plays a vital role in the geometry of almost contact metric manifolds. From (5), we may easily check that the tensor field \(h\) satisfies the following properties:

\[
h \xi = 0, \quad \text{tr} h = 0.
\]

Let \(\nabla\) be the Levi-Civita connection of \(g\). We here recall the following results which characterize quasi contact metric manifold and contact metric manifold ([5] Theorems 3.2 and 4.2).

**Theorem 2.1.** An almost contact metric manifold \(M = (M, \phi, \xi, \eta, g)\) is a quasi contact metric manifold if and only if \(M\) satisfies the below equality:

\[
(\nabla V \phi) U + (\nabla \phi U) \phi V = 2g(U, V) \xi - \eta(V) U - \eta(U) \eta(V) \xi - \eta(V) hU
\]

for any \(U, V \in \mathfrak{X}(M)\).

Further, we also have the following Theorem.

**Theorem 2.2.** A quasi contact metric manifold \(M = (M, \phi, \xi, \eta, g)\) is a contact metric manifold if \(h\) is symmetric with respect to the metric \(g\).

In the remainder of this section, we suppose that \(M = (M, \phi, \xi, \eta, g)\) is a quasi contact metric manifold. Then, the following equalities can be derived from the equality (7):

\[
(\nabla U \eta)(V) + (\nabla \phi U \eta)(V) + 2g(\phi U, V) = 0,
\]

\[
\nabla \xi \phi = 0,
\]

\[
\nabla \xi \xi = 0,
\]

\[
hU = \frac{1}{2} (\nabla \phi U \xi + \phi \nabla \xi U),
\]

for any \(U, V \in \mathfrak{X}(M)\). From (11), taking account of (10), we obtain further the following equalities:

\[
h \phi + \phi h = 0,
\]

\[
\nabla U \xi = -\phi U - \phi hU,
\]

for any \(U \in \mathfrak{X}(M)\). From (12), we also get

\[
\eta \circ h = 0.
\]
The equalities (7) \sim (13) play an essential role in the proof of Theorem A.

3. Proof of Theorem A

Throughout this section, we suppose that \( M = (M, \phi, \xi, \eta, g) \) is a \((2n + 1)\)-dimensional quasi contact metric manifold such that \( \xi \) is a Killing vector field. Here, thanks to account of Remark 1, for the prove Theorem A, it suffices to discuss in the case \( n \geq 2 \). Now, let \( \nabla \xi \) be the \((1,1)\)-tensor field on \( M \) defined by

\[
(\nabla \xi)U = \nabla U \xi
\]

for any \( U \in \mathfrak{X}(M) \). Then, we can check that the \((1,1)\)-tensor field \( \nabla \xi \) is skew-symmetric with respect to \( g \) on \( M \). So, from (13), it follows that \( \phi h \) is also skew-symmetric with respect to \( g \);

\[
g(h(\phi U), V) + g(h(\phi V), U) = 0
\]

for any \( U, V \in \mathfrak{X}(M) \). Here, changing \( V \) by \( \phi V \) in (16), and taking account of (12), we get

\[
g(hU, V) + g(hV, U) = 0
\]

for any \( U, V \in \mathfrak{X}(M) \). Therefore, we obtain the following.

**Lemma 3.1.** Under the hypothesis of Theorem A, the tensor field \( h \) is skew-symmetric with respect to the metric \( g \).

Since the characteristic vector field \( \xi \) is a Killing vector field, the \((1,1)\)-tensor field \( \nabla \xi \) is a skew-symmetric linear endomorphism on the surface \( \{ \xi_p \}^\perp \) in the tangent space \( T_p M \) at any point \( p \in M \). Thus, we may choose a local orthonormal frame field \( \{ e_1, e_1^*, e_2, e_2^*, \ldots, e_n, e_n^* \} \) in a neighborhood of the point \( p \) such that \( e_a, e_a^* \perp \xi \) for any \( a \) \((1 \leq a \leq n) \), and further satisfies the condition:

\[
\nabla \xi (e_a) = -\lambda_a e_a^*, \quad \nabla \xi (e_a^*) = \lambda_a e_a,
\]

for some smooth functions \( \lambda_a \) defined on a neighborhood of the point \( p \), for any \( a \) \((1 \leq a \leq n) \) \([3]\). Thus, from (15) and (18), taking account of (13), we have

\[
\phi(I + h)e_a = \lambda_a e_a^*,
\]

\[
\phi(I + h)e_a^* = -\lambda_a e_a,
\]

