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On the Fekete-Szegö Problem for Starlike Functions of Complex Order

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Abstract. For a non-zero complex number b and for m and n in $\mathbb{N}_0 = \{0, 1, 2, ...\}$ let $\Psi_{n,m}(b)$ denote the class of normalized univalent functions f satisfying the condition $\Re\left[1 + \frac{1}{b}\left(\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right)\right] > 0$ in the unit disk U, where $D^n f(z)$ denotes the Salagean operator of f. Sharp bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ are obtained.

1. Introduction

Fekete and Szegö proved the remarkable result that the estimate

$$\left|a_3 - \lambda a_2^2\right| \leq 1 + 2\exp(\frac{-2\lambda}{1-\lambda})$$

holds with $0 \leq \lambda \leq 1$ for any normalized univalent function

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \tag{1.1}$$

in the open unit disk U. This inequality is sharp for all λ (see [7]). The coefficient functional

$$\phi\left(f(z)\right) = a_3 - \lambda a_2^2 = \frac{1}{6}(f'''(0) - \frac{3\lambda}{2}\left[f''(0)\right]^2)$$

in the unit disk represents various geometric quantities. For example, when $\lambda = 1$, $\phi\left(f(z)\right) = a_3 - \lambda a_2^2$ becomes $S_f(0)/6$ where S_f denotes the Schwarzian derivative

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$(f''/f')' - (f''/f')^2/2$. Note that, if we consider the nth root transform $[f(z^n)]^{1/n} = z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + ...$ of f with the power series (1.1), then $c_{n+1} = a_2/n$ and $c_{2n+1} = a_3/n + (n-1)a_2^2/2n^2$, so that

$$a_3 - \lambda a_2^2 = n(c_{2n+1} - \mu c_{n+1}^2),$$

where $\mu = \lambda n + (n-1)/2$. Moreover, $\phi(f(z))$ behaves well with respect to the rotation, namely $\phi\left((e^{-i\theta} f(e^{i\theta} z)\right) = e^{2i\theta}\phi(f)$, for $\theta \in R$.

This is quite natural to discuss the behavior of $\phi(f(z))$ for subclasses of normalized univalent functions in the unit disk. This is called the Fekete-Szegö problem. Many authors have considered this problem for typical classes of univalent functions (see, for instance [1, 2]).

We denote by A the set of all functions of the from (1.1) that are normalized analytic and univalent in the unit disk U. Also, for $0 \leq \alpha < 1$, let $S^*(\alpha)$ and $C(\alpha)$ denote the classes of starlike and convex univalent functions of order α, respectively, i.e., let

$$S^*(\alpha) = \left\{ f(z) \in A : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in U \right\} \tag{1.2}$$

and

$$C(\alpha) = \left\{ f(z) \in A : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in U \right\}. \tag{1.3}$$

The notions of α-starlikeness and α-convexity were generalized onto a complex order by Nasr and Aouf [4], Wiatrowski [8], and Nasr and Aouf [3].

Observe that $S^*(0) = S^*$ and $C(0) = C$ represent standard starlike and convex univalent functions, respectively.

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be analytic functions in U. The Hadamard product (convolution) of f and g, denoted by $f * g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, z \in U.$$

For a function $f(z)$ in A, we define

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z)$$

and

$$D^n f(z) = D(D^{n-1} f(z)),\ n \in \mathcal{N} = \{0, 1, 2, ...\}.$$

With the above operator D^n, we say that a function $f(z)$ belonging to A is in the class $A(n, m, \alpha)$ if and only if

$$\Re\left\{\frac{D^{n+m} f(z)}{D^n f(z)}\right\} > \alpha \ (n, m \in \mathcal{N}_0 = N \cup \{0\}) \tag{1.4}$$

for some α $(0 \leq \alpha < 1)$, and for all $z \in U$.

