

Variants of Essential Arity for Partial Functions

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ABSTRACT. Generalizations of essential arity of partial functions, based on essential sets of arguments, are introduced, and their interplay with the minor relation of functions is investigated.

1. Introduction

In many-valued logic, the theory of essential arguments of functions is a topic that has been widely studied by many authors [2, 3, 5, 6, 7, 8, 9, 10, 14, 16]. In 1963, Salomaa [14] considered how the number of essential arguments of a Boolean function is affected by identification of arguments. It turned out that the so-called arity gap (i.e., the minimum decrease in the number of essential arguments when a pair of essential arguments is identified) of any Boolean function is at most 2. Willard [17] generalized this result in 1996. He considered the same problem for a function $f: A^n \rightarrow B$ defined on arbitrary finite sets A and B . He found that if $n > |A|$, then the arity gap of f is at most 2.

A function f is a minor of a function g if f can be obtained from g by the operations of identification of arguments, permutation of arguments, introduction of inessential arguments, and deletion of inessential arguments. Minors of functions have been extensively investigated in the past decades [4, 5, 6, 7, 11, 12, 13]. This concept has also been used as a tool to study threshold functions [18]. It appears

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in many papers under different names, for example, “identification minor” in [11], “ \mathcal{J} -minor”, where \mathcal{J} is the set containing just the identity function, in [13], “function obtained by simple variable substitution” in [4] and “simple minor” in [6].

The number of essential arguments (essential arity) can be seen as a kind of measure of the structural complexity of a function. It is compatible with the minor quasiorder; the number of essential arguments of a total function does not increase upon formation of minors. Moreover, the essentially at most unary functions are minimal members of the minor quasiorder, and a total function is constant if and only if it has no essential arguments. This is no longer true for partial functions. As we will see in Section 2, in contrast to total functions, the number of essential arguments of a partial function may increase upon formation of minors, and it is possible that a nonconstant partial function has no essential argument. From this standpoint, essential arity seems slightly unsatisfactory as a measure of structural complexity for partial functions. We would like to modify this notion to be better suitable for partial functions.

In this paper, we introduce five notions that attempt to generalize essential arguments and essential arity to partial functions: essential argument, essential set of arguments, minimal essential set, width of the poset of essential sets, exact essential set. Each one of these notions gives rise to a quantity analogous to essential arity, and we will assess each with regard to the following properties, which are useful and desirable for a “successful” substitute of essential arity for partial functions:

- (1) isotonicity with respect to the minor order,
- (2) equality to 0 precisely for constant functions,
- (3) coincidence with essential arity for total functions.

Introduced in Section 3, the first generalization of the notion of essential argument is that of essential set of arguments. The set of all essential sets of arguments of a function, ordered by inclusion, forms a poset, and we take the minimal essential sets as the second generalization of essential arguments. The next generalization is the width of the poset of essential sets, which is considered in Section 4. We will see that the associated quantities (essential arity, the number of essential sets, the number of minimal essential sets, the width of the poset of essential sets) may increase upon formation of minors, and hence do not have the desired property (1).

As a last variant of essential argument, we define the notion of exact essential set in Section 5. The logarithm of the number of exact essential sets of a partial function is called exact essential hyperarity. We will see that the exact essential hyperarity of a partial function does not increase when we form minors, and it equals 0 precisely for constant functions. Although this notion does not coincide with essential arity for total functions, it seems a suitable measure of the structural complexity for partial functions. We continue in Section 6 with a modification of the arity gap in which we consider the minimum decrease in the exact essential hyperarity of a partial function upon formation of minors. We will see that an

analogue of Willard’s theorem does not hold for the exact hyperarity gap of partial functions. We summarize our findings in the final Section 7.

2. Minors and Essential Arguments of Partial Functions

We denote the set of natural numbers by $\mathbb{N} := \{0, 1, 2, \dots\}$ and the set of positive integers by $\mathbb{N}_+ := \{1, 2, \dots\}$. For $n \in \mathbb{N}$, the set $\{1, \dots, n\}$ is denoted by $[n]$ (note that $[0] = \emptyset$). Let A be a nonempty set and n a positive integer. The usual notation for a tuple in A^n is (a_1, \dots, a_n) where $a_i \in A$ for all $1 \leq i \leq n$. We will use bold letters to denote tuples and corresponding italic letters with subscript to denote their components, that is, $\mathbf{a} = (a_1, \dots, a_n)$. We denote the set $\{0, 1\}$ by \mathbb{B} . We denote constant tuples $(0, \dots, 0)$ and $(1, \dots, 1)$ in \mathbb{B}^n by $\mathbf{0}$ and $\mathbf{1}$, respectively.

For arbitrary sets A and B , a *partial function* (of several arguments) from A to B is a function $f: X \rightarrow B$ where $X \subseteq A^n$, $n \in \mathbb{N}_+$. The number n is called the *arity* of f . The set X is called the *domain* of f . If $A = B$, then f is called a *partial operation on A*. Furthermore, if $X = A^n$, then f is said to be an n -ary (*total*) *function* from A to B . We denote the set of all partial functions from A to B by \mathcal{P}_{AB} and the set of all total functions from A to B by \mathcal{F}_{AB} .

We can view any tuple $\mathbf{a} \in A^m$ as a mapping $\mathbf{a}: [m] \rightarrow A$; hence, $\mathbf{a}(i) = a_i$ for all $i \in [m]$. Then we can compose tuples with other maps. If $\sigma: [n] \rightarrow [m]$ is a map, then the composite $\mathbf{a} \circ \sigma: [n] \rightarrow A$ is a tuple in A^n . For simplicity, we write $\mathbf{a}\sigma$ instead of $\mathbf{a} \circ \sigma$. Any map $\sigma: [n] \rightarrow [m]$ gives rise to a map $\underline{\sigma}: A^m \rightarrow A^n$ defined by $\underline{\sigma}(\mathbf{a}) = \mathbf{a}\sigma$ for all $\mathbf{a} \in A^m$.

Let $X \subseteq A^m$ and $Y \subseteq A^n$ for some natural numbers m and n . A partial function $f: X \rightarrow B$ is said to be a *minor* of a partial function $g: Y \rightarrow B$ if there exists a map $\sigma: [n] \rightarrow [m]$ such that $X = \underline{\sigma}^{-1}(Y) = \{\mathbf{a} \in A^m : \mathbf{a}\sigma \in Y\}$ and $f = g \circ \underline{\sigma}|_X$. The notation $f \leq g$ is used to designate the fact that f is a minor of g . For notational simplicity, we omit the “domain restriction” and write simply $f = g \circ \underline{\sigma}$ for $f = g \circ \underline{\sigma}|_X$. If we want to emphasise the map σ that makes the above condition hold, we say that f is a minor of g (*arising*) *via* σ , briefly $f \leq g$ *via* σ .

It is not difficult to see that the minor relation \leq is a quasiorder (a reflexive and transitive relation) on \mathcal{P}_{AB} . The quasiorder \leq on \mathcal{P}_{AB} induces an equivalence relation \equiv on this set by the following: $f \equiv g$ if and only if $f \leq g$ and $g \leq f$. We say that f and g are *equivalent* if $f \equiv g$. Furthermore, the quasiorder \leq on \mathcal{P}_{AB} induces a partial order \leq on the quotient \mathcal{P}_{AB}/\equiv by the following rule: $f/\equiv \leq g/\equiv$ if and only if $f \leq g$.

Let $f: X \rightarrow B$ be a partial function, where $X \subseteq A^n$. We say that the argument $i \in [n]$ is *essential* in f , or f *depends* on argument i , if there exist tuples $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in the domain of f such that $a_j = b_j$ for all $j \in [n] \setminus \{i\}$ and $f(\mathbf{a}) \neq f(\mathbf{b})$. We call such a pair (\mathbf{a}, \mathbf{b}) a *witness (of essentiality)* of argument i in f . We denote by $\text{Ess } f$ the set of essential arguments of f , that is,

$$\text{Ess } f := \{i \in [n] : \text{argument } i \text{ is essential in } f\}.$$

The number of essential arguments in f is called the *essential arity* of f and denoted

by $\text{ess } f$, that is, $\text{ess } f = |\text{Ess } f|$.

The above definition of essential arguments is a natural and straightforward generalization of the usual definition of essential arguments of a total function (see [17]). For total functions, the essential arity behaves nicely with respect to minors, as described in the following lemma.

