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Generalized Integration Operator between the Bloch-type Space and Weighted Dirichlet-type Spaces

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Abstract. Let $H(\mathbb{D})$ be the space of all holomorphic functions on the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. In this paper, we investigate the boundedness and compactness of the generalized integration operator

$$I_{g,\varphi}^{(n)}(f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)\, d\xi, \quad z \in \mathbb{D},$$

between Bloch-type and weighted Dirichlet-type spaces, where φ is a holomorphic self-map of $\mathbb{D}$, $n \in \mathbb{N}$ and $g \in H(\mathbb{D})$.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$. For $\alpha \in (0, \infty)$, the α-Bloch space $\mathcal{B}^\alpha$ is the space of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

These are collectively referred to as Bloch-type spaces. The little Bloch-type space

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$\mathcal{B}_0^\alpha$ consists of those functions $f \in \mathcal{B}^\alpha$ for which

$$\lim_{|z|\longrightarrow 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

The space $\mathcal{B}^\alpha$ is a Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha},$$

and $\mathcal{B}_0^\alpha$ is a closed subspace of $\mathcal{B}^\alpha$.

For $p \in (0,\infty)$ and $\beta > -1$, $\mathcal{A}_\beta^p$ denotes the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{A}_\beta^p}^p = \int_{\mathbb{D}} |f(z)|^p \left(\log \frac{1}{|z|}\right)^\beta \, dA(z) < \infty,$$

where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. The space $\mathcal{A}_\beta^p$ is called the weighted Bergman space. The weighted Bergman space $\mathcal{A}_\beta^p$ is a Banach space for $p \geq 1$ and a Hilbert space for $p = 2$. It is well-known that $f \in \mathcal{A}_\beta^p$ if and only if

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta \, dA(z) < \infty.$$

For $p \in (0,\infty)$ and $\beta > -1$, the weighted Dirichlet-type space $\mathcal{D}_\beta^p$ is the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_\beta^p}^p = \int_{\mathbb{D}} |f'(z)|^p \left(\log \frac{1}{|z|}\right)^\beta \, dA(z) < \infty.$$

We note that $f \in \mathcal{D}_\beta^p$ if and only if $f' \in \mathcal{A}_\beta^p$.

Let u be a holomorphic function on $\mathbb{D}$ and φ a nonconstant holomorphic self-map of $\mathbb{D}$. The weighted composition operator uC_φ induced by u and φ is defined on $H(\mathbb{D})$ as follows:

$$uC_\varphi(f) = ufo\varphi.$$

Putting $u = 1, uC_\varphi$ reduces to the composition operator C_φ. For general background on composition operators, we refer to [3, 14] and for weighted composition operators acting on Bloch-type spaces and Dirichlet-type spaces we refer for example to [2, 5, 13, 18].

In this paper, we consider an integration operator $I_{g,\varphi}^{(n)}$ which is defined on $H(\mathbb{D})$ by

$$I_{g,\varphi}^{(n)}(f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)\, d\xi, \quad z \in \mathbb{D},$$

where φ is a holomorphic self-map of $\mathbb{D}$, $n \in \mathbb{N}$ and $g \in H(\mathbb{D})$.

This operator, which was introduced in [15], is called the generalized integration operator. It is a generalization of the Riemann-Stieltjes operator I_g induced by g, defined by

$$I_g f(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta, \qquad z \in D.$$

Y. Yu and Y. Liu in [20] characterized the boundedness and compactness of Riemann-Stieltjes operator I_g from weighted Bloch spaces into Bergman-type spaces. The essential norm of the integral operator I_g on some spaces of holomorphic functions was studied by L. Liu, Z. Lou and C. Xiong in [10].

The operator $I_{g,\varphi}^{(n)}$ induces some known operators. For example, when $n = 1$, $I_{g,\varphi}^{(n)}$ reduces to an integration operator recently studied by S. Li and S. Stevic in [6, 7, 8]. Taking $n = 1$ and $g(z) = \varphi'(z)$, we obtain the composition operator C_φ defined by $C_\varphi f = f(\varphi) - f(\varphi(0)), f \in H(D)$.

Recently, S. D. Sharma and A. Sharma in [15] characterized the boundedness and compactness of generalized integration operator $I_{g,\varphi}^{(n)}$ from Bloch-type spaces to weighted BMOA.

The boundedness and compactness of Riemann-Stieltjes operators from mixed norm spaces to Zygmund-type spaces on the unit ball was studied by Y. Liu and Y. Yu in [11]. X. Zhu in [24] investigated the boundedness and compactness of generalized integration operators from H^∞ to Zygmund-type spaces. Z. He and G. Cao in [4] investigated the boundedness and compactness of generalized integration operators between Bloch-type spaces and $F(p, q, s)$ spaces. For related integral-type operators on unit disc and also in $\mathbb{C}^n$, see for example [1, 9, 16]. Motivated by the above results, in this article we give an equivalent conditions for the boundedness and compactness of the generalized integration operator $I_{g,\varphi}^{(n)}$ between the Bloch-type and weighted Dirichlet-type spaces.

