KYUNGPOOK Math. J. 60(2020), 455-465
https://doi.org/10.5666/KMJ.2020.60.3.455
pISSN 1225-6951 eISSN 0454-8124
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Purities of Ordered Ideals of Ordered Semirings

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Abstract. We introduce the concepts of the left purity, right purity, quasi-purity, bi-purity, left weak purity and right weak purity of ordered ideals of ordered semirings and use them to characterize regular ordered semirings, left weakly regular ordered semirings, right weakly regular ordered semirings and fully idempotent ordered semirings.

1. Introduction

In 1989, Ahsan and Takahashi [2] introduced the notions of pure ideals and purely prime ideals of semigroups. Later, Changphas and Sanborisoot [3] defined the notions of left pure, right pure, left weakly pure and right weakly pure ideals in ordered semigroups and gave some of their characterizations. In 2007, Shabir and Iqbal [10] studied the concepts of pure ideals on semirings and characterized left and right weakly regular semirings using their pure ideals. As a special case of pure ideals of semirings, Jagatap [5] defined left pure k-ideals and right pure k-ideals of Γ-semirings and use them to characterize left and right weakly k-regular Γ-semirings, respectively. In 2017, Senarat and Pibaljommee [9] characterized left and right weakly ordered k-regular semirings using left pure and right pure ordered k-ideals, respectively.

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Received October 30, 2019; revised April 2, 2020; accepted April 21, 2020.
2020 Mathematics Subject Classification: 16Y60, 06F25.
Key words and phrases: pure ordered ideal, ordered semiring, regular ordered semiring.
This work was supported by Faculty of Science, Khon Kaen University, Khon Kaen, Thailand.

An ordered semiring defined by Gan and Jiang [4] is a generalization of a semiring; indeed, it is a semiring together with a partially ordered relation connected by the compatibility property. In this work, we present the notions of left pure, right pure, quasi-pure and bi-pure ordered ideals of ordered semirings, investigate some of their properties and characterize regular, left weakly regular and right weakly regular ordered semirings by their pure ordered ideals. Furthermore, we introduce the notions of left and right weakly pure ordered ideals and use them to characterize fully idempotent ordered semirings.

2. Preliminaries

An *ordered semiring* [4] $(S,+,\cdot,\leq)$ is a semiring $(S,+,\cdot)$ together with a binary relation $\leq$ on S such that $(S,\leq)$ is a poset satisfying the following condition; if $a\leq b$, then $a+c\leq b+c$, $c+a\leq c+b$, $ac\leq bc$ and $ca\leq cb$ for all $a,b,c\in S$. An element 0 of an ordered semiring S is called an *absorbing zero* [1] if $0+x=x=x+0$ and $0x=0=x0$ for all $x\in S$. Throughout this work, *we always assume that S is an additively commutative ordered semiring with an absorbing zero* 0, i.e., $a+b=b+a$ for all $a,b\in S$.

For nonempty subsets A and B of S, we denote that

$$\begin{aligned} A+B &= \{a+b \mid a\in A, b\in B\},\\ AB &= \{ab \mid a\in A, b\in B\},\\ \Sigma A &= \left\{\sum_{i\in I} a_i \mid a_i\in A \text{ and } I \text{ is a finite subset of } \mathbb{N}\right\} \text{ and}\\ (A] &= \{x\in S \mid x\leq a \text{ for some } a\in A\}. \end{aligned}$$

In a particular case of $a\in S$, we write Σa and $(a]$ instead of $\Sigma\{a\}$ and $(\{a\}]$, respectively. If $I=\emptyset$, we set $\sum_{i\in I} a_i = 0$ for all $a_i\in S$.

For the basic properties of the finite sums Σ and the operator ( ] of nonempty subsets of ordered semirings, we refer to [7, 8].

A nonempty subset A of an ordered semiring S such that $A+A\subseteq A$ is called a *left* (resp. *right*) *ordered ideal* if $SA\subseteq A$ (resp. $AS\subseteq A$) and $A=(A]$. If A is both a left and a right ordered ideal of S, then A is called an *ordered ideal* [4] of S. A nonempty subset Q of S such that $Q+Q\subseteq Q$ is called an *ordered quasi-ideal* [7] of S if $(\Sigma QS]\cap(\Sigma SQ]\subseteq Q$. A subsemiring B of S (i.e., $B+B\subseteq B$ and $B^2\subseteq B$) is called an *ordered bi-ideal* [7] if $BSB\subseteq B$.