Lemma 2.1.([12, Lemma 2.2.4]) *Let $f: A^m \rightarrow B$ and $g: A^n \rightarrow B$ be total functions such that $f \leq g$ via σ . Then the following statements hold.*

- (i) $\text{Ess } f \subseteq \text{Im } \sigma$.
- (ii) For every $i \in \text{Ess } f$, we have that $\sigma^{-1}(i) \cap \text{Ess } g \neq \emptyset$.
- (iii) $\text{ess } f \leq \text{ess } g$.

As shown by the following example, conditions (ii) and (iii) do not hold in general for partial functions. Nevertheless, implication (ii) \Rightarrow (iii) holds also for partial functions.

Example 2.2. Let X and Y be sets given by

$$X = \{(0, 0), (0, 1), (1, 0)\} \quad \text{and} \quad Y = \{(0, 0, 0), (0, 0, 1), (1, 1, 0)\}.$$

Define $f: X \rightarrow \mathbb{B}$ and $g: Y \rightarrow \mathbb{B}$ by

$$\begin{array}{ccc} & f & \\ (0, 0) & \mapsto & 0 \\ (0, 1) & \mapsto & 1 \\ (1, 0) & \mapsto & 1 \end{array} \quad \begin{array}{ccc} & g & \\ (0, 0, 0) & \mapsto & 0 \\ (0, 0, 1) & \mapsto & 1 \\ (1, 1, 0) & \mapsto & 1. \end{array}$$

We see that $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, $\text{Ess } f = \{1, 2\}$ and $\text{Ess } g = \{3\}$. Now, $1 \in \text{Ess } f$ but $\sigma^{-1}(1) \cap \text{Ess } g = \{1, 2\} \cap \{3\} = \emptyset$, and $\text{ess } f > \text{ess } g$.

Lemma 2.3.([12, Lemma 2.2.4(vi)]) *Let $f: A^m \rightarrow B$, $g: A^n \rightarrow B$ be total functions, and assume that $f \leq g$ via σ . Then $f \equiv g$ if and only if $\text{ess } f = \text{ess } g$.*

Fact 2.4. *Let f and g be total functions. Then the following statements hold.*

- (i) $\text{ess } f = 0$ if and only if f is constant.
- (ii) $f < g$ implies $\text{ess } f < \text{ess } g$.

We note that Fact 2.4(ii) follows from Lemma 2.1(iii) and Lemma 2.3. These facts do not hold for partial functions. Before giving a counterexample below, in Example 2.5, we introduce some notation.

Let $n \in \mathbb{N}_+$. For $S \subseteq [n]$, the *characteristic tuple* of S is $\mathbf{e}_S := (a_1, \dots, a_n)$ where

$$a_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

for every $i \in [n]$.

Example 2.5. Let $n \in \mathbb{N}_+$. We define a partial function $f: X \rightarrow \mathbb{B}$ of arity n , where

$$X := \{\mathbf{e}_{\{i\}} : i \in [n]\} \cup \{\mathbf{0}\},$$

by

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0}, \\ 1 & \text{if } \mathbf{x} \in X \setminus \{\mathbf{0}\}, \end{cases}$$

for all $\mathbf{x} \in X$. We define a partial function $g: Y \rightarrow \mathbb{B}$ of arity $2n$, where

$$Y := \{\mathbf{e}_{\{2i-1, 2i\}} : i \in [n]\} \cup \{\mathbf{0}\},$$

by

$$g(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y} = \mathbf{0}, \\ 1 & \text{if } \mathbf{y} \in Y \setminus \{\mathbf{0}\}, \end{cases}$$

for all $\mathbf{y} \in Y$. Then $f \leq g$ via the map $\sigma: [2n] \rightarrow [n]$ given by $\sigma(i) = \lceil i/2 \rceil$ for all $i \in [2n]$, where $\lceil \cdot \rceil$ is the ceiling function. We can further see that $\text{ess } f = n$ and $\text{ess } g = 0$.

Let $f: X \rightarrow B$, where $X \subseteq A^n$, be a partial function and $I \in \binom{[n]}{2}$, where $\binom{[n]}{2}$ designates the set of all two-element subsets on $[n]$. A partial function $f_I: \delta_I^{-1}(X) \rightarrow B$ defined by the rule $f_I(\mathbf{a}) = f(\mathbf{a}\delta_I)$ for all $\mathbf{a} \in \delta_I^{-1}(X)$, where $\delta_I: [n] \rightarrow [n-1]$ is defined as

$$\delta_I(i) = \begin{cases} i & \text{if } i < \max I, \\ \min I & \text{if } i = \max I, \\ i - 1 & \text{if } i > \max I, \end{cases}$$

is called an *identification minor* of f . In explicit terms, if $I = \{i, j\}$ with $i < j$, then

$$f_I(a_1, \dots, a_{n-1}) = f(a_1, \dots, a_{j-1}, a_i, a_j, \dots, a_{n-1})$$

for all $(a_1, \dots, a_{n-1}) \in \delta_I^{-1}(X)$.

The identification minors f_I are perhaps the most relevant among all minors of f , because every minor of f is equivalent to one that is obtained by successively identifying just a single pair of arguments at a time (see [12, Lemma 2.2.11, Fact 2.2.12]). As illustrated by Example 2.5, the essential arity of a minor of a partial function can be much larger than the essential arity of the given function itself. On the other hand, Proposition 2.7 below shows that the essential arity can only increase by at most 1 when a single pair of arguments is identified. The large increase in the essential arity seen in Example 2.5 is possible because many arguments are identified.

Lemma 2.6. *Let f be a partial function of arity n and $I \in \binom{[n]}{2}$. If $k \in \text{Ess } f_I \setminus \delta_I(I)$, then $\delta_I^{-1}(k) \in \text{Ess } f \setminus I$.*

Proof. Let $k \in \text{Ess } f_I \setminus \delta_I(I)$ and let (\mathbf{a}, \mathbf{b}) be a witness of essentiality of argument k in f_I . Then $\mathbf{a}\delta_I = \mathbf{c}$ and $\mathbf{b}\delta_I = \mathbf{d}$ for some $\mathbf{c}, \mathbf{d} \in X$. That is,

$$(c_1, \dots, c_n) = (a_{\delta_I(1)}, \dots, a_{\delta_I(n)})$$

and

$$(d_1, \dots, d_n) = (b_{\delta_I(1)}, \dots, b_{\delta_I(n)}).$$

Since $k \in \text{Ess } f_I$, we have that $a_l = b_l$ for all $l \in [n-1] \setminus \{k\}$. Then $a_{\delta_I(p)} = b_{\delta_I(p)}$ for all $p \in [n] \setminus \{\delta_I^{-1}(k)\}$. Thus, $c_r = d_r$ for all $r \in [n] \setminus \{\delta_I^{-1}(k)\}$. Then $f(\mathbf{c}) = f(\mathbf{a}\delta_I) = f_I(\mathbf{a}) \neq f_I(\mathbf{b}) = f(\mathbf{b}\delta_I) = f(\mathbf{d})$. This shows that (\mathbf{c}, \mathbf{d}) is a witness of essentiality of argument $\delta_I^{-1}(k)$ in f . Since $k \notin \delta_I(I)$, we have that $\delta_I^{-1}(k) \notin I$. Thus, $\delta_I^{-1}(k) \in \text{Ess } f \setminus I$. \square

Proposition 2.7. *Let f be a partial function of arity n and $I \in \binom{[n]}{2}$. Then $\text{ess } f_I \leq \text{ess } f + 1$.*

Proof. By Lemma 2.6, we have

$$\text{ess } f_I \leq |\text{Ess } f_I \setminus \delta_I(I)| + 1 \leq |\text{Ess } f \setminus I| + 1 \leq \text{ess } f + 1. \quad \square$$

Note that Example 2.2 shows that the inequality in Proposition 2.7 can hold as an equality.

Let $X \subseteq A^m$, $Y \subseteq A^n$, $f: X \rightarrow B$, $g: Y \rightarrow B$ and $f \leq g$ via σ . Let $i \in \text{Ess } f$. Note that $\sigma^{-1}(i)$ is a nonempty set since $\text{Ess } f \subseteq \text{Im } \sigma$. Let \leq be a linear order of the set $\sigma^{-1}(i)$, say $\sigma^{-1}(i) = \{l_1, \dots, l_r\}$ with $l_1 < l_2 < \dots < l_r$. For each $i \in \text{Ess } f$ and a witness of essentiality (\mathbf{a}, \mathbf{b}) of the i -th argument in f , we define $C_{\leq}^{(\mathbf{a}, \mathbf{b})}(i) := \{\mathbf{c}^0, \mathbf{c}^1, \dots, \mathbf{c}^r\}$, where

- (i) $\mathbf{c}^0 := \mathbf{a}\sigma$,
- (ii) for each $j = 1, \dots, r$, let \mathbf{c}^j be the tuple obtained from \mathbf{c}^{j-1} by changing the l_j -th entry from a_i to b_i .