The notation $a \preceq b$ means that there exists a positive constant C such that $a \leq Cb$. If both $a \preceq b$ and $b \preceq a$ occur, then $a \sim b$.

2. Boundednes and Compactness of the Operator $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \longrightarrow \mathcal{D}^p_\beta$

In this section we characterize the boundedness and compactness of the generalized integration operator $I_{g,\varphi}^{(n)}$ from the Bloch-type space $\mathcal{B}^\alpha$ into Dirichlet-type space $\mathcal{D}^p_\beta$.

Let $\alpha > 0$. From [12] it follows that there are two holomorphic functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that

$$\frac{C}{(1 - |z|^2)^\alpha} \leq |f_1'(z)| + |f_2'(z)|, \quad z \in \mathbb{D},$$

where C ia a positive constant.

If we define $h_1(z) = f_1(z) - zf_1'(0)$ and $h_2(z) = f_2(z) - zf_2'(0)$, using the following relation from [22],

$$(1 - |z|^2)^{\alpha+1}|f''(z)| + |f'(0)| \sim (1 - |z|^2)^{\alpha+1}|f'(z)|,$$

it can be shown that $h_1, h_2 \in \mathcal{B}^\alpha$ and

$$\frac{C}{(1 - |z|^2)^{\alpha+1}} \leq |h_1''(z)| + |h_2''(z)|, \quad z \in \mathbb{D}.$$

By repeating the above method, we have the following:

Lemma 2.1.([4, 23]) *Let $\alpha > 0$ and $n \in \mathbb{N}$. There exist two holomorphic functions $h_1, h_2 \in \mathcal{B}^\alpha$ such that*

$$\frac{C}{(1-|z|^2)^{\alpha+n-1}} \leq |h_1^{(n)}(z)| + |h_2^{(n)}(z)|, \quad z \in \mathbb{D},$$

where C is a positive constant.

For achieving the boundedness of $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \longrightarrow \mathcal{D}_\beta^p$ we need to the following result from [23]:

Lemma 2.2. *Let $\alpha > 0$ and $n \in \mathbb{N}$. If $f \in \mathcal{B}^\alpha$, then*

$$|f(z)| \leq C \begin{cases} \|f\|_{\mathcal{B}^\alpha} & 0 < \alpha < 1 \\ \|f\|_{\mathcal{B}^\alpha} \ln \frac{2}{1-|z|^2} & \alpha = 1 \\ \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & \alpha > 1 \end{cases}$$

and

$$|f^{(n)}(z)| \leq \frac{C\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+n-1}},$$

where C is a positive constant.

Theorem 2.3. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ be a holomorphic self-map of $\mathbb{D}$, $0 < \alpha, p < \infty$ and $\beta > -1$. Then the following statements are equivalent:*

(i) *$I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \longrightarrow \mathcal{D}_\beta^p$ is bounded.*

(ii) *$I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \longrightarrow \mathcal{D}_\beta^p$ is bounded.*

(iii)

$$M = \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} \left(\log \frac{1}{|z|}\right)^\beta \, dA(z) < \infty.$$

Proof. (i) $\Longrightarrow$ (ii) is trivial, since $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(ii) $\Longrightarrow$ (iii). First, note that if $h \in \mathcal{B}^\alpha$, then by defining h_s as $h_s(z) = h(sz)$ for every $z \in \mathbb{D}$ and $s \in (0,1)$, $h_s \in \mathcal{B}_0^\alpha$ and $\|h_s\|_{\mathcal{B}_0^\alpha} \leq \|h\|_{\mathcal{B}^\alpha}$. By Lemma 2.1, there are two holomorphic functions $h_1, h_2 \in \mathcal{B}^\alpha$ such that the following inequality holds:

$$\frac{C}{(1-|z|^2)^{\alpha+n-1}} \leq |h_1^{(n)}(z)| + |h_2^{(n)}(z)|, \quad z \in \mathbb{D}. \tag{2.1}$$

So, by (2.1),

$$\begin{aligned}\int_{\mathbb{D}} \frac{|sg(z)|^p}{(1-|s\varphi(z)|^2)^{p(\alpha+n-1)}} \left(\log\frac{1}{|z|}\right)^{\beta} dA(z) &\\ \leq\ & C\int_{\mathbb{D}} \left|h_{1s}^{(n)}(\varphi(z))\right|^p |sg(z)|^p \left(\log\frac{1}{|z|}\right)^{\beta} dA(z)\\ & + C\int_{\mathbb{D}} \left|h_{1s}^{(n)}(\varphi(z))\right|^p |sg(z)|^p \left(\log\frac{1}{|z|}\right)^{\beta} dA(z)\\ \leq\ & C\left(\|I_{g,\varphi}^{(n)}(h_{1s})\|_{\mathcal{D}_\beta^p}^p + \|I_{g,\varphi}^{(n)}(h_{2s})\|_{\mathcal{D}_\beta^p}^p\right),\end{aligned}$$

for every $s \in (0,1)$. Since boundedness of $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \longrightarrow \mathcal{D}_\beta^p$ implies that $\|I_{g,\varphi}^{(n)}(h_{1s})\|_{\mathcal{D}_\beta^p}^p < \infty$ and $\|I_{g,\varphi}^{(n)}(h_{2s})\|_{\mathcal{D}_\beta^p}^p < \infty$, so, by an application of Fatou's Lemma,

$$M = \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} \left(\log\frac{1}{|z|}\right)^{\beta} dA(z) < \infty.$$