We note that $A(0,1,\alpha) = S^*(\alpha)$ is the class of starlike functions of order α, $A(1,1,\alpha) = C(\alpha)$ is the class of convex functions of order α, and that $A(1,0,\alpha) = A_n(\alpha)$ is the class of functions defined by Salagean [6].

Definition 1.1. Let b be a nonzero complex number, and let f be an univalent function of the form (1.1), such that $D^{n+m}f(z) \neq 0$ for $z \in U \setminus \{0\}$. We say that f belongs to $\Psi_{n,m}(b)$ if

$$\Re\left\{1 + \frac{1}{b}\left(\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right)\right\} > 0, \ z \in U. \tag{1.5}$$

By giving specific values to n, m and b, we obtain the following important subclasses studied by various researchers in earlier works, for instance, $\Psi_{1,0}(b) = S^*(1-b)$ (Nasr and Aouf [4]), $\Psi_{1,1}(b) = C(1-b)$ (Wiatrowski [8], Nasr and Aouf [3]).

2. Main Results

We denote by $\mathcal{P}$ class of the analytic functions in U with $p(0) = 1$ and $\Re(p(z)) > 0$. We shall require the following:

Lemma 2.1.([5], p.166) *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$, then*

$$|c_n| \leq 2, \quad \textit{for } n \geq 1.$$

If $|c_1| = 2$ then $p(z) \equiv p_1(z) = (1 + \gamma_1 z)/(1 - \gamma_1 z)$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore we have

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|}{2}.$$

If $|c_1| < 2$ and $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + \frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 + \gamma_2 z}}{1 - \frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 + \gamma_2 z}},$$

and $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$. Conversely if $p(z) \equiv p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$, then $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|}{2}$.

Theorem 2.2. *Let $n, m \geq 0$ and let b be non-zero complex number. If f of the form* (1.1) *is in $\Psi_{n,m}(b)$, then*

$$|a_2| \leq \frac{2|b|}{2^n(2^m - 1)}, \tag{2.1}$$

$$|a_3| \le \frac{2|b|}{3^n(3^m-1)} \max\left\{1, \frac{|(2^m-1)+2b|}{(2^m-1)}\right\}, \tag{2.2}$$

and

$$\left|a_3 - \frac{2^2 n\,(2^m-1)}{3^n(3^m-1)} a_2^2\right| \le \frac{2|b|}{3^n(3^m-1)}.$$

Equality in (2.1) *holds if* $D^{n+m}f(z)/D^n f(z) = 1 + b\,[p_1(z)-1]$, *and in* (2.2) *if* $D^{n+m}f(z)/D^n f(z) = 1 + b\,[p_2(z)-1]$, *where* p_1, p_2 *are given in Lemma 2.1.*

Proof. Denote $F(z) = D^n f(z) = z + A_2 z^2 + A_3 z^3 + ...$, $D^m F(z) = D^{n+m} f(z) = z + 2^m A_2 z^2 + 3^m A_3 z^3 + ...$, where $A_2 = 2^n a_2$ and $A_3 = 3^n a_3, ...$, then

$$2^m A_2 = 2^{n+m} a_2, \qquad 3^m A_3 = 3^{n+m} a_3. \tag{2.3}$$

By the definition of the class $\Psi_{n,m}(b)$ there exists $p \in \mathcal{P}$ such that $\frac{D^m F(z)}{F(z)} = 1 - b + bp(z)$, so that

$$\frac{z + 2^m A_2 z^2 + 3^m A_3 z^3 + ...}{z + A_2 z^2 + A_3 z^3 + ...} = 1 - b + b(1 + c_1 z + c_2 z^2 + ...),$$

which implies the equality

$$z + 2^m A_2 z^2 + 3^m A_3 z^3 + ... = z + (A_2 + bc_1) z^2 + (A_3 + bc_1 A_2 + bc_2) z^3 + ...$$