Lemma 2.8. *Let $X \subseteq A^m$, $Y \subseteq A^n$, $f: X \rightarrow B$, $g: Y \rightarrow B$ and $f \leq g$ via σ . For each $i \in \text{Ess } f$, if there exists a witness (\mathbf{a}, \mathbf{b}) of essentiality of the i -th argument of f and for some linear order \leq of $\sigma^{-1}(i)$ it holds that $C_{\leq}^{(\mathbf{a}, \mathbf{b})}(i) \subseteq Y$, then $\sigma^{-1}(i) \cap \text{Ess } g \neq \emptyset$.*

Proof. Let $i \in \text{Ess } f$. Then by Lemma 2.1(i), $\sigma^{-1}(i) \neq \emptyset$. Suppose that $\sigma^{-1}(i) = \{l_1, \dots, l_r\}$ with $l_1 < l_2 < \dots < l_r$. Let $\mathbf{a}, \mathbf{b} \in X$ be a witness of essentiality of the i -th argument in f . By our procedure, we have that $\mathbf{c}^0, \mathbf{c}^1, \dots, \mathbf{c}^r \in A^n$, $\mathbf{c}^0 = \mathbf{a}\sigma$ and $\mathbf{c}^r = \mathbf{b}\sigma$. By our assumption, we now know that $\mathbf{c}^0, \mathbf{c}^1, \dots, \mathbf{c}^r \in Y$. Then

$$g(\mathbf{a}\sigma) = f(\mathbf{a}) \neq f(\mathbf{b}) = g(\mathbf{b}\sigma).$$

That is, there exists $k \in \{0, \dots, r-1\}$ such that $g(\mathbf{c}^k) \neq g(\mathbf{c}^{k+1})$. This means that \mathbf{c}^k and \mathbf{c}^{k+1} witness the essentiality of the l_{k+1} -th argument in g . Thus, $l_{k+1} \in \text{Ess } g$. That is, $\sigma^{-1}(i) \cap \text{Ess } g \neq \emptyset$. \square

The following example shows that the converse of Lemma 2.8 does not hold.

Example 2.9. Let X and Y be sets given by

$$X = \{(0, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

and

$$Y = \{(0, 0, 0, 0, 0), (0, 1, 1, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 1), (0, 1, 0, 1, 1)\}.$$

Define $f: X \rightarrow \mathbb{B}$ and $g: Y \rightarrow \mathbb{B}$ by

	f		g
$(0, 0, 0)$	$\mapsto 0$	$(0, 0, 0, 0, 0)$	$\mapsto 0$
$(0, 1, 0)$	$\mapsto 1$	$(0, 1, 1, 1, 0)$	$\mapsto 1$
$(0, 0, 1)$	$\mapsto 1$	$(0, 0, 0, 0, 1)$	$\mapsto 1$
		$(0, 1, 0, 0, 1)$	$\mapsto 0$
		$(0, 1, 0, 1, 1)$	$\mapsto 1$.

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix}$. Now, $\sigma^{-1}(2) \cap \text{Ess } g = \{2, 3, 4\} \cap \{4, 5\} \neq \emptyset$ and $\sigma^{-1}(3) \cap \text{Ess } g = \{5\} \cap \{4, 5\} \neq \emptyset$. But for $2 \in \text{Ess } f$, $C_{\leq}^{((0,0,0),(0,1,0))}(2) \not\subseteq Y$ for every linear order \leq on $\sigma^{-1}(2)$.

Corollary 2.10. *Let $X \subseteq A^m$, $Y \subseteq A^n$, $f: X \rightarrow B$, $g: Y \rightarrow B$ and $f \leq g$ via σ . For each $i \in \text{Ess } f$, if there exists a witness (\mathbf{a}, \mathbf{b}) of essentiality of the i -th argument of f and for some linear order \leq of $\sigma^{-1}(i)$ it holds that $C_{\leq}^{(\mathbf{a}, \mathbf{b})}(i) \subseteq Y$, then $\text{ess } f \leq \text{ess } g$.*

Proof. Apply Lemma 2.1 and Lemma 2.8. □

3. Essential Sets of Arguments

Essential arguments do not seem to adequately describe the behaviour of partial functions. A partial function might be very complex yet it depends on very few arguments. In particular, in stark contrast to total functions, the essential arity of partial functions can increase when minors are formed and a non-constant function might be essentially nullary (see Fact 2.4 and Examples 2.2 and 2.5).

In this section, we define an alternative notion of “essentiality” that would better express the idea of dependence of a (partial) function on several arguments collectively, not just on a single argument.

Definition 3.1. Let $f: X \rightarrow B$ with $X \subseteq A^n$ be a partial function. A subset $S \subseteq [n]$ is *essential* in f if there exist tuples $\mathbf{a}, \mathbf{b} \in X$ such that $a_i = b_i$ for all $i \in [n] \setminus S$ and $f(\mathbf{a}) \neq f(\mathbf{b})$.

The pair (\mathbf{a}, \mathbf{b}) is called a *witness of essentiality* of S in f . We denote the collection of essential sets in f by $\text{ES}(f)$ and $\text{es } f := |\text{ES}(f)|$.

A set $M \in \text{ES}(f)$ is called a *minimal element* of $\text{ES}(f)$ if $N \subseteq M$ implies $M = N$ for all $N \in \text{ES}(f)$. We denote the set of all minimal elements of $\text{ES}(f)$ by $\text{MES}(f)$ and we set $\text{mes}(f) := |\text{MES}(f)|$. We can see that for every partial function f , $\text{ES}(f)$ is an upper set of $(\mathcal{P}([n]); \subseteq)$.

These notions seem reasonable because for total functions they coincide with the classical notion of essential arguments. More generally, for a partial function $f: X \rightarrow B$ with $X \subseteq A^n$ and $i \in [n]$ the following are equivalent:

- (1) argument i is essential in f .
- (2) $\{i\}$ is an essential set in f .
- (3) $\{i\}$ is a minimal essential set in f .

Example 3.2. Let $X = \{(0, 0, 0), (0, 0, 1), (1, 1, 0)\}$. Define $f: X \rightarrow \mathbb{B}$ by

$$\begin{array}{ccc} & f & \\ (0, 0, 0) & \mapsto & 0 \\ (0, 0, 1) & \mapsto & 1 \\ (1, 1, 0) & \mapsto & 1. \end{array}$$

Then $\{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ and $\{1, 2, 3\}$ are essential sets in f . Hence, $\text{MES}(f) = \{\{3\}, \{1, 2\}\}$ and $\text{mes}(f) = 2$.

Recall from Lemma 2.1 that for total functions f and g , $\text{ess } f \leq \text{ess } g$ whenever $f \leq g$; moreover, $\text{ess } f = \text{mes}(f)$ and $\text{ess } g = \text{mes}(g)$, that is, $\text{mes}(f) \leq \text{mes}(g)$ whenever $f \leq g$. In contrast, for partial functions if f is a minor of g , then it does not necessarily hold that $\text{mes}(f) \leq \text{mes}(g)$, as shown by the following example.

Example 3.3. Let X and Y be sets given by

$$X = \{(0, 0, 0, 0), (1, 1, 1, 0), (0, 1, 1, 1)\}$$

and

$$Y = \{(0, 0, 0, 0, 0), (0, 1, 0, 0, 0), (1, 1, 1, 1, 0), (0, 1, 1, 1, 1)\}.$$

Define $f: X \rightarrow \mathbb{B}$ and $g: Y \rightarrow \mathbb{B}$ by

$$\begin{array}{ccc} f & & g \\ (0, 0, 0, 0) & \mapsto & 0 & (0, 0, 0, 0, 0) & \mapsto & 0 \\ (1, 1, 1, 0) & \mapsto & 1 & (0, 1, 0, 0, 0) & \mapsto & 1 \\ (0, 1, 1, 1) & \mapsto & 1 & (1, 1, 1, 1, 0) & \mapsto & 1 \\ & & & (0, 1, 1, 1, 1) & \mapsto & 1. \end{array}$$

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 \end{pmatrix}$.