For an element a of an ordered semiring S, we denote $L(a)$, $R(a)$, $J(a)$, $Q(a)$ and $B(a)$ to be the left ordered ideal, right ordered ideal, ordered ideal, ordered quasi-ideal and ordered bi-ideal of S generated by a, respectively. We recall their constructions which occur in [7] as follows.

Lemma 2.1. *Let $\emptyset\neq A\subseteq S$. Then*

(i) $L(a) = (\Sigma a + Sa]$;

(ii) $R(a) = (\Sigma a + aS]$;

(iii) $J(a) = (\Sigma a + Sa + aS + \Sigma SaS]$;

(iv) $Q(a) = (\Sigma a + ((aS] \cap (Sa])]$;

(v) $B(a) = (\Sigma a + \Sigma a^2 + aSa]$.

An ordered semiring S is called *regular* [6] if $a \in (aSa]$ for all $a \in S$ (i.e., for each $a \in S$, $a \leq axa$ for some $x \in S$). If for all $a \in S$, $a \in (\Sigma SaSa]$ (resp. $a \in (\Sigma aSaS]$), then S is called *left weakly regular* (resp. *right weakly regular*) [5].

3. Main Results

In this section, we present the notions of left pure, right pure, quasi-pure, bi-pure, left weakly pure and right weakly pure ordered ideals of ordered semirings and use them to characterize regular, left weakly regular, right weakly regular and fully idempotent ordered semirings.

Definition 3.1. An ordered ideal A of an ordered semiring S is called *left pure* (resp. *right pure*) if $x \in (Ax]$ (resp. $x \in (xA]$) for all $x \in A$.

Theorem 3.2. *Let A be an ordered ideal of an ordered semiring S. Then the following statements hold:*

(i) *A is left pure if and only if $A \cap L = (AL]$ for every left ordered ideal L of S;*

(ii) *A is right pure if and only if $R \cap A = (RA]$ for every right ordered ideal R of S.*

Proof. (i) Assume that A is left pure. Let L be a left ordered ideal of S. If $x \in A \cap L$, then $x \in (Ax] \subseteq (AL]$. So, $A \cap L \subseteq (AL]$. Clearly, $(AL] \subseteq A \cap L$. Hence, $A \cap L = (AL]$.

Conversely, let $x \in A$. Using assumption and Lemma 2.1, we get

$$\begin{aligned} x \in A \cap L(x) = (AL(x)] = (A(\Sigma x + Sx]] &\subseteq ((\Sigma Ax + ASx]] \\ &\subseteq (Ax + Ax] \subseteq (Ax]. \end{aligned}$$

Hence, A is a left pure ordered ideal of S.

(ii) It can be proved similarly. □

Definition 3.3. An ordered ideal A of an ordered semiring S is called *quasi-pure* if $x \in (xA] \cap (Ax]$ for all $x \in A$.

The following remark is directly obtained by Definition 3.1 and 3.3.

Remark 3.4 Every quasi-pure ordered ideal of an ordered semiring is both left pure and right pure.

Now, we give an example of a left pure ordered ideal of an ordered semiring which is not quasi-pure.

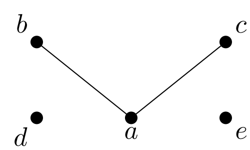
Example 3.5. Let $S = \{a, b, c, d, e\}$. Define binary operations $+$ and $\cdot$ on S by the following tables;

$+$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	b
c	a	a	a	a	c
d	a	a	a	a	d
e	a	b	c	d	e

and

$\cdot$	a	b	c	d	e
a	a	a	a	a	e
b	a	b	c	a	e
c	a	a	a	a	e
d	a	a	a	a	e
e	e	e	e	e	e

Define a binary relation $\leq$ of S by $\leq\ =: \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c)\}$.