for \( a = 1, 2, \ldots, n \). Thus, from (19), taking account of (12), we get respectively

\[
\phi he_a = \lambda_a e_a^* - \phi e_a,
\]

\[
\phi he_a^* = -\lambda_a e_a - \phi e_a^*,
\]
for any \( 1 \leq a \leq n \). Thus, from (20), taking account of (1) and (14), we have respectively
\[
he_a = -e_a - \lambda_a \phi e_a^*, \\
he_a^* = -e_a^* + \lambda_a \phi e_a,
\]
for any \( 1 \leq a \leq n \). From the equalities of (21), taking account of (1) and (14), we have
\[
g(h^e_a, e_a^*) = 0, \\
g(h^e_a^*, e_a) = 0.
\]
Therefore, from (21), taking account of (20) together with (12) and (21), we have
\[
h^2 e_a = -he_a + \lambda_a \phi he_a^* \\
= e_a + \lambda_a \phi e_a^* + \lambda_a (\lambda_a e_a - \phi e_a^*) \\
= (1 - \lambda_a^2) e_a,
\]
\[
h^2 e_a^* = -he_a^* - \lambda_a \phi he_a \\
= e_a^* - \lambda_a \phi e_a - \lambda_a (\lambda_a e_a^* - \phi e_a) \\
= (1 - \lambda_a^2) e_a^*.
\]
for any \( 1 \leq a \leq n \). Thus, from (23) taking account of (12), we see that the linear endomorphism \( h^2 \) is symmetric with respect to \( g, \phi \)-invariant, and can be block-diagonalized as follows:
\[
h^2(e_1, e_1^*, e_2, e_2^*, \ldots, e_n, e_n^*) \\
= (h^2 e_1, h^2 e_1^*, h^2 e_2, h^2 e_2^*, \ldots, h^2 e_n, h^2 e_n^*) \\
= (e_1, e_1^*, \ldots, e_n, e_n^*) \begin{pmatrix}
1 - \lambda_1^2 & 0 & 0 \\
0 & 1 - \lambda_1^2 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 - \lambda_n^2 & 0 \\
0 & 0 & 0 & 0 & 1 - \lambda_n^2
\end{pmatrix}
\]
Here, from (23), taking account of Lemma 3.1, we have
\[
0 \leq g(h^e_a, he_a) = -g(e_a, h^2 e_a) = \lambda_a^2 - 1, \\
0 \leq g(h^e_a^*, he_a^*) = -g(e_a^*, h^2 e_a^*) = \lambda_a^2 - 1,
\]
and hence, \( \lambda_a^2 \geq 1 \) for any \( a (1 \leq a \leq n) \). Thus, from (24) and (25), taking account of (20)~(23), we may check that eigenvalues of the skew-symmetric linear endomorphism \( h \) are \( i \sqrt{\lambda_a^2 - 1} \) or \( -i \sqrt{\lambda_a^2 - 1} \) (possibly zero) for any \( a (1 \leq a \leq n) \). Now, we here arrange the notational convention as follows:
\[
\varepsilon_1 = e_1, \quad \varepsilon_2 = e_1^*, \quad \varepsilon_3 = e_2, \quad \varepsilon_4 = e_2^*, \ldots, \varepsilon_{2n-1} = e_n, \quad \varepsilon_{2n} = e_n^*.
\]
Taking account of the arrangement of notational convention (26), we may identify \( \{\xi_p^\perp, g_p\} \) with a 2\( n \)-dimensional Euclidean space \( \mathbb{R}^{2n} \) with orthonormal basis \( \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}\} \) in the natural way. We also denote by \( H = (H_{jk})(j,k = 1,2,\ldots,2n) \) the \( 2n \times 2n \) real matrix defined by

\[
H_{jk} = g(h_{\varepsilon_j}, \varepsilon_k) \quad (j,k = 1,2,\ldots,2n).
\]  

(27)

Then, from (26) and (27), taking account of Lemma 3.1, the matrix \( H = (H_{jk}) \) is a \( 2n \times 2n \) skew-symmetric matrix. Thus, from the above discussion and the result in the paper [10] (Proposition 2.1, and Theorem 2.5), we have the following.