Now, we see that $\text{MES}(f) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$ and $\text{MES}(g) = \{\{2\}\}$, that is, $\text{mes}(f) = 2 > 1 = \text{mes}(g)$.

Lemma 3.4. *Let $X \subseteq A^m, Y \subseteq A^n, f: X \rightarrow B, g: Y \rightarrow B$ and $f \leq g$ via σ . We have that for all $S \subseteq [m]$, if $S \in \text{ES}(f)$, then $\sigma^{-1}(S) \in \text{ES}(g)$.*

Proof. Let $S \subseteq [m]$ be such that $S \in \text{ES}(f)$. Then there exist $\mathbf{a}, \mathbf{b} \in X$ such that (\mathbf{a}, \mathbf{b}) is a witness of essentiality of S in f . Let $\mathbf{c} := \mathbf{a}\sigma$ and $\mathbf{d} := \mathbf{b}\sigma$. Then

$$\begin{aligned} (a_{\sigma(1)}, \dots, a_{\sigma(n)}) &= (c_1, \dots, c_n), \\ (b_{\sigma(1)}, \dots, b_{\sigma(n)}) &= (d_1, \dots, d_n). \end{aligned}$$

Since $a_i = b_i$ for all $i \in [m] \setminus S$, we have that $a_{\sigma(i)} = b_{\sigma(i)}$ for all $i \in [n] \setminus \sigma^{-1}(S)$. Thus, $c_i = d_i$ for all $i \in [n] \setminus \sigma^{-1}(S)$; moreover,

$$g(\mathbf{c}) = g(\mathbf{a}\sigma) = f(\mathbf{a}) \neq f(\mathbf{b}) = g(\mathbf{b}\sigma) = g(\mathbf{d}).$$

We conclude that the pair (\mathbf{c}, \mathbf{d}) is a witness of essentiality of $\sigma^{-1}(S)$ in g . That is, $\sigma^{-1}(S) \in \text{ES}(g)$. □

We observe that the converse of Lemma 3.4 does not hold as shown by Example 3.3. In fact, if we let $S = \{2\}$, then $\sigma^{-1}(S) = \{2, 3\} \in \text{ES}(g)$ but $S \notin \text{ES}(f)$. On the other hand, if $S \in \text{ES}(g)$, then it is not necessarily true that $\sigma(S) \in \text{ES}(f)$ by Example 3.3. Moreover, if S is a minimal essential set in f , then $\sigma^{-1}(S)$ may not be a minimal essential set in g , as shown by the following example.

Example 3.5. Let X and Y be sets given by

$$X = \{(0, 0, 0), (1, 1, 0), (0, 1, 1)\}$$

and

$$Y = \{(0, 0, 0, 0), (1, 1, 1, 0), (0, 0, 1, 1), (1, 0, 0, 0)\}.$$

Define $f: X \rightarrow \mathbb{B}$ and $g: Y \rightarrow \mathbb{B}$ by

	f		g		
$(0, 0, 0)$	\mapsto	0	$(0, 0, 0, 0)$	\mapsto	0
$(1, 1, 0)$	\mapsto	1	$(1, 1, 1, 0)$	\mapsto	1
$(0, 1, 1)$	\mapsto	1	$(0, 0, 1, 1)$	\mapsto	1
			$(1, 0, 0, 0)$	\mapsto	1 .

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Now, we can see that $\text{MES}(f) = \{\{1, 2\}, \{2, 3\}\}$ and $\text{MES}(g) = \{\{1\}, \{3, 4\}\}$. It is clear that $\{1, 2\} \in \text{MES}(f)$ but $\sigma^{-1}(\{1, 2\}) = \{1, 2, 3\} \notin \text{MES}(g)$.

The above examples show that the number of minimal essential sets may increase or decrease upon formation of minors. It is not clear to us under which conditions this quantity is nonincreasing.

Open problem 1. Provide necessary and sufficient conditions for functions f and g with $f \leq g$ to satisfy $\text{mes}(f) \leq \text{mes}(g)$.

As a partial answer to the above problem, the following theorem provides a sufficient condition.

Theorem 3.6. *Let f and g be partial functions such that $f \leq g$ via σ . Assume that $\sigma^{-1}(S) \in \text{MES}(g)$ for all $S \in \text{MES}(f)$. Then $\text{mes}(f) \leq \text{mes}(g)$.*

Proof. The map $\sigma^{-1}: \text{MES}(f) \rightarrow \text{MES}(g)$ is clearly injective, which implies $\text{mes}(f) \leq \text{mes}(g)$. \square

Next, we introduce a particular kind of minor that forces the hypothesis of Theorem 3.6. Let $f: X \rightarrow B$ and $g: Y \rightarrow B$, where $X \subseteq A^m$ and $Y \subseteq A^n$, be partial functions. A partial function f is said to be a *surjective minor* of a partial function g if $f \leq g$ via σ for some $\sigma: [n] \rightarrow [m]$, and for every $\mathbf{a} \in Y$, there exists $\mathbf{b} \in X$ such that $\mathbf{b}\sigma = \mathbf{a}$; in other words, the map $\sigma|_X: X \rightarrow Y$ is surjective. We designate the fact that f is a surjective minor of g by $f \leq_s g$, or if we want to specify a map σ , we write $f \leq_s g$ via σ .

Example 3.7. Let X and Y be sets given by

$$X = \{(0,0), (0,1), (1,0)\} \quad \text{and} \quad Y = \{(0,0,0), (0,0,1), (1,1,0)\}.$$

Define $f: X \rightarrow \mathbb{B}$ and $g: Y \rightarrow \mathbb{B}$ by

$$\begin{array}{lll} & f & \\ (0,0) & \mapsto & 0 \\ (0,1) & \mapsto & 1 \\ (1,0) & \mapsto & 1 \end{array} \quad \begin{array}{lll} & g & \\ (0,0,0) & \mapsto & 0 \\ (0,0,1) & \mapsto & 1 \\ (1,1,0) & \mapsto & 1. \end{array}$$

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$. We consider $(0,0,0) = (0,0)\sigma$, $(0,0,1) = (0,1)\sigma$ and $(1,1,0) = (1,0)\sigma$. Thus, $f \leq_s g$ via σ .

Example 3.8. Let X and Y be sets given by

$$X = \{(0,0,0), (0,1,0), (0,0,1)\}$$

and

$$Y = \{(0,0,0,0), (0,1,1,0), (0,0,1,0), (0,0,0,1)\}.$$

Define $f: X \rightarrow \mathbb{B}$ and $g: Y \rightarrow \mathbb{B}$ by

$$\begin{array}{lll} & f & \\ (0,0,0) & \mapsto & 0 \\ (0,1,0) & \mapsto & 1 \\ (0,0,1) & \mapsto & 1 \end{array} \quad \begin{array}{lll} & g & \\ (0,0,0,0) & \mapsto & 0 \\ (0,1,1,0) & \mapsto & 1 \\ (0,0,1,0) & \mapsto & 1 \\ (0,0,0,1) & \mapsto & 1. \end{array}$$

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}$. Non-surjectivity is obvious for cardinality reasons. This shows that f is not a surjective minor of g .

Lemma 3.9. *Let f and g be partial functions such that $f \leq_s g$ via σ . If $S \in \text{ES}(g)$, then $\sigma(S) \in \text{ES}(f)$.*

Proof. Since $S \in \text{ES}(g)$, there exist $\mathbf{a}, \mathbf{b} \in Y$ such that (\mathbf{a}, \mathbf{b}) is a witness of essentiality of S in g . By assumption there exist $\mathbf{c}, \mathbf{d} \in X$ such that $\mathbf{c}\sigma = \mathbf{a}$ and $\mathbf{d}\sigma = \mathbf{b}$, that is,

$$\begin{aligned} (c_{\sigma(1)}, \dots, c_{\sigma(n)}) &= (a_1, \dots, a_n), \\ (d_{\sigma(1)}, \dots, d_{\sigma(n)}) &= (b_1, \dots, b_n). \end{aligned}$$

Since $a_i = b_i$ for all $i \in [n] \setminus S$, we have that $c_{\sigma(i)} = d_{\sigma(i)}$ for all $i \in [n] \setminus S$. This implies that $c_i = d_i$ for all $i \in [m] \setminus \sigma(S)$; moreover,

$$f(\mathbf{c}) = g(\mathbf{c}\sigma) = g(\mathbf{a}) \neq g(\mathbf{b}) = g(\mathbf{d}\sigma) = f(\mathbf{d}).$$