(iii) $\Longrightarrow$ (i). From Lemma 2.2 we have

$$\left|f^{(n)}(z)\right| \leq C\frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+n-1}}\ , \tag{2.2}$$

for every $f \in \mathcal{B}^\alpha$. This implies that

$$\begin{aligned}\|I_{g,\varphi}^{(n)} f\|_{\mathcal{D}_\beta^p}^p &= \int_{\mathbb{D}} \left|f^{(n)}(\varphi(z))\right|^p |g(z)|^p \left(\log\frac{1}{|z|}\right)^{\beta} dA(z)\\ &\leq C\|f\|_{\mathcal{B}^\alpha}^p \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}^{p} \left(\log\frac{1}{|z|}\right)^{\beta} dA(z)\\ &= CM\|f\|_{\mathcal{B}^\alpha}^p < \infty.\end{aligned}$$

Therefore, $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \longrightarrow \mathcal{D}_\beta^p$ is bounded. □

Now, we investigate the compactness of $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \longrightarrow \mathcal{D}_\beta^p$. For this investigation we need to the following Lemma which can be found for example in [17].

Lemma 2.4. *Let X and Y be Banach spaces of holomorphic functions on $\mathbb{D}$. Suppose that*

(i) *The point evaluation functions on X are continuous.*

(ii) *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*

(iii) *$T : X \longrightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \longrightarrow 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

For $X = \mathcal{B}^\alpha$ and $Y = \mathcal{D}^p_\beta$, the above Lemma can be applied. So it follows that:

Lemma 2.5. *Let $T : \mathcal{B}^\alpha \longrightarrow \mathcal{D}^p_\beta$ be a bounded operator. Then, T is compact if and only if given a bounded sequence $\{f_n\}$ in $\mathcal{B}^\alpha$ such that $f_n \longrightarrow 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of $\mathcal{D}^p_\beta$.*

By the following result we characterize the compactness of $I^{(n)}_{g,\varphi} : \mathcal{B}^\alpha \longrightarrow \mathcal{D}^p_\beta$.

Theorem 2.6. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ be a holomorphic self-map of $\mathbb{D}$, $0 < \alpha, p < \infty$ and $\beta > -1$. Then the following statements are equivalent:*

(i) $I^{(n)}_{g,\varphi} : \mathcal{B}^\alpha \longrightarrow \mathcal{D}^p_\beta$ *is compact.*

(ii) $I^{(n)}_{g,\varphi} : \mathcal{B}^\alpha_0 \longrightarrow \mathcal{D}^p_\beta$ *is compact.*

(iii) $I^{(n)}_{g,\varphi} : \mathcal{B}^\alpha_0 \longrightarrow \mathcal{D}^p_\beta$ *is weakly compact.*

(iv)

$$M = \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \infty, \tag{2.3}$$

and

$$\lim_{t \longrightarrow 1} \int_{|\varphi(z)|>t} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} \left(\log \frac{1}{|z|}\right)^\beta dA(z) = 0. \tag{2.4}$$

Proof. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longleftrightarrow$ (iii). Clearly, $I^{(n)}_{g,\varphi} : \mathcal{B}^\alpha_0 \longrightarrow \mathcal{D}^p_\beta$ is weakly compact if and only if its adjoint, i.e. $\left(I^{(n)}_{g,\varphi}\right)^* : \left(\mathcal{D}^p_\beta\right)^* \longrightarrow (\mathcal{B}^\alpha_0)^*$ is weakly compact. According to [21], $(\mathcal{B}^\alpha_0)^* = \mathcal{A}^1_\beta$. Since $\mathcal{A}^1_\beta$ satisfies in the Schur property, $\left(I^{(n)}_{g,\varphi}\right)^* : \left(\mathcal{D}^p_\beta\right)^* \longrightarrow (\mathcal{B}^\alpha_0)^*$ is compact. Thus $I^{(n)}_{g,\varphi} : \mathcal{B}^\alpha_0 \longrightarrow \mathcal{D}^p_\beta$ is compact.