Then $(S, +, \cdot, \leq)$ is an ordered semiring. We have that $A = \{a, b, c, e\}$ is a left pure ordered ideal of S. However, A is not right pure because $c \notin \{a, e\} = (cA]$. So, $c \notin (cA] \cap (Ac]$ and thus A is not quasi-pure.

As a duality of Example 3.5, we now give an example of a right pure ordered ideal of an ordered semiring which is not quasi-pure.

Example 3.6. Let $S = \{a, b, c, d, e\}$ together with the operation $+$ and the relation $\leq$ defined in Example 3.5. Define a binary operation $\cdot$ on S by the following table;

$\cdot$	a	b	c	d	e
a	a	a	a	a	e
b	a	b	a	a	e
c	a	c	a	a	e
d	a	a	a	a	e
e	e	e	e	e	e

Then $(S, +, \cdot, \leq)$ is an ordered semiring. We have that $A = \{a, b, c, e\}$ is a right pure ordered ideal of S. However, A is not left pure because $c \notin \{a, e\} = (Ac]$. So, $c \notin (cA] \cap (Ac]$ and thus A is not quasi-pure.

As a consequence of Example 3.5 and 3.6, the concepts of left pure and right pure ordered ideals of ordered semirings are independent.

Theorem 3.7. *An ordered ideal A of an ordered semiring S is quasi-pure if and only if $A \cap Q = (QA] \cap (AQ]$ for every ordered quasi-ideal Q of S.*

Proof. Assume that A is quasi-pure. Let Q be an ordered quasi-ideal of S. If

$x \in A \cap Q$, then $x \in (xA] \cap (Ax] \subseteq (QA] \cap (AQ]$. So, $A \cap Q \subseteq (QA] \cap (AQ]$. Clearly, $(QA] \cap (AQ] \subseteq A \cap Q$. Hence, $A \cap Q = (QA] \cap (AQ]$.

Conversely, let $x \in A$. Using assumption and Lemma 2.1, we get

$$\begin{aligned} x \in A \cap Q(x) &= (Q(x)A] \cap (AQ(x)] \\ &= ((\Sigma x + ((xS] \cap (Sx]))]A] \cap (A(\Sigma x + ((xS] \cap (Sx]))]] \\ &\subseteq ((\Sigma x + (xS]]A] \cap (A(\Sigma x + (Sx]]] \\ &\subseteq ((\Sigma x + xS]A] \cap (A(\Sigma x + Sx]] \\ &\subseteq (\Sigma xA + xSA] \cap (\Sigma Ax + ASx] \\ &\subseteq (xA + xA] \cap (Ax + Ax] \\ &\subseteq (xA] \cap (Ax]. \end{aligned}$$

Hence, A is a quasi-pure ordered ideal of S. □

Definition 3.8. An ordered ideal A of an ordered semiring S is called *bi-pure* if $x \in (xAx]$ for all $x \in A$.

The following remark is directly obtained by Definition 3.3 and 3.8.

Remark 3.9 Every bi-pure ordered ideal of an ordered semiring is quasi-pure.

Now, we give an example of a quasi-pure ordered ideal of an ordered semiring which is not bi-pure.

Example 3.10. Let $S = \{a, b, c, d, e\}$ together with the operation $+$ and the relation $\leq$ defined in Example 3.5. Define a binary operation $\cdot$ on S by the following table;

$\cdot$	a	b	c	d	e
a	a	a	a	a	e
b	a	b	c	a	e
c	a	c	a	a	e
d	a	a	a	a	e
e	e	e	e	e	e

Then $(S, +, \cdot, \leq)$ is an ordered semiring. We have that $A = \{a, b, c, e\}$ is a quasi-pure ordered ideal of S. However, A is not bi-pure because $c \notin \{a, e\} = (cAc]$.

Theorem 3.11. *An ordered ideal A of an ordered semiring S is bi-pure if and only if $A \cap B = (BAB]$ for every ordered bi-ideal B of S.*

Proof. Assume that A is bi-pure. Let B be an ordered bi-ideal of S. If $x \in A \cap B$, then $x \in (xAx] \subseteq (BAB]$. So, $A \cap B \subseteq (BAB]$. Clearly, $(BAB] \subseteq A \cap B$. Hence, $A \cap B = (BAB]$.