Therefore, (\mathbf{c}, \mathbf{d}) is a witness of essentiality of $\sigma(S)$ in f , that is, $\sigma(S) \in \text{ES}(f)$. \square

Proposition 3.10. *Let f and g be partial functions such that $f \leq_s g$ via σ . If $S \in \text{MES}(g)$, then $\sigma(S) \in \text{MES}(f)$.*

Proof. Let $S \in \text{MES}(g)$. Then by Lemma 3.9, $\sigma(S) \in \text{ES}(f)$. Suppose that there exists $K \in \text{MES}(f)$ such that $K \subseteq \sigma(S)$. Then by Lemma 3.4, $\sigma^{-1}(K) \in \text{ES}(g)$ and $\sigma^{-1}(K) \subseteq S$. By the minimality of S , we obtain that $\sigma^{-1}(K) = S$. This implies that $K = \sigma(S)$. Therefore, $\sigma(S) \in \text{MES}(f)$. \square

Proposition 3.11. *Let f and g be partial functions such that $f \leq_s g$ via σ . If $S \in \text{MES}(f)$, then $\sigma^{-1}(S) \in \text{MES}(g)$.*

Proof. Since $S \in \text{MES}(f)$, by Lemma 3.4 we have that $\sigma^{-1}(S) \in \text{ES}(g)$. Suppose that there exists $K \in \text{MES}(g)$ such that $K \subseteq \sigma^{-1}(S)$. Then $\sigma(K) \in \text{MES}(f)$ by Proposition 3.10 and $\sigma(K) \subseteq S$. By the minimality of S , we have that $\sigma(K) = S$. Thus, $K = \sigma^{-1}(S)$, that is, $\sigma^{-1}(S) \in \text{MES}(g)$. \square

Thus, the hypothesis of Theorem 3.6 holds whenever f is a surjective minor of g . In fact, in this case, an even stronger conclusion holds.

Theorem 3.12. *Let f and g be partial functions. If $f \leq_s g$ via σ , then $\text{mes}(f) = \text{mes}(g)$.*

Proof. It suffices to show that the map $\psi : \text{MES}(f) \rightarrow \text{MES}(g)$ given by $\psi(A) = \sigma^{-1}(A)$ for all $A \in \text{MES}(f)$ is bijective.

The map ψ is well defined by Proposition 3.11, and it is clearly injective. The surjectivity of ψ follows from Proposition 3.10. \square

4. The Width of the Poset of Essential Sets

In this section, we will consider our next variant of essential arity: the width of the poset of essential sets. Recall that a poset $(P; \leq)$ is called an *antichain* if for each distinct $x, y \in P$, we have $x \not\leq y$ and $y \not\leq x$. We call an antichain Q a *largest antichain* in P if Q is a partially ordered subset of P and $|R| \leq |Q|$ for every

antichain R in P . Let P be a poset. The *width* of P , denoted by $w(P)$, is the cardinality of the largest antichain in P (see [15]).

Let $f: X \rightarrow B$, where $X \subseteq A^n$, be a partial function. Then we can see that the poset $(\text{ES}(f); \subseteq)$ is a partially ordered subset of $(\mathcal{P}([n]); \subseteq)$. We are interested in $w(\text{ES}(f))$, the width of the poset of essential sets of f . Unfortunately, as shown by the following example, also this quantity may increase upon formation of minors.

Example 4.1. Let X and Y be sets given by

$$X = \{(0, 0, 0), (0, 0, 1), (1, 0, 0), (1, 0, 1)\}$$

and

$$Y = \{(0, 0, 0), (1, 1, 0)\}.$$

Define $f: X \rightarrow \mathbb{B}$ and $g: Y \rightarrow \mathbb{B}$ by

	f		g
$(0, 0, 0)$	\mapsto	0	$(0, 0, 0)$
$(0, 0, 1)$	\mapsto	0	$(1, 1, 0)$
$(1, 0, 0)$	\mapsto	1	\mapsto
$(1, 0, 1)$	\mapsto	1	1.

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$. We see that $\text{ES}(f) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ and $\text{ES}(g) = \{\{1, 2\}, \{1, 2, 3\}\}$. Then the largest antichain in $(\text{ES}(f); \subseteq)$ is $\{\{1, 2\}, \{1, 3\}\}$ and $\{\{1, 2, 3\}\}$ is a largest antichain in $(\text{ES}(g); \subseteq)$, that is, $w(\text{ES}(f)) = 2 > 1 = w(\text{ES}(g))$. We observe that $\mathcal{F} = \{\{1, 2\}, \{1, 3\}\}$ is an antichain in $(\text{ES}(f); \subseteq)$, but $\sigma^{-1}(\mathcal{F}) = \{\{1, 2\}, \{1, 2, 3\}\}$ is not an antichain in $(\text{ES}(g); \subseteq)$.

We will still provide a sufficient condition for functions f and g with $f \leq g$ to satisfy $w(\text{ES}(f)) \leq w(\text{ES}(g))$. For this, we are going to make a few observations about minors arising via nonsurjective maps σ . Note that the following two statements do not mean the same thing:

- (1) f is a minor of g via surjective σ ,
- (2) f is a surjective minor of g via σ .

(All combinations are possible: surjective minor via surjective σ (Example 2.2), nonsurjective minor via surjective σ (Example 3.8), surjective minor via nonsurjective σ (Example 4.1), nonsurjective minor via nonsurjective σ (modify Example 4.1 by adding $(0, 1, 1) \mapsto 1$ to the definition of g .)

Lemma 4.2. *Let $f \leq g$ via σ , where σ is surjective, and let \mathcal{S} be an antichain in $(\text{ES}(f); \subseteq)$. Then $\sigma^{-1}(\mathcal{S})$ is an antichain in $(\text{ES}(g); \subseteq)$. Moreover, $|\mathcal{S}| = |\sigma^{-1}(\mathcal{S})|$.*

Proof. First, we observe that $\sigma^{-1}(\mathcal{S}) \subseteq \text{ES}(g)$ by Lemma 3.4. Let K and L be distinct members of $\sigma^{-1}(\mathcal{S})$. Then $K = \sigma^{-1}(S)$ and $L = \sigma^{-1}(T)$ for some distinct $S, T \in \mathcal{S}$. Suppose that $K \subseteq L$. Then by the surjectivity of σ , $S = \sigma(\sigma^{-1}(S)) \subseteq$

$\sigma(\sigma^{-1}(T)) = T$. This is a contradiction to the fact that \mathcal{S} is an antichain in $(\text{ES}(f); \subseteq)$. Similarly, if $L \subseteq K$, then a contradiction arises for the same reason. Therefore, $\sigma^{-1}(\mathcal{S})$ is an antichain in $(\text{ES}(g); \subseteq)$. By the surjectivity of σ , we can define a bijective map $\psi: \mathcal{S} \rightarrow \sigma^{-1}(\mathcal{S})$ by $\psi(S) = \sigma^{-1}(S)$ for all $S \in \mathcal{S}$. Thus, $|\mathcal{S}| = |\sigma^{-1}(\mathcal{S})|$. \square

Corollary 4.3. *Let $f \leq g$ via σ , where σ is surjective. Then $(\text{ES}(g); \subseteq)$ has an antichain of cardinality equal to the cardinality of the largest antichain in $(\text{ES}(f); \subseteq)$.*

Proof. Let \mathcal{S} be a largest antichain in $(\text{ES}(f); \subseteq)$. Then by Lemma 4.2, $\sigma^{-1}(\mathcal{S})$ is an antichain in $(\text{ES}(g); \subseteq)$ and $|\mathcal{S}| = |\sigma^{-1}(\mathcal{S})|$. \square

Theorem 4.4. *If $f \leq g$ via σ , where σ is surjective, then $w(\text{ES}(f)) \leq w(\text{ES}(g))$.*

Proof. Let \mathcal{S} be the largest antichain in $(\text{ES}(f); \subseteq)$. We observe that for each antichain \mathcal{K} in $\text{ES}(g)$, $|\mathcal{K}| \leq |\mathcal{L}|$ where \mathcal{L} is a largest antichain in $(\text{ES}(g); \subseteq)$. Thus, by Corollary 4.3, $|\mathcal{S}| = |\sigma^{-1}(\mathcal{S})| \leq |\mathcal{T}|$ where \mathcal{T} is the largest antichain in $(\text{ES}(g); \subseteq)$. Therefore, $w(\text{ES}(f)) \leq w(\text{ES}(g))$. \square

5. Exact Essential Sets

We have seen in Sections 2, 3 and 4 that the number of essential arguments, the number of essential sets, the number of minimal essential sets, and the width of the poset of essential sets are perhaps not the most suitable notions for describing the structural complexity of partial functions, because they are neither monotone increasing nor decreasing with respect to the minor order. In this section we will introduce a variant of essential arity which is somewhat better in that it does not increase upon formation of minors.