(iii) $\Longrightarrow$ (iv). Assume that $I^{(n)}_{g,\varphi} : \mathcal{B}^\alpha_0 \longrightarrow \mathcal{D}^p_\beta$ is (weakly) compact. Then Theorem 2.3 implies that (2.3) holds. Let $f_k(z) = \frac{z^k}{k^{1-\alpha}}$ for $k \in \mathbb{N}$ and $z \in \mathbb{D}$. Then $\{f_k\} \subset \mathcal{B}^\alpha_0$ is a norm bounded sequence and $f_k \longrightarrow 0$ as $k \longrightarrow \infty$ for every $k \in \mathbb{N}$ on any compact subset of $\mathbb{D}$. Thus, by Lemma 2.5, we have

$$\lim_{k \longrightarrow \infty} \|I^{(n)}_{g,\varphi} f_k\|_{\mathcal{D}^p_\beta} = 0. \tag{2.5}$$

Hence, for every $\varepsilon > 0$ there is an N such that for every $k \geq N$,

$$\lim \left(\frac{k^{\alpha}(k-1)!}{(k-n)!} \right)^p \int_{\mathbb{D}} |\varphi(z)|^{p(k-n)} |g(z)|^p \log \left(\frac{1}{|z|} \right)^{\beta} dA(z) < \varepsilon. \tag{2.6}$$

Thus, for each $r \in (0,1)$,

$$\left(\frac{N^{\alpha}(N-1)!}{(N-n)!} \right)^p r^{p(N-n)} \int_{|\varphi(z)|>r} |g(z)|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) < \varepsilon. \tag{2.7}$$

If we choose $r \geq \left(\frac{(N-n)!}{(N-1)!} \right)^{\frac{1}{(N-n)}} N^{-\frac{\alpha}{N-n}}$, then we have

$$\int_{|\varphi(z)|>r} |g(z)|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) < \varepsilon. \tag{2.8}$$

Let $f \in \mathbb{B}_{\mathcal{B}_0^{\alpha}}$, where $\mathbb{B}_{\mathcal{B}_0^{\alpha}}$ is the unit ball of $\mathcal{B}_0^{\alpha}$. The compactness of $I_{g,\varphi}^{(n)} : \mathcal{B}_0^{\alpha} \longrightarrow \mathcal{D}_{\beta}^p$ implies that for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that

$$\int_{\mathbb{D}} \left| \left(I_{g,\varphi}^{(n)}(f - f_t) \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) < \varepsilon, \tag{2.9}$$

where $f_t(z) = f(tz)$, $z \in \mathbb{D}$.

So, (2.8) and (2.9) imply that

$$\begin{aligned}
&\int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)} f \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) \\
\leq \;& C \int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)}(f - f_t) \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) \\
& + C \int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)}(f_t) \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) \\
\leq \;& C \int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)}(f - f_t) \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) \\
& + C \int_{|\varphi(z)|>r} \left| f_t^{(n)}(\varphi(z)) \right|^p |g(z)|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) \\
\leq \;& C\varepsilon + C\varepsilon \sup_{z \in \mathbb{D}} |f_t^{(n)}(z)|^p \\
= \;& C\varepsilon (1 + \sup_{z \in \mathbb{D}} |f_t^{(n)}(z)|^p),
\end{aligned}$$

where C is a positive constant. Thus, for every $f \in \mathbb{B}_{\mathcal{B}_0^{\alpha}}$ and every $\varepsilon > 0$, there is a $\delta = \delta(f, \varepsilon)$ (depended on f and ε) such that for every $r \in [\delta, 1)$ we have

$$\int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)} f \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^{\beta} dA(z) < \varepsilon. \tag{2.10}$$

The compactness of $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \longrightarrow \mathcal{D}_\beta^p$ leads that $I_{g,\varphi}^{(n)}\left(\mathbb{B}_{\mathcal{B}_0^\alpha}\right)$ is a relatively compact subset of $\mathcal{D}_\beta^p$. Hence, for every $\varepsilon > 0$ there exists a finite family of functions $f_1, \dots, f_N \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ such that for every $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$, $\|I_{g,\varphi}^{(n)} f - I_{g,\varphi}^{(n)} f_i\|_{\mathcal{D}_\beta^p} < \varepsilon$ for $i \in \{1, \dots, N\}$. i.e.,

$$\int_{\mathbb{D}} \left| \left(I_{g,\varphi}^{(n)}(f - f_i) \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) < \varepsilon. \tag{2.11}$$

Hence, putting $\delta = \max_{1 \le i \le N} \delta(f_i, \varepsilon)$, for any $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ we have

$$\int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)} f_i \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) < C\varepsilon, \tag{2.12}$$

if $r \in [\delta, 1)$.

Applying (2.12) to functions $(f_i)_s(z) = f_i(sz)$ for $i = 1, 2$ (the functions are as in Lemma 2.1), we obtain

$$\begin{aligned}
&\int_{|\varphi(z)|>r} \frac{|sg(z)|^p}{(1 - |s\varphi(z)|^2)^{p(\alpha+n-1)}} \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) \\
\le\ & C \int_{|\varphi(z)|>r} \left| f_1^{(n)}(s\varphi(z)) \right|^p |sg(z)|^p \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) \\
& + C \int_{|\varphi(z)|>r} \left| f_2^{(n)}(s\varphi(z)) \right|^p |sg(z)|^p \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) \\
\le\ & \frac{C}{\|f_{1s}\|_{\mathcal{B}_0^\alpha}^p} \int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)} f_{1s} \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) \\
& + \frac{C}{\|f_{2s}\|_{\mathcal{B}_0^\alpha}^p} \int_{|\varphi(z)|>r} \left| \left(I_{g,\varphi}^{(n)} f_{2s} \right)'(z) \right|^p \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) \\
<\ & C\varepsilon,
\end{aligned}$$

for all $r \in [\delta, 1)$. By Fatou's Lemma, this estimate implies (2.4).