Equating the coefficients of both sides we have

$$A_2 = \frac{b}{2^m-1} c_1, \qquad A_3 = \frac{b}{3^m-1}\left[c_2 + \frac{bc_1^2}{2^m-1}\right], \tag{2.4}$$

so that, on account of $D^{n+m}f(z)$

$$a_2 = \frac{bc_1}{2^n(2^m-1)}, \qquad a_3 = \frac{b}{3^n(3^m-1)}\left[c_2 + \frac{b}{(2^m-1)} c_1^2\right]. \tag{2.5}$$

Taking into account (2.5) and Lemma 2.1, we obtain

$$|a_2| = \left|\frac{b}{2^n(2^m-1)} c_1\right| \le \frac{2|b|}{2^n(2^m-1)},$$

and

$$\begin{aligned}
|a_3| &= \left|\frac{b}{3^n(3^m-1)}\left[c_2 - \frac{c_1^2}{2} + \frac{(2^m-1)+2b}{2\,(2^m-1)} c_1^2\right]\right| \\
&\le \frac{|b|}{3^n(3^m-1)}\left[2 - \frac{|c_1^2|}{2} + \frac{|(2^m-1)+2b|}{2\,(2^m-1)}\left|c_1^2\right|\right] \\
&= \frac{|b|}{3^n(3^m-1)}\left[2 + \frac{|(2^m-1)+2b| - (2^m-1)}{2\,(2^m-1)}\left|c_1^2\right|\right] \\
&\le \frac{2|b|}{3^n(3^m-1)} \max\left\{1, 1 + \frac{|(2^m-1)+2b| - (2^m-1)}{(2^m-1)}\right\}.
\end{aligned}$$

Thus

$$|a_3| \leq \frac{2|b|}{3^n(3^m-1)} \max\left\{1, \frac{|(2^m-1)+2b|}{(2^m-1)}\right\}. \tag{2.6}$$

Moreover

$$\begin{aligned}\left|a_3 - \frac{2^{2n}(2^m-1)}{3^n(3^m-1)}a_2^2\right| &= \left|\frac{b}{3^n(3^m-1)}\left[c_2 + \frac{b}{(2^m-1)}c_1^2\right] - \frac{b^2c_1^2}{[2^n(2^m-1)]^2}\frac{2^{2n}(2^m-1)}{3^n(3^m-1)}\right| \\ &= \left|\frac{bc_2}{3^n(3^m-1)}\right| \\ &\leq \frac{2|b|}{3^n(3^m-1)},\end{aligned}$$

as asserted. □

Remark 2.3. Putting $b = 1-\alpha$ in the above theorem we get the following corollary.

Corollary 2.4. *Let $n, m \geq 0$. If f of the form (1.1) is in $A(n, m, \alpha)$, then*

$$|a_2| \leq \frac{2(1-\alpha)}{2^n(2^m-1)},$$

$$|a_3| \leq \frac{2(1-\alpha)}{3^n(3^m-1)} \max\left\{1, \frac{(2^m+1-2\alpha)}{(2^m-1)}\right\},$$

and

$$\left|a_3 - \frac{2^{2n}(2^m-1)}{3^n(3^m-1)}a_2^2\right| \leq \frac{2(1-\alpha)}{3^n(3^m-1)}.$$

Remark 2.5. In the above Theorem 2.2 and Corollary 2.4 a special case of Fekete-Szegö problem e.g. for real $\mu = 2^{2n}(2^m-1)/3^n(3^m-1)$ occurred very naturally and simple estimate was obtained.

Now, we consider functional $\left|a_3 - \mu a_2^2\right|$ for complex μ.

Theorem 2.6. *Let b be a non-zero complex number and let $f \in \Psi_{n,m}(b)$. Then for $\mu \in \mathbb{C}$*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2|b|}{3^n(3^m-1)} \max\left\{1, \left|1 + \frac{2b}{(2^m-1)} - 2\mu b\frac{3^n(3^m-1)}{[2^n(2^m-1)]^2}\right|\right\}.$$

For each μ there is a function in $\Psi_{n,m}(b)$ such that equality holds.