Conversely, let $x \in A$. Using assumption and Lemma 2.1, we get

$$\begin{aligned} x \in A \cap B(x) = (B(x)AB(x)] &= ((\Sigma x + \Sigma x^2 + xSx]A(\Sigma x + \Sigma x^2 + xSx]] \\ &\subseteq (\Sigma xAx] = (xAx]. \end{aligned}$$

Hence, A is a bi-pure ordered ideal of S. □

Now, we characterize regular, left weakly regular and right weakly regular semirings using left pure, right pure, quasi-pure and bi-pure ordered ideals.

Lemma 3.12. *Let S be an ordered semiring. Then the following statements hold:*

(i) *if $a \in (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]$ for any $a \in S$, then S is left weakly regular;*

(ii) *if $a \in (\Sigma a^2 + aSa + a^2S + \Sigma aSaS]$ for any $a \in S$, then S is right weakly regular.*

Proof. (i) Let $a \in S$. Assume that

$$a \in (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa] \tag{3.1}$$
$$\subseteq (Sa + \Sigma SaSa]. \tag{3.2}$$

Using (3.2), we obtain that

$$a^2 = aa \in (Sa + \Sigma SaSa](Sa + \Sigma SaSa] \subseteq (\Sigma SaSa]. \tag{3.3}$$

Using (3.2) again, we obtain that

$$aSa \subseteq (Sa + \Sigma SaSa]S(Sa + \Sigma SaSa] \subseteq (\Sigma SaSa]. \tag{3.4}$$

Using (3.3), we obtain that

$$Sa^2 \subseteq S(\Sigma SaSa] \subseteq (\Sigma SSaSa] \subseteq (\Sigma SaSa]. \tag{3.5}$$

Using (3.1), (3.3), (3.4) and (3.5), we obtain that

$$\begin{aligned} a &\in (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa] \\ &\subseteq (\Sigma(\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa] + \Sigma SaSa] \\ &\subseteq ((\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa]] \\ &\subseteq ((\Sigma SaSa + \Sigma SaSa + \Sigma SaSa + \Sigma SaSa]] \\ &= (\Sigma SaSa]. \end{aligned}$$

Therefore, S is left weakly regular.

(ii) It can be proved in a similar way of (i). □

Theorem 3.13. *An ordered semiring S is left (resp. right) weakly regular if and only if every ordered ideal of S is left (resp. right) pure.*

Proof. Assume that S is left weakly regular. Let I be an ordered ideal of S and let $x \in I$. By assumption, $x \in (\Sigma SxSx] \subseteq (\Sigma SISx] \subseteq (\Sigma Ix] = (Ix]$. Hence, I is left pure.

Conversely, let $a \in S$. By assumption, we obtain that $J(a)$ is left pure. Using Lemma 2.1 and Theorem 3.2(i), we obtain that

$$a \in J(a) \cap L(a) = (J(a)L(a)] = ((\Sigma a + aS + Sa + \Sigma SaS](\Sigma a + Sa]] \subseteq (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa].$$

By Lemma 3.12(i), we get that S is left weakly regular. □

As a consequence of Theorem 3.13 and the fact that every quasi-pure ordered ideal is both left pure and right pure, we directly obtain the following corollary.

Corollary 3.14 *An ordered semiring S is both left and right weakly regular if and only if every ordered ideal of S is quasi-pure.*

We note that an ordered ideal of an ordered semiring is bi-pure if and only if it is a regular subsemiring. Accordingly, we obtain the following remark.

Remark 3.15 An ordered semiring S is regular if and only if every ordered ideal of S is bi-pure.

Proof. Assume that S is regular. Let I be an ordered ideal of S and let $x \in I$. By the regularity of S, we have that $x \in (xSx] \subseteq (xSxSx] \subseteq (xSISx] \subseteq (xSIx] \subseteq (xIx]$. Hence, I is bi-pure.

The converse is obvious since S itself is a bi-pure ordered ideal and so S is regular. □

Now, we introduce the notions of left and right weakly pure ordered ideals of ordered semirings and then we characterize fully idempotent ordered semirings using their left and right weakly pure ordered ideals.