Definition 5.1. Let $f: X \rightarrow B$, where $X \subseteq A^n$, be a partial function, and let S be a nonempty subset of $[n]$. A witness (\mathbf{a}, \mathbf{b}) of essentiality of S in f is called an *exact witness (of essentiality)* of S in f if (\mathbf{a}, \mathbf{b}) is not a witness of essentiality of $S \setminus \{k\}$ in f for any $k \in S$.

Example 5.2. Consider the following partial function f .

$$\begin{array}{rcl} & f & \\ (0, 0, 0) & \mapsto & 0 \\ (1, 1, 0) & \mapsto & 1 \end{array}$$

We see that $\text{ES}(f) = \{\{1, 2\}, \{1, 2, 3\}\}$, and the pair $((0, 0, 0), (1, 1, 0))$ is a witness of essentiality of $\{1, 2\}$ and $\{1, 2, 3\}$ in f . It is easy to see that $((0, 0, 0), (1, 1, 0))$ is not a witness of essentiality of $\{1, 2\} \setminus \{1\} = \{2\}$ and $\{1, 2\} \setminus \{2\} = \{1\}$ in f . Thus, $((0, 0, 0), (1, 1, 0))$ is an exact witness of essentiality of $\{1, 2\}$ in f . But, $((0, 0, 0), (1, 1, 0))$ is not an exact witness of essentiality of $\{1, 2, 3\}$ in f since $((0, 0, 0), (1, 1, 0))$ is a witness of essentiality of $\{1, 2, 3\} \setminus \{3\} = \{1, 2\}$ in f .

The following proposition provides a possibly clearer, equivalent condition describing exact witnesses of essentiality.

Proposition 5.3. *Let $f: X \rightarrow B$, where $X \subseteq A^n$, be a partial function. Assume that the pair (\mathbf{a}, \mathbf{b}) is a witness of essentiality of a nonempty set $S \subseteq [n]$ in f . Then the following statements are equivalent:*

- (i) (\mathbf{a}, \mathbf{b}) is not a witness of essentiality of $S \setminus \{k\}$ in f for any $k \in S$,
- (ii) for all $i \in [n]$, $a_i = b_i$ if and only if $i \notin S$.

Proof. (i) \Rightarrow (ii): Let $i \in [n]$. Suppose that $a_i = b_i$. If $i \in S$, then (\mathbf{a}, \mathbf{b}) is a witness of essentiality of $S \setminus \{i\}$ in f . This contradicts our presumption; hence $i \notin S$. On the other hand, if $i \notin S$, then it is clear that $a_i = b_i$ by the fact that (\mathbf{a}, \mathbf{b}) is a witness of essentiality of S in f .

(ii) \Rightarrow (i): Suppose, to the contrary, that there exists $k \in S$ such that (\mathbf{a}, \mathbf{b}) is a witness of essentiality of $S \setminus \{k\}$ in f . This implies that $a_k = b_k$. But this means that $k \notin S$. This is a contradiction to our assumption. Therefore, we have that (\mathbf{a}, \mathbf{b}) is a witness of essentiality of $S \setminus \{k\}$ in f . \square

Let $\mathbf{a} := (a_1, \dots, a_n) \in A^n$. Let $i \in [n]$ and $b \in A$. We denote the tuple

$$(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \in A^n$$

by $\mathbf{a}(i \leftarrow b)$.

Let $f: X \rightarrow B$, where $X \subseteq A^n$, be a partial function. The i -th argument is *fictitious* in f if for all $\mathbf{a} \in A^n$, one of the following conditions holds:

- (i) $\mathbf{a}(i \leftarrow b) \notin X$ for all $b \in A$,
- (ii) $\mathbf{a}(i \leftarrow b) \in X$ for all $b \in A$ and $f(\mathbf{a}(i \leftarrow b)) = f(\mathbf{a}(i \leftarrow c))$ for all $b, c \in A$.

Denote the set of all fictitious arguments of f by $\text{Fic } f$.

Remark 5.4. In a total function, an argument is fictitious if and only if it is not essential. For partial functions this equivalence does not hold in general; an argument may be essential, fictitious, or neither of the two, but not both essential and fictitious.

Let $f: X \rightarrow B$ and $g: Y \rightarrow B$, where $X \subseteq A^m$ and $Y \subseteq A^n$, be partial functions. We observe that if $f \leq g$ via σ , then $[m] \setminus \text{Im } \sigma \subseteq \text{Fic } f$.

Example 5.5. Consider the following partial function f .

$$\begin{array}{rcl} & f & \\ (0, 0, 0) & \mapsto & 0 \\ (0, 0, 1) & \mapsto & 0 \\ (1, 1, 0) & \mapsto & 1 \\ (1, 1, 1) & \mapsto & 1 \end{array}$$

We see that the 3rd argument is fictitious in f since $(0, 1, 0), (0, 1, 1), (1, 0, 0)$ and $(1, 0, 1)$ are not in the domain, and moreover $f(0, 0, 0) = f(0, 0, 1)$ and $f(1, 1, 0) = f(1, 1, 1)$.

Definition 5.6. Let $f: X \rightarrow B$, where $X \subseteq A^n$, be a partial function. A subset $S \subseteq [n]$ is called an *exact essential set* in f if S is the empty set, or S does not contain any fictitious argument of f and there exists a pair (\mathbf{a}, \mathbf{b}) , where $\mathbf{a}, \mathbf{b} \in A^n$, such that (\mathbf{a}, \mathbf{b}) is an exact witness of essentiality of S in f . We denote the set of all exact essential sets in f by $\text{XES}(f)$ and we set $\text{xes } f := |\text{XES}(f)|$.

Remark 5.7. We observe that $\text{XES}(f) \neq \emptyset$ since $\emptyset \in \text{XES}(f)$, every minimal essential set in f is an exact essential set in f , and every exact essential set in f is an essential set in f , but the converse is not true by Example 5.2.

We note that if S is an exact essential set in a total function f , then every element of S is an essential argument by Definition 5.6 and Remark 5.4.

Since the number of essential sets of an n -ary function can be as large as 2^n , a number exponential to the arity, we find it convenient to rescale this quantity logarithmically so as to stay in the same order of magnitude as the arity.

Definition 5.8. The *exact essential hyperarity* of a partial function f is defined as $\beta f := \log_2 \text{xes } f$.

Example 5.9. Consider the partial function f of Example 5.5. We see that $\{1, 2\}$ is an exact essential set in f since $\{1, 2\}$ does not contain any fictitious argument of f , and the pair $((0, 0, 0), (1, 1, 0))$ is an exact witness of essentiality of $\{1, 2\}$ in f . However, $\{1, 2, 3\}$ is not an exact essential set in f since it contains a fictitious argument of f .

In general, if $f \leq g$ via σ and S is a nonempty exact essential set in f , then $\sigma^{-1}(S)$ is not necessarily an exact essential set in g .

Example 5.10. Consider the following partial functions.

f		g			
$(0, 0)$	\mapsto	0	$(0, 0, 0)$	\mapsto	0
$(1, 1)$	\mapsto	1	$(0, 0, 1)$	\mapsto	0
			$(1, 1, 0)$	\mapsto	1
			$(1, 1, 1)$	\mapsto	1

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$. We see that $S = \{1, 2\} \in \text{XES}(f)$ but $\sigma^{-1}(S) = \{1, 2, 3\} \notin \text{XES}(g)$. Observe that the 3rd argument is fictitious in g and $\sigma^{-1}(S) \setminus \{3\} \in \text{XES}(g)$.

Theorem 5.11. Let $f \leq g$ via σ . If S is a nonempty exact essential set in f , then $\sigma^{-1}(S) \setminus \text{Fic } g$ is an exact essential set in g .