(iv) $\Longrightarrow$ (i). Let $\{f_k\}$ be a bounded sequence in $\mathcal{B}^\alpha$ converges to zero on compact subsets of $\mathbb{D}$ as $k \longrightarrow \infty$. Cauchy's estimate implies that for any $n \in \mathbb{N}$, $\{f_k^{(n)}\}$ also converges to zero on compact subset of $\mathbb{D}$ as $k \longrightarrow \infty$. In particular

$$\lim_{k \longrightarrow \infty} \sup_{|w| \le r} \left| f_k^{(n)}(w) \right| = 0. \tag{2.13}$$

By hypothesis, for every $\varepsilon > 0$ there is $r \in (0, 1)$ such that,

$$\int_{|\varphi(z)|>r} \frac{|g(z)|^p}{(1 - |\varphi(z)|^2)^{p(\alpha+n-1)}} \left(\log \frac{1}{|z|} \right)^\beta \, dA(z) < \varepsilon. \tag{2.14}$$

Taking the function $f(z)=z^n$, boundedness of $I_{g,\varphi}^{(n)}$ implies that

$$L=\int_{\mathbb{D}}|g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta\,dA(z)<\infty. \tag{2.15}$$

So, using Lemma 2.2 and relations (2.14) and (2.15),

$$\begin{aligned}\|I_{g,\varphi}^{(n)}f_k\|_{\mathcal{D}_\beta^p}^p &= \int_{\mathbb{D}}\left|\left(I_{g,\varphi}^{(n)}f_k\right)'(z)\right|^p\left(\log\frac{1}{|z|}\right)^\beta\,dA(z)\\ &= \int_{|\varphi(z)|\le r}\left|f_k^{(n)}(\varphi(z))\right|^p|g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta\,dA(z)\\ &\quad+\int_{r<|\varphi(z)|<1}\left|f_k^{(n)}(\varphi(z))\right|^p|g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta\,dA(z)\\ &\le L\sup_{|\varphi(z)|\le r}\left|f_k^{(n)}(\varphi(z))\right|^p+C\varepsilon\|f_k\|_{\mathcal{B}^\alpha}^p.\end{aligned}$$

Letting $k\longrightarrow\infty$ and using (2.13), we conclude that $\|I_{g,\varphi}^{(n)}f_k\|_{\mathcal{D}_\beta^p}\longrightarrow 0$. Thus, by Lemma 2.5, $I_{g,\varphi}^{(n)}:\mathcal{B}^\alpha\longrightarrow\mathcal{D}_\beta^p$ is compact. □

3. Boundednes and Compactness of the Operator $I_{g,\varphi}^{(n)}:\mathcal{D}_\beta^p\longrightarrow\mathcal{B}^\alpha$

In this section we study the boundedness and compactness of the generalized integration operator $I_{g,\varphi}^{(n)}:\mathcal{D}_\beta^p\longrightarrow\mathcal{B}^\alpha$.

For every $a\in\mathbb{D}$ the holomorphic mapping from $\mathbb{D}$ onto $\mathbb{D}$ is defined by $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$.

Lemma 3.1.([19]) *Let $\beta>-1$, $0<p<\infty$ and $f\in\mathcal{A}_\beta^p$. Then*

$$|f(z)|\left(1-|z|^2\right)^{\frac{2+\beta}{p}}\le\left((1+\beta)\int_{\mathbb{D}}|f(z)|^p\left(1-|z|^2\right)^\beta\,dA(z)\right)^{\frac{1}{p}}\quad z\in\mathbb{D},$$

with equality if and only if f is a constant multiple of the function $f_a(z)=(-\sigma_a'(z))^{\frac{2+\beta}{p}}$.

We recall the following fundamental lemma from [21]:

Lemma 3.2.([21, Lemma 4.2.2]) *Suppose $z\in\mathbb{D}$, c is real, $t>-1$ and*

$$I_{c,t}(z)=\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^t}{|1-z\overline{w}|^{2+t+c}}\,dA(w).$$

(a) *If $c<0$, then as a function of z, $I_{c,t}(z)$ is bounded on $\mathbb{D}$.*

(b) *If* $c > 0$, *then*

$$I_{c,t}(z) \sim \frac{1}{\left(1-|z|^2\right)^c}, \qquad |z| \longrightarrow 1^-.$$

(c) *If* $c = 0$, *then*

$$I_{0,t}(z) \sim \log \frac{1}{1-|z|^2}, \qquad |z| \longrightarrow 1^-.$$

Let $0 < p < \infty$, $\beta > -1$ and $f \in \mathcal{A}^p_\beta$. Then

$$|f(z)| \le \frac{\|f\|_{\mathcal{A}^p_\beta}}{\left(1-|z|^2\right)^{\frac{2+\beta}{p}}}, \tag{3.1}$$

for $z \in \mathbb{D}$ ([21]). Also, for $f \in \mathcal{A}^p_\beta$ and $z \in \mathbb{D}$, we have

$$f(z) = (\beta+1)\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^\beta f(w)}{(1-z\overline{w})^{2+\beta}}\, dA(w), \tag{3.2}$$