Definition 3.16. An ordered ideal A of S is called *left weakly pure* (resp. *right weakly pure*) if $A \cap I = (\Sigma AI]$ (resp. $I \cap A = (\Sigma IA]$) for every ordered ideal I of S.

Remark 3.17. Every left (resp. right) pure ordered ideal of an ordered semiring is left (resp. right) weakly pure.

Now, we give an example of a left weakly pure ordered ideal of an ordered semiring which is not left pure.

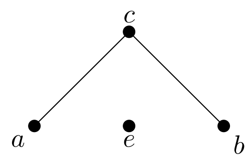
Example 3.18. Let $S = \{a, b, c, e\}$. Define two binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c	e
a	a	b	c	a
b	b	b	c	b
c	c	c	c	c
e	a	b	c	e

and

$\cdot$	a	b	c	e
a	a	a	c	e
b	a	a	c	e
c	a	a	c	e
e	e	e	e	e

Define a binary relation $\leq$ on S by $\leq := \{(a, a), (b, b), (c, c), (e, e), (a, c), (b, c)\}$.



Then $(S, +, \cdot, \leq)$ is an ordered semiring with only two ordered ideals S and $\{e\}$. Since $S \cap \{e\} = \{e\} = (\Sigma S\{e\}]$ and $S \cap S = S = (\Sigma SS]$, S itself is a left weakly pure ordered ideal. However, S is not a left pure ordered ideal because $b \notin \{a, e\} = (Sb]$.

As a duality of Example 3.18, we now give an example of a right weakly pure ordered ideal of an ordered semiring which is not right pure.

Example 3.19. Let $S = \{a, b, c, e\}$. Define a binary operation $\cdot$ on S by the following table:

$\cdot$	a	b	c	e
a	a	a	a	e
b	a	a	a	e
c	c	c	c	e
e	e	e	e	e

Then $(S, +, \cdot, \leq)$ is an ordered semiring together with the operation $+$ and the relation $\leq$ defined in Example 3.18. We have that S has only two ordered ideals S and $\{e\}$. Since $\{e\} \cap S = \{e\} = (\Sigma\{e\}S]$ and $S \cap S = S = (\Sigma SS]$, S itself is a right weakly pure ordered ideal. However, S is not a right pure ordered ideal because $b \notin \{a, e\} = (bS]$.

Definition 3.20. An ordered semiring S is called *fully idempotent* if $I = (\Sigma I^2]$ for every ordered ideal I of S.

Lemma 3.21. *Let S be an ordered semiring. Then the following statements are equivalent:*

(i) *S is fully idempotent;*

(ii) *$a \in (\Sigma SaSaS]$ for all $a \in S$;*

(iii) *$A \subseteq (\Sigma SASAS]$ for all $\emptyset \neq A \subseteq S$.*

Proof. (i) $\Rightarrow$ (ii): Assume that S is fully idempotent and let $a \in S$. Using Lemma 2.1, we obtain that

$$\begin{aligned} a \in J(a) = (\Sigma J(a)^2] &= (\Sigma(\Sigma a + Sa + aS + \Sigma SaS](\Sigma a + Sa + aS + \Sigma SaS]] \\ &\subseteq (\Sigma(\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]] \\ &\subseteq (\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]. \end{aligned} \tag{3.6}$$

Using (3.6), we obtain

$$a \in (aS + \Sigma SaS + \Sigma SaSaS], \tag{3.7}$$
$$a \in (Sa + \Sigma SaS + \Sigma SaSaS]. \tag{3.8}$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned} a^2 = aa &\in (aS + \Sigma SaS + \Sigma SaSaS](Sa + \Sigma SaS + \Sigma SaSaS] \\ &\subseteq (aSa + \Sigma aSaS + \Sigma SaSa + \Sigma SaSaS]. \end{aligned} \tag{3.9}$$

Using (3.7) and (3.8), we obtain

$$aSa \subseteq (Sa + \Sigma SaS + \Sigma SaSaS]S(aS + \Sigma SaS + \Sigma SaSaS] \subseteq (\Sigma SaSaS]. \tag{3.10}$$

Using (3.7), we obtain

$$\Sigma SaSa \subseteq \Sigma(SaS(aS + \Sigma SaS + \Sigma SaSaS]) \subseteq (\Sigma SaSaS]. \tag{3.11}$$