Proof. Applying (2.5) we have

$$
\begin{aligned}
a_3-\mu a_2^2 &= \frac{b}{3^n(3^m-1)}\left[c_2+\frac{b}{(2^m-1)}c_1^2\right]-\mu\frac{b^2c_1^2}{\left[2^n(2^m-1)\right]^2}\\
&= \frac{b}{3^n(3^m-1)}\left[c_2+\frac{b}{(2^m-1)}c_1^2-\mu b\frac{3^n(3^m-1)}{\left[2^n(2^m-1)\right]^2}c_1^2\right]\\
&= \frac{b}{3^n(3^m-1)}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(1+\frac{2b}{(2^m-1)}-2\mu b\frac{3^n(3^m-1)}{\left[2^n(2^m-1)\right]^2}\right)\right].
\end{aligned}
$$

Then, with the aid of Lemma 2.1, we obtain

$$
\begin{aligned}
\left|a_3-\mu a_2^2\right| &\le \frac{|b|}{3^n(3^m-1)}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left|1+\frac{2b}{(2^m-1)}-2\mu b\frac{3^n(3^m-1)}{\left[2^n(2^m-1)\right]^2}\right|\right]\\
&= \frac{|b|}{3^n(3^m-1)}\left[2+\frac{|c_1|^2}{2}\left(\left|1+\frac{2b}{(2^m-1)}-2\mu b\frac{3^n(3^m-1)}{\left[2^n(2^m-1)\right]^2}\right|-1\right)\right]\\
&\le \frac{2\,|b|}{3^n(3^m-1)}\max\left\{1,\left|1+\frac{2b}{(2^m-1)}-2\mu b\frac{3^n(3^m-1)}{\left[2^n(2^m-1)\right]^2}\right|\right\}.
\end{aligned}
$$

An examination of the proof shows that equality is attained for the first case, when $c_1=0$, $c_2=2$, then the functions in $\Psi_{n,m}(b)$ is given by

$$
\frac{D^{n+m}f(z)}{D^nf(z)}=\frac{1+(2b-1)z}{1-z}, \tag{2.7}
$$

and, for the second case, when $c_1=c_2=2$, so that

$$
\frac{D^{n+m}f(z)}{D^nf(z)}=\frac{1+(2b-1)z^2}{1-z}, \tag{2.8}
$$

respectively. □

Remark 2.7. Putting $b=1-\alpha$ in the above theorem we get the following corollary.

Corollary 2.8. *Let $f\in A(n,m,\alpha)$. Then for $\mu\in\mathbb{C}$*

$$
\left|a_3-\mu a_2^2\right|\le\frac{2(1-\alpha)}{3^n(3^m-1)}\max\left\{1,\left|1+\frac{2(1-\alpha)}{(2^m-1)}-2\mu(1-\alpha)\frac{3^n(3^m-1)}{\left[2^n(2^m-1)\right]^2}\right|\right\}.
$$

For each μ there is a function in $A(n,m,\alpha)$ such that equality holds.

We next consider the case, when μ and b are real. Then we have:

Theorem 2.9. *Let $b>0$ and let $f\in\Psi_{n,m}(b)$. Then for $\mu\in R$ we have*

$$
\left|a_3-\mu a_2^2\right|\le\begin{cases}
\frac{2b}{3^n(3^m-1)}\left[1+\frac{2b}{(2^m-1)}\left(1-\mu\frac{3^n(3^m-1)}{2^{2n}(2^m-1)}\right)\right] & \text{if } \mu\le\frac{2^{2n}(2^m-1)}{3^n(3^m-1)},\\
\frac{b}{3^n(3^m-1)} & \text{if } \frac{2^{2n}(2^m-1)}{3^n(3^m-1)}\le\mu\le\frac{2^{2n}(2^m-1)[(2^m-1)+2b]}{2b[3^n(3^m-1)]},\\
\frac{2b}{3^n(3^m-1)}\left[2\mu b\frac{3^n(3^m-1)}{[2^n(2^m-1)]^2}-1-\frac{2b}{(2^m-1)}\right] & \text{if } \mu\ge\frac{2^{2n}(2^m-1)[(2^m-1)+2b]}{2b[3^n(3^m-1)]}.
\end{cases}
$$

For each there is a function in $\Psi_{n,m}(b)$ such that equality holds.