See 4.2.1 of [21]. Differentiating under the integral sign n times, we obtain a constant $K_n > 0$ such that

$$f^{(n)}(z) = K_n \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^\beta}{(1-z\overline{w})^{n+2+\beta}}\overline{w}^n f(w)\, dA(w). \tag{3.3}$$

Lemma 3.3. *Let* $0 < p < \infty$, $\beta > -1$, $m \in \mathbb{N}$ *and* $f \in \mathcal{A}^p_\beta$. *Then there exists a constant* $C > 0$ *such that*

$$|f^{(m)}(z)| \le C\frac{\|f\|_{\mathcal{A}^p_\beta}}{\left(1-|z|^2\right)^{\frac{2+\beta}{p}+m}}. \tag{3.4}$$

Proof. By (3.1), (3.3) and setting $t = \beta - \frac{2+\beta}{p}$ in Lemma 3.2, we have

$$\begin{aligned}
|f^{(m)}(z)| &\le K_m \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^\beta |\overline{w}^m|}{|1-z\overline{w}|^{2+m+\beta}}|f(w)|\, dA(w)\\
&\le K_m \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^\beta}{|1-z\overline{w}|^{2+m+\beta}} \cdot \frac{\|f\|_{\mathcal{A}^p_\beta}}{\left(1-|w|^2\right)^{\frac{2+\beta}{p}}}\, dA(w)\\
&= K_m \|f\|_{\mathcal{A}^p_\beta} \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^t}{|1-z\overline{w}|^{2+t+(m+\frac{2+\beta}{p})}}\, dA(w)\\
&\sim K_m \frac{\|f\|_{\mathcal{A}^p_\beta}}{\left(1-|z|^2\right)^{\frac{2+\beta}{p}+m}}.
\end{aligned}$$

So, there exists a constant C such that

$$|f^{(m)}(z)| \leq C \frac{\|f\|_{\mathcal{A}^p_\beta}}{(1-|z|^2)^{\frac{2+\beta}{p}+m}}. \qquad \square$$

Theorem 3.4. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$ and φ be a holomorphic self-map of $\mathbb{D}$, $0 < \alpha, p < \infty$ and $\beta > -1$. Then the following statements are equivalent:*

(i) $I^{(n)}_{g,\varphi} : \mathcal{D}^p_\beta \longrightarrow \mathcal{B}^\alpha$ *is bounded.*

(ii)

$$M = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}} < \infty. \tag{3.5}$$

Proof. (ii) $\Longrightarrow$ (i). Let $f \in \mathcal{D}^p_\beta$. Then $f' \in \mathcal{A}^p_\beta$ and from Lemma 3.3, there is a constant $C > 0$ such that

$$|f^{(n)}(z)| = \left|(f'(z))^{(n-1)}\right| \leq C \frac{\|f'\|_{\mathcal{A}^p_\beta}}{(1-|z|^2)^{\frac{2+\beta}{p}+n-1}} \leq C \frac{\|f\|_{\mathcal{D}^p_\beta}}{(1-|z|^2)^{\frac{2+\beta}{p}+n-1}}, \tag{3.6}$$

for $z \in \mathbb{D}$. By (3.6),

$$\begin{aligned} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|\left(I^{(n)}_{g,\varphi} f\right)'(z)\right| &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f^{(n)}(\varphi(z))\right| |g(z)| \\ &\leq C \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}} \|f\|_{\mathcal{D}^p_\beta} \\ &\leq CM\|f\|_{\mathcal{D}^p_\beta}. \end{aligned}$$

Hence, $I^{(n)}_{g,\varphi} : \mathcal{D}^p_\beta \longrightarrow \mathcal{B}^\alpha$ is bounded.

(i) $\Longrightarrow$ (ii). Assume that (i) holds. Taking the function $f(z) = \frac{z^n}{n!}$, the boundedness of $I^{(n)}_{g,\varphi}$ implies that

$$L = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |g(z)| < \infty. \tag{3.7}$$

Define the functions f_a for every $a \in \mathbb{D}$ as follows:

$$f_a(z) = \int_0^z \left(\frac{1-|a|^2}{(1-\overline{a}w)^2}\right)^{\frac{\beta+2}{p}} dw.$$

Then Lemma 3.1 implies that $f_a \in \mathcal{D}^p_\beta$ and$\|f_a\|_{\mathcal{D}^p_\beta} \sim 1$. The boundedness of $I^{(n)}_{g,\varphi} : \mathcal{D}^p_\beta \longrightarrow \mathcal{B}^\alpha$ implies that there exists a constant $C > 0$ such that $\|I^{(n)}_{g,\varphi} f_a\|_{\mathcal{B}^\alpha} \leq$