Using (3.8), we obtain

$$\Sigma aSaS \subseteq \Sigma((Sa + \Sigma SaS + \Sigma SaSaS]SaS) \subseteq (\Sigma SaSaS]. \tag{3.12}$$

Using (3.9), (3.10), (3.11) and (3.12), we obtain

$$\begin{aligned} a^2 &\in (aSa + \Sigma aSaS + \Sigma SaSa + \Sigma SaSaS] \\ &\subseteq ((\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS]] \\ &\subseteq ((\Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS]] \\ &= (\Sigma SaSaS]. \end{aligned} \tag{3.13}$$

Using (3.13), we obtain

$$a^2S \subseteq (\Sigma SaSaS]S \subseteq (\Sigma SaSaS], \tag{3.14}$$
$$Sa^2 \subseteq S(\Sigma SaSaS] \subseteq (\Sigma SaSaS], \tag{3.15}$$
$$Sa^2S \subseteq S(\Sigma SaSaS]S \subseteq (\Sigma SaSaS]. \tag{3.16}$$

Using (3.6), (3.13), (3.10),(3.14), (3.12), (3.15), (3.11) and (3.16), we obtain

$$\begin{aligned} a &\in (\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS] \\ &\subseteq ((\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] \\ &\quad + (\Sigma SaSaS] + \Sigma SaSaS] \\ &\subseteq ((\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] \\ &\quad + (\Sigma SaSaS] + (\Sigma SaSaS]] \\ &\subseteq (\Sigma SaSaS]. \end{aligned}$$

(ii) $\Rightarrow$ (iii): It is obvious.

(iii) $\Rightarrow$ (i): Assume that (iii) holds and let I be an ordered ideal of S. Clearly, $(\Sigma I^2] \subseteq (\Sigma I] = I$. On the other hand, $I \subseteq (\Sigma SISIS] \subseteq (\Sigma ISI] \subseteq (\Sigma I^2]$. So, $I = (\Sigma I^2]$. Therefore, S is fully idempotent. □

By the proof of Lemma 3.21, we can directly obtain the following corollary.

Corollary 3.22. *Let S be an ordered semiring. If*

$$a \in (\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]$$

for all $a \in S$, then S is fully idempotent.

Theorem 3.23. *Let S be an ordered semiring. Then the following statements hold:*

(i) *if S is fully idempotent, then every ordered ideal of S is both left and right weakly pure;*

(ii) *if every ordered ideal of S is left weakly pure (right weakly pure), then S is fully idempotent.*

Proof. (i) Assume that S is fully idempotent. Let A and I be any ordered ideals of S. By assumption and Lemma 3.21, it turns out that if $x \in A \cap I$, then

$$x \in (\Sigma SxSxS] \subseteq (\Sigma SASIS] \subseteq (\Sigma ASI] \subseteq (\Sigma AI] \text{ and}$$
$$x \in (\Sigma SxSxS] \subseteq (\Sigma SISAS] \subseteq (\Sigma ISA] \subseteq (\Sigma IA].$$

So, $A \cap I \subseteq (\Sigma AI]$ and $A \cap I \subseteq (\Sigma IA]$. Clearly, $(\Sigma AI] \subseteq A \cap I$ and $(\Sigma IA] \subseteq A \cap I$. Hence, $A \cap I = (\Sigma AI] = (\Sigma IA]$ and thus A is both left and right weakly pure.

(ii) Assume that every ordered ideal of S is left weakly pure. Let $a \in S$. Then $J(a)$ is left weakly pure. It follows that $J(a) = (\Sigma J(a)J(a)]$. By Lemma 2.1, we obtain that

$$\begin{aligned} a \in J(a) = (\Sigma J(a)J(a)] &= (\Sigma(\Sigma a + Sa + aS + \Sigma SaS](\Sigma a + Sa + aS + \Sigma SaS]] \\ &= (\Sigma(\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]] \\ &= (\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]. \end{aligned}$$

By Corollary 3.22, we obtain that S is fully idempotent.

It can be proved analogously if every ordered ideal of S is right weakly pure. □

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