Proof. Assume that f is m -ary and g is n -ary. Denote by X and Y the domain of f and the domain of g , respectively. Since S is an exact essential set in f , there exist tuples $\mathbf{a}, \mathbf{b} \in X$ such that the pair (\mathbf{a}, \mathbf{b}) is an exact witness of essentiality of S in f ; this means, by Proposition 5.3, that $f(\mathbf{a}) \neq f(\mathbf{b})$ and for all $i \in [m]$, $a_i = b_i$ if and only if $i \notin S$. Without loss of generality, assume that $\text{Fic } g = [\ell]$ for some

$\ell \in \mathbb{N}$. Let $\mathbf{u} := (b_{\sigma(1)}, \dots, b_{\sigma(\ell)}, a_{\sigma(\ell+1)}, \dots, a_{\sigma(n)})$. By the definition of fictitious arguments, we have $\mathbf{u} \in Y$ and $f(\mathbf{a}\sigma) = f(\mathbf{u})$; hence, the pair $(\mathbf{u}, \mathbf{b}\sigma)$ is a witness of essentiality of $\sigma^{-1}(S) \setminus \text{Fic } g$ in g .

It remains to show that $(\mathbf{u}, \mathbf{b}\sigma)$ is an exact witness. Let $k \in [n]$. Consider first the case where $k \in \sigma^{-1}(S) \setminus \text{Fic } g$. Then $\sigma(k) \in S$, so $a_{\sigma(k)} \neq b_{\sigma(k)}$ because (\mathbf{a}, \mathbf{b}) is an exact witness of S in f ; therefore $(\mathbf{a}\sigma)(k) = a_{\sigma(k)} \neq b_{\sigma(k)} = (\mathbf{b}\sigma)(k) = \mathbf{u}(k)$. Consider then the case where $k \notin \sigma^{-1}(S) \setminus \text{Fic } g$. If $k \in \text{Fic } g$, then $\mathbf{u}(k) = b_{\sigma(k)} = (\mathbf{b}\sigma)(k)$. Otherwise ($k \notin \sigma^{-1}(S)$ and $k \notin \text{Fic } g$) we have $\sigma(k) \notin S$, so $\mathbf{u}(k) = a_{\sigma(k)} = b_{\sigma(k)} = (\mathbf{b}\sigma)(k)$. It now follows from Proposition 5.3 that the pair $(\mathbf{u}, \mathbf{b}\sigma)$ is an exact witness of essentiality of $\sigma^{-1}(S) \setminus \text{Fic } g$ in g . \square

Corollary 5.12. *Let $f \leq g$ via σ . Then $\text{xes } f \leq \text{xes } g$. Equivalently, $\beta f \leq \beta g$.*

Remark 5.13. Let $f: X \rightarrow B$ be a partial function, where $X \subseteq A^n$. We can conclude immediately from definitions the following:

- (i) $i \in \text{Ess } f$ if and only if $\{i\} \in \text{XES}(f)$.
- (ii) $\text{ess } f \leq \text{xes } f$.
- (iii) f is constant if and only if $\text{xes } f = 1$, or, equivalently, $\beta f = 0$.

Proof. $\text{xes } f = 1 \Leftrightarrow$ The empty set is the only exact essential set in f .
 \Leftrightarrow For every nonempty subset $S \subseteq [n]$, there is no exact witness of essentiality of S in f .
 $\Leftrightarrow f(\mathbf{a}) = f(\mathbf{b})$ for any $\mathbf{a}, \mathbf{b} \in X$.
 $\Leftrightarrow f$ is constant. \square

- (iv) $\text{xes } f \leq |\text{ES}(f)|$.
- (v) $\text{mes}(f) \leq \text{xes } f$.

We observe that if f and g are total functions such that $f \leq g$ via σ , then for every $i \in \text{Ess } f$, $\sigma^{-1}(\{i\}) \setminus \text{Fic } g = \sigma^{-1}(i) \cap \text{Ess } g \neq \emptyset$.

Lemma 5.14. *Let $f: A^m \rightarrow B$, $g: A^n \rightarrow B$ be total functions, and assume that $f \leq g$ via σ . Then the mapping $\varphi: \text{XES}(f) \setminus \{\emptyset\} \rightarrow \text{XES}(g) \setminus \{\emptyset\}$ defined by $\varphi(S) = \sigma^{-1}(S) \cap \text{Ess } g$ is injective.*

Proof. By Theorem 5.11, it is clear that φ is well defined. Let $S, S' \in \text{XES}(f)$ such that $S \neq S'$. It is clear that $S, S' \subseteq \text{Ess } f$. Without loss of generality, assume that $i \in S$ and $i \notin S'$. By Lemma 2.1(ii), $\sigma^{-1}(i) \cap \text{Ess } g \neq \emptyset$. Then there exists $i' \in \sigma^{-1}(i) \cap \text{Ess } g$. We have $i' \in \sigma^{-1}(S) \cap \text{Ess } g$ and $i' \notin \sigma^{-1}(S') \cap \text{Ess } g$, which shows that φ is injective. \square

As a consequence, an analogue of Lemma 2.3 is true for the exact essential hyperarity of total functions.

Proposition 5.15. *Let $f: A^m \rightarrow B$, $g: A^n \rightarrow B$ be total functions, and assume that $f \leq g$ via σ . Then we have that $f \equiv g$ if and only if $\text{xes } f = \text{xes } g$ (equivalently,*

$\mathfrak{B} f = \mathfrak{B} g$).

Proof. By Corollary 5.12, $f \equiv g$ implies $\text{xes } f = \text{xes } g$. For the converse implication, assume $\text{xes } f = \text{xes } g$. By Lemma 5.14, the map $\varphi: \text{XES}(f) \setminus \{\emptyset\} \rightarrow \text{XES}(g) \setminus \{\emptyset\}$, $\varphi(S) = \sigma^{-1}(S) \cap \text{Ess } g$ is injective, and it is, in fact, bijective because $\text{xes } f = \text{xes } g$. Suppose, to the contrary, that $f \not\equiv g$. Then $f < g$, so $\text{ess } f < \text{ess } g$ by Fact 2.4(ii). In view of Remark 5.13(i), there exists $i \in \text{Ess } g$ (i.e., $\{i\} \in \text{XES}(g)$) such that $S := \varphi^{-1}(\{i\})$ has at least two elements, say $\{a, b\} \subseteq S$, $a \neq b$. We have $\{a, b\} \subseteq \text{Ess } f$ by Remark 5.7; hence $\sigma^{-1}(a) \cap \text{Ess } g \neq \emptyset$ and $\sigma^{-1}(b) \cap \text{Ess } g \neq \emptyset$ by Lemma 2.1(ii), so there exist $a' \in \sigma^{-1}(a) \cap \text{Ess } g$ and $b' \in \sigma^{-1}(b) \cap \text{Ess } g$. Since $\sigma^{-1}(a) \cap \sigma^{-1}(b) = \emptyset$, we have $a' \neq b'$. Consequently,

$$\{a', b'\} \subseteq \sigma^{-1}(\{a, b\}) \cap \text{Ess } g \subseteq \sigma^{-1}(S) \cap \text{Ess } g = \varphi(S) = \{i\},$$

which yields the desired contradiction, and we conclude that $f \equiv g$. □

The following example shows that Proposition 5.15 does not hold for partial functions in general.

Example 5.16. Consider the following partial functions.

	f		g		
$(0, 0)$	\mapsto	0	$(0, 0, 0)$	\mapsto	0
$(0, 1)$	\mapsto	1	$(0, 0, 1)$	\mapsto	1
$(1, 1)$	\mapsto	1	$(1, 1, 1)$	\mapsto	1

Then $f \leq g$ via $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$. We see that $\text{xes } f = 2 = \text{xes } g$ but g is not a minor of f . Hence $f \not\equiv g$.

6. Exact Hyperarity Gap

As pointed out in Fact 2.4(ii), the essential arity of a total function decreases upon formation of minors. One may ask how large this decrease of essential arity must be for a given function. In this way one arrives at the notion of the arity gap of a function. This question was first considered by Salomaa [14].

Let $f: A^n \rightarrow B$ be a function that depends on all of its variables, $n \geq 2$. The *arity gap* of f is defined by the following

$$\text{gap } f := \min_{I \in \binom{[n]}{2}} \text{ess } f - \text{ess } f_I.$$

The arity gap of a total function is clearly at least 1. Salomaa showed that the arity gap can be as large as $|A| = k$, where $k \geq 2$, by providing an example of a function $f: A^k \rightarrow A$ such that $\text{ess } f = k$ and $\text{ess } f_I = 0$ for every $I \in \binom{[k]}{2}$. He also showed that the arity gap of any Boolean function is at most 2 (see [14]).