$C\|f_a\|_{\mathcal{D}^p_\beta} \leq C$. Also, it is easy to see that for any $n \in \mathbb{N}$ and $z \in \mathbb{D}$,

$$f_a^{(n+1)}(z) = \frac{c_{n+1}\overline{a}^n \left(1-|a|^2\right)^{\frac{\beta+2}{p}}}{(1-\overline{a}z)^{\frac{2(\beta+2)}{p}+n}}, \tag{3.8}$$

where $c_{n+1} = \Pi_{m=0}^{n-1}(\frac{2(\beta+2)}{p}+m)$. Since

$$f_a'(z) = \frac{\left(1-|a|^2\right)^{\frac{\beta+2}{p}}}{(1-\overline{a}z)^{\frac{2(\beta+2)}{p}}},$$

so, $f_a'(z)$ follows from (3.8) by letting $n=0$ and $c_1 = 1$.
Since $I_{g,\varphi}^{(n)} f_a(0) = 0$, letting $a = \varphi(z)$ and using (3.8),

$$\begin{aligned}
C &\geq \|I_{g,\varphi}^{(n)} f_{\varphi(z)}\| \geq \|I_{g,\varphi}^{(n)} f_{\varphi(z)}\|_{\mathcal{B}^\alpha} \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left| f_{\varphi(z)}^{(n)}(\varphi(z))\right| |g(z)| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left| \frac{c_n|\overline{\varphi(z)}|^{n-1}\left(1-|\varphi(z)|^2\right)^{\frac{\beta+2}{p}}}{(1-|\varphi(z)|^2)^{\frac{2(\beta+2)}{p}+n-1}}\right| |g(z)| \\
&\geq \frac{c_n|\varphi(z)|^{n-1}\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}}
\end{aligned}$$

This shows that

$$\sup_{z\in\mathbb{D}} \frac{|\varphi(z)|^{n-1}\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}} < \infty. \tag{3.9}$$

For any $\delta, 0<\delta<1$, by (3.9),

$$\sup_{|\varphi(z)|>\delta} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}} < \infty.$$

For $z \in \mathbb{D}$ such that $|\varphi(z)| \leq \delta$, we have

$$\frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}} \leq \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-\delta^2)^{\frac{\beta+2}{p}+n-1}}. \tag{3.10}$$

Hence, from (3.7) and (3.10), we have

$$\sup_{|\varphi(z)|\leq\delta} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}} < \infty.$$

So,

$$\sup_{z\in\mathcal{D}} \frac{\left(1-|z|^2\right)^{\alpha}|g(z)|}{\left(1-|\varphi(z)|^2\right)^{\frac{\beta+2}{p}+n-1}} < \infty.$$

Thus, (3.5) holds and the proof of the theorem is completed. □

Now, we investigate the compactness of $I_{g,\varphi}^{(n)} : \mathcal{D}_{\beta}^{p} \longrightarrow \mathcal{B}^{\alpha}$. We use the following lemma which can be obtained from 2.4 by taking $X = \mathcal{D}_{\beta}^{p}$ and $Y = \mathcal{B}^{\alpha}$.

Lemma 3.5. *Let $T : \mathcal{D}_{\beta}^{p} \longrightarrow \mathcal{B}^{\alpha}$ be a bounded operator. Then, T is compact if and only if given a bounded sequence $\{f_n\}$ in $\mathcal{D}_{\beta}^{p}$ such that $f_n \longrightarrow 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of $\mathcal{B}^{\alpha}$.*

Theorem 3.6. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ be a holomorphic self-map of $\mathbb{D}$, $0 < \alpha, p < \infty$ and $\beta > -1$. If $\|\varphi\|_{\infty} < 1$ and $I_{g,\varphi}^{(n)} : \mathcal{D}_{\beta}^{p} \longrightarrow \mathcal{B}^{\alpha}$ is bounded, Then $I_{g,\varphi}^{(n)}$ is compact.*

Proof. Since $I_{g,\varphi}^{(n)} : \mathcal{D}_{\beta}^{p} \longrightarrow \mathcal{B}^{\alpha}$ is bounded, Theorem 3.4 implies that

$$M = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^{\alpha}|g(z)|}{\left(1-|\varphi(z)|^2\right)^{\frac{\beta+2}{p}+n-1}} < \infty.$$

Let $\{f_k\}$ be a bounded sequence in the unit ball of $\mathcal{D}_{\beta}^{p}$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \longrightarrow \infty$. Then, Cauchy's estimate implies that $\{f_k^{(n)}\}$ for $n \in \mathbb{N}$ also converges uniformly to 0 on compact subset of $\mathbb{D}$ as $k \longrightarrow \infty$. This implies that,

$$\lim_{k\longrightarrow\infty} \sup_{w\in\varphi(\mathbb{D})} \left|f_k^{(n)}(w)\right| = 0.$$