Proof. First, let $\mu \leq \frac{2^{2n}(2^m-1)}{3^n(3^m-1)}$ In this case, (2.5) and Lemma 2.1 give

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{b}{3^n(3^m-1)}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(1 + \frac{2b}{(2^m-1)} - 2\mu b\frac{3^n(3^m-1)}{[2^n(2^m-1)]^2}\right)\right] \\ &\leq \frac{2b}{3^n(3^m-1)}\left[1 + \frac{2b}{(2^m-1)}\left(1 - \mu\frac{3^n(3^m-1)}{2^{2n}(2^m-1)}\right)\right].\end{aligned}$$

Let, now $\frac{2^{2n}(2^m-1)}{3^n(3^m-1)} \leq \mu \leq \frac{2^{2n}(2^m-1)[(2^m-1)+2b]}{2b[3^n(3^m-1)]}$. Then, using the above calculations, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{b}{3^n(3^m-1)}.$$

Finally, if $\mu \geq \frac{2^{2n}(2^m-1)[(2^m-1)+2b]}{2b[3^n(3^m-1)]}$, then

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{b}{3^n(3^m-1)}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(2\mu b\frac{3^n(3^m-1)}{[2^n(2^m-1)]^2} - 1 - \frac{2b}{(2^m-1)}\right)\right] \\ &= \frac{b}{3^n(3^m-1)}\left[2 + \frac{|c_1|^2}{2}\left(2\mu b\frac{3^n(3^m-1)}{[2^n(2^m-1)]^2} - 2 - \frac{2b}{(2^m-1)}\right)\right] \\ &\leq \frac{2b}{3^n(3^m-1)}\left[2\mu b\frac{3^n(3^m-1)}{[2^n(2^m-1)]^2} - 1 - \frac{2b}{(2^m-1)}\right].\end{aligned}$$

Equality is attained for the second case on choosing $c_1 = 0, c_2 = 2$ in (2.7) and in (2.8) by choosing $c_1 = 2, c_2 = 2$ *and* $c_1 = 2i, c_2 = -2$ for the first and third case, respectively. Thus the proof is complete. □

Remark 2.10. Put $b = 1 - \alpha$ in the above theorem we get the following corollary.

Corollary 2.11. *Let $f \in A(n, m, \alpha)$. Then for $\mu \in R$ we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2(1-\alpha)}{3^n(3^m-1)}\left[1 + \frac{2(1-\alpha)}{(2^m-1)}\left(1 - \mu\frac{3^n(3^m-1)}{2^{2n}(2^m-1)}\right)\right] & \text{if } \mu \leq \frac{2^{2n}(2^m-1)}{3^n(3^m-1)}, \\ \frac{(1-\alpha)}{3^n(3^m-1)} & \text{if } \frac{2^{2n}(2^m-1)}{3^n(3^m-1)} \leq \mu \leq \frac{2^{2n}(2^m-1)[2^m+1-2\alpha]}{2(1-\alpha)[3^n(3^m-1)]}, \\ \frac{2(1-\alpha)}{3^n(3^m-1)}\left[2\mu(1-\alpha)\frac{3^n(3^m-1)}{[2^n(2^m-1)]^2} - 1 - \frac{2(1-\alpha)}{(2^m-1)}\right] & \text{if } \mu \geq \frac{2^{2n}(2^m-1)[2^m+1-2\alpha]}{2(1-\alpha)[3^n(3^m-1)]}. \end{cases}$$

For each there is a function in $A(n, m, \alpha)$ such that equality holds.

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