Later, Willard [17] extended this result to functions on arbitrary finite domains. He showed that the arity gap of any function $f: A^n \rightarrow B$, where $k = |A|$ and $n > k$, is at most 2.

Partial functions may have a negative arity gap. By Proposition 2.7,

$$\text{gap } f = \min_{I \in \binom{[n]}{2}} \text{ess } f - \text{ess } f_I \geq \text{ess } f - (\text{ess } f + 1) = -1.$$

See Example 2.2 for an example of a function f with $\text{gap } f = -1$.

The notion of exact essential hyperarity suggests an analogue of arity gap. Let $f: X \rightarrow B$ be a partial function, where $X \subseteq A^n$. Define the *exact hyperarity gap* of f by

$$\text{xgap } f := \min_{I \in \binom{[n]}{2}} \beta f - \beta f_I.$$

Obviously, for any partial function $f: X \rightarrow B$, where $X \subseteq A^n$, we have that $0 \leq \text{xgap } f \leq n$.

By Example 5.16, we see that the exact hyperarity gap of a partial function can be as small as 0. The following example shows that the exact hyperarity gap can be as large as the arity n .

Example 6.1. Let $n \geq 4$ and $A := [k]$ with $k \geq n - 1$. Define the sets

$$X_0 = \{(1, 2, \dots, n - 1)\delta_{\{l, l+1\}} : 1 \leq l \leq n - 1\}$$

and

$$X_1 = \{\mathbf{a} \in A^n : \mathbf{a}(l) \neq \mathbf{a}(l + 1) \text{ for all } 1 \leq l \leq n - 1\}.$$

Define a partial function $f: X_0 \cup X_1 \rightarrow \mathbb{B}$ where $f^{-1}(0) = X_0$ and $f^{-1}(1) = X_1$. Then $\text{xes } f = 2^n$ and $\text{xes } f_I = 1$ for all $I \in \binom{[n]}{2}$. Thus,

$$\text{xgap } f = \min_{I \in \binom{[n]}{2}} \beta f - \beta f_I = n.$$

Example 6.2. Let $n \geq 2$. Define a Boolean (total) function $f: \mathbb{B}^n \rightarrow \mathbb{B}$ by

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\text{xes } f = 2^n$ and $\text{xes } f_I = 2^{n-1}$ for all $I \in \binom{[n]}{2}$. Thus,

$$\text{xgap } f = \min_{I \in \binom{[n]}{2}} \beta f - \beta f_I = 1.$$

Definition 6.3.([1]) Let $\text{oddsupp}: \bigcup_{n \geq 1} A^n \rightarrow \mathcal{P}(A)$ be the mapping defined by

$$\text{oddsupp}(a_1, \dots, a_n) = \{a \in A : |\{i \in [n] : a_j = a\}| \text{ is odd}\}.$$

A function $f: A^n \rightarrow B$ is *determined by* oddsupp if there exists a mapping $f^*: \mathcal{P}(A) \rightarrow B$ such that $f = f^* \circ \text{oddsupp}|_{A^n}$.

Example 6.4. Suppose that $\{0, 1, 2\} \subseteq A$ and $\mathbb{B} \subseteq B$. Define a function $f^*: \mathcal{P}(A) \rightarrow B$ by

$$f^*(X) = \begin{cases} 0 & \text{if } X = \emptyset \text{ or } X = \{0\}, \\ 1 & \text{otherwise.} \end{cases}$$

Let $f: A^n \rightarrow B$ be a function defined by $f := f^* \circ \text{oddsupp}|_{A^n}$. Let S be a nonempty subset of $[n]$. We must consider two different cases according to the parity of $|S|$.

- (1) If $|S|$ is odd, then the pair $(\mathbf{0}, \mathbf{a})$ is an exact witness of essentiality of S in f , where $\mathbf{0} = (0, \dots, 0) \in A^n$ and $\mathbf{a} \in A^n$ is defined by

$$\mathbf{a}(i) = \begin{cases} 0 & \text{if } i \notin S, \\ 1 & \text{if } i \in S. \end{cases}$$

- (2) If $|S|$ is even, then S is a disjoint union of odd subsets S_1 and S_2 of S , and the pair $(\mathbf{0}, \mathbf{b})$ is an exact witness of essentiality of S in f , where $\mathbf{0} = (0, \dots, 0) \in A^n$ and $\mathbf{b} \in A^n$ is defined by

$$\mathbf{b}(i) = \begin{cases} 0 & \text{if } i \notin S, \\ 1 & \text{if } i \in S_1, \\ 2 & \text{if } i \in S_2. \end{cases}$$

We conclude that every subset of $[n]$ is an exact essential set in f , that is, $\text{xes } f = 2^n$. For any $I \in \binom{[n]}{2}$, the argument $\delta_I(I)$ is fictitious in f_I and f_I is equivalent to $f^* \circ \text{oddsupp}|_{A^{n-2}}$. Thus, every subset of $[n-1]$ which does not contain $\delta_I(I)$ is an exact essential set in f_I , that is, $\text{xes } f_I = 2^{n-2}$ for all $I \in \binom{[n]}{2}$. Thus,

$$\text{xgap } f = \min_{I \in \binom{[n]}{2}} \beta f - \beta f_I = 2.$$

Willard’s upper bound for the arity gap can be stated as follows.

Theorem 6.5.([17, Lemma 1.2, Corollary 2.3]) *Let A and B be arbitrary finite nonempty sets, and let $k = |A|$. Let $f: A^n \rightarrow B$, and assume that $\text{ess } f = n$. If $n > k$, then $\text{gap } f \leq 2$. Moreover, if $n > \max(k, 3)$, then $\text{gap } f = 2$ if and only if f is determined by oddsupp .*

The analogue of the above for the exact hyperarity gap of partial functions does not hold. The condition $\text{ess } f = n$ means that f has no fictitious arguments. As shown by the following example, there exist partial functions f without fictitious arguments with $\text{xgap } f > 2$.

attributes \ variants	ess f	es f	mes(f)	$w(\text{ES}(f))$	βf
(1)	–	–	–	–	✓
(2)	–	✓	✓	✓	✓
(3)	✓	–	✓	–	–

Table 1: Variants of essential arity and their attributes. See Section 7 for a list of the attributes.

Example 6.6. Let $k \in \mathbb{N}$ with $k \geq 2$, let $A := [k]$, and assume $\mathbb{B} \subseteq B$. Let $\mathbf{a} := (1, 2, \dots, k)$ and

$$\mathbf{u} := (\underbrace{1, \dots, 1}_k, \underbrace{2, \dots, 2}_k, \dots, \underbrace{k, \dots, k}_k) \in A^{k^2}.$$

Let $X_0 := \{\mathbf{u}\}$ and

$$X_1 := \{(\mathbf{a}\sigma_1, \mathbf{a}\sigma_2, \dots, \mathbf{a}\sigma_k) \in A^{k^2} \mid \sigma_1, \sigma_2, \dots, \sigma_k \text{ permutations of } [k]\}.$$

Then X_0 and X_1 are disjoint subsets of A^{k^2} , and we can define a partial function $f: X_0 \cup X_1 \rightarrow B$ by $f^{-1}(0) = X_0$ and $f^{-1}(1) = X_1$. It is easy to see that the nonempty (exact) essential sets of f are precisely the subsets S of $[k^2]$ satisfying $|S \cap \{ik + 1, ik + 2, \dots, ik + k\}| = k - 1$ for all $0 \leq i \leq k - 1$. It follows that $\beta f = \log_2(k^k + 1)$ and $\beta f_I = 0$ for all $I \in \binom{[k^2]}{k}$; thus, $\text{xgap } f = \log_2(k^k + 1) > 2$.

Open problem 2. Provide a (sharp and nontrivial) upper bound for the exact hyperarity gap of n -ary functions in \mathcal{P}_{AB} . (A trivial upper bound for an n -ary function is n , but is this sharp?)

7. Concluding Remarks

We have defined some variants of essential arity for partial functions: the number of essential sets, the number of minimal essential sets, the width of the poset of essential sets, and the exact essential hyperarity. For these measures of structural complexity of a function, the following attributes are desirable:

- (1) isotonicity with respect to the minor order,
- (2) equality to 0 precisely for constant functions,
- (3) coincidence with essential arity for total functions.

Our investigations show that only the exact essential hyperarity is isotone with respect to the minor order and that all measures but the essential arity are equal to 0 precisely for constant functions. The number of minimal essential sets coincides with essential arity for total functions; the other measures do not. This is summarized in Table 1.

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