So,

$$\begin{aligned}
\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}^{\alpha}} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} \left|\left(I_{g,\varphi}^{(n)} f_k\right)'(z)\right| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} \left|f_k^{(n)}(\varphi(z))\right| |g(z)| \\
&\le M \sup_{z\in\mathbb{D}} \left|f_k^{(n)}(\varphi(z))\right| \longrightarrow 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Hence, by Lemma 3.5, $I_{g,\varphi}^{(n)} : \mathcal{D}_{\beta}^{p} \longrightarrow \mathcal{B}^{\alpha}$ is compact. □

Theorem 3.7. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ be a holomorphic self-map of $\mathbb{D}$, $0 < \alpha, p < \infty$ and $\beta > -1$. If $\|\varphi\|_{\infty} = 1$, then the following statements are equivalent:*

(i) $I_{g,\varphi}^{(n)} : \mathcal{D}_{\beta}^{p} \longrightarrow \mathcal{B}^{\alpha}$ *is compact.*

(ii) $I_{g,\varphi}^{(n)} : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ *is bounded and*

$$\lim_{|\varphi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n-1}} = 0. \tag{3.11}$$

Proof. (i) $\Longrightarrow$ (ii). Suppose that $I_{g,\varphi}^{(n)} : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ is compact. Obviously, it is bounded. We consider the function f_a for $a \in \mathbb{D}$ defined as in Theorem 3.4. This function converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \longrightarrow 1$.

Now, pick the sequence $\{z_m\} \subseteq \mathbb{D}$ such that $|\varphi(z_m)| \longrightarrow 1$ as $m \longrightarrow \infty$. Using the test function $f_m(z) = f_{\varphi(z_m)}(z)$, we obtain

$$\begin{aligned}
\|I_{g,\varphi}^{(n)} f_m\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|\left(I_{g,\varphi}^{(n)} f_m\right)'(z)\right| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f_m^{(n)}(\varphi(z))\right| |g(z)| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f_{\varphi(z_m)}^{(n)}(\varphi(z))\right| |g(z)| \\
&\geq \left(1-|z_m|^2\right)^\alpha \left| \frac{|\overline{\varphi(z_m)}|^n \left(1-|\varphi(z_m)|^2\right)^{\frac{\beta+2}{p}}}{\left(1-\overline{\varphi(z_m)}\varphi(z_m)\right)^{\frac{2(\beta+2)}{p}+n}} \right| |g(z)| \\
&\geq \left(1-|z_m|^2\right)^\alpha \left| \frac{|\varphi(z_m)|^{n-1}}{(1-|\varphi(z_m)|^2)^{\frac{\beta+2}{p}+n-1}} \right| |g(z_m)|.
\end{aligned}$$

As we mentioned above, since $f_m = f_{\varphi(z_m)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|\varphi(z_m)| \longrightarrow 1$, from Lemma 3.5 it follows that $\|I_{g,\varphi}^{(n)} f_m\|_{\mathcal{B}^\alpha} \longrightarrow 0$ as $|\varphi(z_m)| \longrightarrow 1$ and so, (3.11) holds.

(ii) $\Longrightarrow$ (i). Let $\{f_k\}$ be a bounded sequence in the unit ball of $\mathcal{D}_\beta^p$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \longrightarrow \infty$. The relation (3.11) implies that for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that

$$\sup_{\{z:\delta<|\varphi(z)|<1\}} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}+n}} < \varepsilon. \tag{3.12}$$

Also, the uniform convergence of $\{f_k\}$ on compact subset of $\mathbb{D}$ together with Cauchy's estimate, implies that $\{f_k^{(n)}\}$ for $n \in \mathbb{N}$ converges to 0 on compact subset of $\mathbb{D}$ as $k \longrightarrow \infty$. This implies that

$$\lim_{k\longrightarrow\infty} \sup_{|w|\leq\delta} \left|f_k^{(n)}(w)\right| = 0. \tag{3.13}$$

Then, by (3.6), (3.7) and (3.12) we get the following:

$$
\begin{aligned}
\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left| \left(I_{g,\varphi}^{(n)} f_k\right)'(z)\right| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left| f_k^{(n)}(\varphi(z))\right| |g(z)| \\
&= \sup_{|\varphi(z)|\le\delta} \left(1-|z|^2\right)^\alpha \left| f_k^{(n)}(\varphi(z))\right| |g(z)| \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \left(1-|z|^2\right)^\alpha \left| f_k^{(n)}(\varphi(z))\right| |g(z)| \\
&\le L \sup_{|w|\le\delta} \left| f_k^{(n)}(w)\right| \\
&\quad + C\|f_k\|_{\mathcal{D}_\beta^p} \sup_{\delta<|\varphi(z)|<1} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{\left(1-|\varphi(z)|^2\right)^{\frac{\beta+2}{p}+n-1}} \\
&\le L \sup_{|\varphi(z)|\le\delta} \left| f_k^{(n)}(\varphi(z))\right| + C\varepsilon\|f_k\|_{\mathcal{D}_\beta^p}.
\end{aligned}
$$

Letting $k \longrightarrow \infty$ and using (3.13), it folloes that $\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}^\alpha} \longrightarrow 0$. Thus, Lemma 3.5 implies that $I_{g,\varphi}^{(n)} : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ is compact. □

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