**Lemma 3.2.** There exists a \( 2n \times 2n \) orthogonal matrix \( A \in O(2n) \) satisfying the equality:

\[
AHA^{-1} = \begin{pmatrix}
0 & \sqrt{\lambda_1^2 - 1} & & & & \\
-\sqrt{\lambda_1^2 - 1} & 0 & \sqrt{\lambda_2^2 - 1} & & & \\
& -\sqrt{\lambda_2^2 - 1} & 0 & \sqrt{\lambda_3^2 - 1} & & \\
& & & \ddots & \ddots & \ddots \\
& & & & -\sqrt{\lambda_{2n}^2 - 1} & 0 \\
& & & & & 0
\end{pmatrix}.
\]  

(28)

We here denote by \( \bar{H} = (\bar{H}_{jk})(j,k = 1,2,\ldots,2n) \) the \( 2n \times 2n \) real matrix of the right-hand side of the equality (28), which is called the normal real form for the skew-symmetric \( H = (H_{jk}) \). Then, we can check that \( \bar{H} = (\bar{H}_{jk}) \) is also a \( 2n \times 2n \) skew-symmetric matrix. Now, we denote by \( \alpha \) the orthogonal transformation on the \( 2n \)-dimensional Euclidean space \( \mathbb{R}^{2n} = \{\xi_p^\perp, g_p\} \) corresponding to the matrix \( A \), we now set

\[
\alpha e_a = \bar{e}_a, \quad \alpha e_a^* = \bar{e}_a^* \quad (a = 1,2,\ldots,n),
\]  

(29)

and also

\[
\alpha \varepsilon_l = \bar{\varepsilon}_l \quad (l = 1,2,\ldots,2n).
\]  

(30)

Further, we denote by \( \bar{h} \) the linear endomorphism on \( \mathbb{R}^{2n} = \{\xi_p^\perp, g_p\} \) corresponding to the \( 2n \times 2n \) skew-symmetric matrix \( \bar{H} \) defined by

\[
\bar{H}_{jk} = g(\bar{h}_{\varepsilon_j}, \varepsilon_k) \quad (j,k = 1,2,\ldots,2n).
\]  

(31)

Then, from (29) ~ (31), we may easily check that

\[
\alpha \bar{h} \alpha^{-1} = \bar{h},
\]  

(32)

and hence \( \alpha \bar{h} = \bar{h} \alpha \) holds. Thus, from (28) ~ (32), taking account of (27), we have

\[
\alpha \bar{e}_a = \bar{h} \alpha e_a = \bar{h} \bar{e}_a.
\]
and hence
\[ he_a = \alpha^{-1} (he_a) \]
\[ = \alpha^{-1} (-\sqrt{\lambda^2 - 1} e_a) \]
\[ = -\sqrt{\lambda^2 - 1} e_a \]
for any \(a \ (1 \leq a \leq n)\). Similarly we have also
\[ he_a^* = \sqrt{\lambda^2 - 1} e_a \]
for any \(a \ (1 \leq a \leq n)\). Thus, from (34) and (22), we get
\[ \sqrt{\lambda^2 - 1} = 0 \]
for any \(a \ (1 \leq a \leq n)\), and similarly from (33) and (22), we get also
\[ \sqrt{\lambda^2 - 1} = 0 \]
for any \(a \ (1 \leq a \leq n)\). Therefore, from (23), taking account of (35) and (36), we see that \(h^2 e_a = 0\), and also \(h^2 e_a^* = 0\) for any \(a \ (1 \leq a \leq n)\), and hence \(h^2 = 0\) everywhere on \(M\) since \(h^2 \xi = 0\). We here recall that \(h\) is skew-symmetric with respect to the metric \(g\) by Lemma 3.1, and hence, it follows that \(\|h\|^2 = -trh^2 = 0\). Thus, we see that \(h\) vanishes identically on \(M\). Especially, \(h\) is symmetric with respect to the metric \(g\). Therefore, \(M = (M, \phi, \xi, \eta, g)\) is a contact metric manifold by virtue of Theorem 2.2, and hence, a K-contact manifold. This completes the proof of Theorem A.

References


Jihong Bae
Department of Mathematics
Sungkyunkwan University
Suwon 16419, Korea
Email address: baeji0904@skku.edu

Yeongjae Jang
Department of Mathematics
Sungkyunkwan University
Suwon 16419, Korea
Email address: ja3156@skku.edu

JeongHyeong Park
Department of Mathematics
Sungkyunkwan University
Suwon 16419, Korea
Email address: parkj@skku.edu

Kouei Sekigawa
Department of Mathematics
Niigata University
Niigata 950-2181, Japan
Email address: sekigawa@math.sc.niigata-u.ac.jp