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Submanifolds of Sasaki-like Almost Contact Manifolds with B-metric

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Abstract. In this paper, we introduce the geometry of contact CR submanifolds and radical transversal lightlike submanifolds of Sasaki-like almost contact manifolds with B-metric. We obtain some new results that establish a relationship between these two submanifolds.

1. Introduction

Bejancu [1] initiated the study of CR-submanifolds with Kaehler manifolds. Yano and Kon [10] introduced the concept of odd dimensional manifolds called the contact CR submanifolds of a Sasakian manifold. They proved some basic results for contact CR submanifolds of a Sasakian manifold with definite metric. Further, Matsumoto [8] studied such manifolds and obtained some fundamental results. Ganchev et al. [6] introduced the geometry of almost contact B metric manifolds which are a natural extension, to the odd dimensional case, of the geometry of the almost complex manifolds with B-metric.

Duggal and Sahin [5] defined and studied lightlike submanifolds of indefinite Sasakian manifolds and introduced radical tranversal lightlike submanifolds of indefinite Sasakian manifolds. Recently, Nakova [9] studied submanifolds of almost complex manifolds with Norden metric which are non-degenerate with respect to one Norden metric and lightlike with respect to other Norden metric on the manifold and introduced radical transversal lightlike submanifold of almost complex manifolds with the Norden metric. Ivanov et al[7] defined Sasaki-like almost contact Complex Riemannian manifolds that resemble with the Sasakian manifold and thus motivated us to study such manifolds.

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In this paper, we study the geometry of contact CR submanifolds and radical transversal lightlike submanifolds of Sasaki-like almost contact manifold with B-metric. We investigate conditions for the integrability of distributions of contact CR-submanifolds and radical transversal lightlike submanifolds of Sasaki like almost contact manifold with B-metric. We find the necessary and sufficient condition for integrability of screen distribution of radical transversal lightlike submanifold with B-metric. Further, we obtain some new results that establish relationship between the concerned geometric objects of both the submanifolds of Sasaki-like almost contact manifold with B-metric. Finally, we prove that for a contact CR-product submanifold of Sasaki like almost contact manifold with B-metric, the induced connection $\tilde{\nabla}$ of radical transversal lightlike submanifold with Norden metric is a metric connection.

2. Almost Contact Manifold with B-metric

Let $(\bar{M}^{2n+1}, \varphi, \zeta, \eta)$ be an almost contact manifold with B-metric [6], that is, let (φ, ζ, η) be an almost contact structure consisting of a tensor field φ of type $(1,1)$ a vector field ζ, a 1-form η and a metric $\bar{g}$ on $\bar{M}$ satisfying the following algebraic conditions for arbitrary vector fields X and Y on $\bar{M}$:

$$\varphi^2(X) = -X + \eta(X)\zeta, \quad \eta(\zeta) = 1, \quad \varphi\zeta = 0, \quad \eta o\zeta = 0. \tag{2.1}$$

$$\bar{g}(\varphi X, \varphi Y) = -\bar{g}(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

The following identities are valid for an almost contact manifold with B-metric.

$$\eta(X) = g(X, \zeta), \quad g(\varphi X, Y) = g(X, \varphi Y). \tag{2.3}$$

The associated metric $\bar{\bar{g}}$ of $\bar{g}$ on $\bar{M}$ defined by

$$\bar{\bar{g}}(X, Y) = \bar{g}(\varphi X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

is also a B-metric on $\bar{M}$ and the manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{\bar{g}})$ is also called an almost contact manifold with B-metric. Both the metrics $\bar{g}$ and $\bar{\bar{g}}$ are indefinite of signature $(n+1, n)$.

Let $\bar{\nabla}$ and $\bar{\bar{\nabla}}$ be the Levi-Civita connection of $\bar{g}$ and $\bar{\bar{g}}$ respectively. The tensor field F of type $(0,3)$ is defined on $\bar{M}$ by

$$F(X, Y, Z) = \bar{g}((\bar{\nabla}_X \varphi)Y, Z)$$

and the following general properties hold [6]:

$$\begin{aligned} F(X, Y, Z) &= F(X, Z, Y) \\ F(X, \varphi Y, \varphi Z) &= F(X, Y, Z) + \eta(Y)F(X, \zeta, Z) + \eta(Z)F(X, Y, \zeta), \end{aligned} \tag{2.5}$$

for any $X, Y, Z \in T\bar{M}$. The relations of F with $\nabla\zeta$ and $\nabla\eta$ are given by :

$$(\nabla_X \eta)Y = g(\nabla_X \zeta, Y) = F(X, \varphi Y, \zeta), \quad \eta(\nabla_X \zeta) = 0, \quad \varphi(\nabla_X \varphi)\zeta = \nabla_X \zeta \tag{2.6}$$

Let $\{e_i, \zeta\}$, $(i = 1, 2, ..., 2n)$ be a basis of $T_m M$ and (g^{ij}) be the inverse matrix of g_{ij} then for $X \in T_m\bar{M}$, the following 1-forms are associated with F :

$$\theta(X) = g^{ij}F(e_i, e_j, X), \quad \theta^*(X) = g^{ij}F(e_i, \varphi e_j, X), \quad \omega(X) = F(\zeta, \zeta, X), \tag{2.7}$$

Using above equation, we have

$$\omega(\zeta) = 0 \quad \theta^*(\varphi X) = -\theta(\varphi^2 X) - \omega(X).$$

The Nijenhuis tensor N of the almost complex structure is defined by

$$N = [\varphi, \varphi] + d\eta \otimes \zeta, \tag{2.8}$$

$$N(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + d\eta \otimes \zeta.$$

An almost contact structure (φ, ζ, η) is said to be *normal* if and only if Nijenhuis tensor denoted by N vanishes [2] and such manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ is called normal almost contact manifold.

In [6] Ganchev et al. defined eleven basic classes $F_i(i = 1, 2, ...11)$ of almost contact manifolds with B-metric and gave a classification of almost contact manifolds with B-metric with respect to tensor F. The special class F_0 defined by the condition $F(X, Y, Z) = 0$ belongs to everyone of the basic classes. Throughout this paper, we will consider the class F_0.

Definition 2.1.([7]) An almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B- metric is *Sasaki-like* if the structure tensors $\varphi, \zeta, \eta, \bar{g}$ satisfy the following equalities

$$F(X, Y, Z) = F(\zeta, Y, Z) = F(\zeta, \zeta, Z) = 0, \tag{2.9}$$

$$F(X, Y, \zeta) = -g(X, Y). \tag{2.10}$$

Also, the covariant derivative $\bar{\nabla}\varphi$ satisfies the following equality

$$(\bar{\nabla}_X \varphi)Y = -\bar{g}(X, Y)\zeta - \eta(Y)X + 2\eta(X)\eta(Y)\zeta \tag{2.11}$$

In this paper, we refer to these manifolds as indefinite Sasaki-like almost contact manifolds with B-metric.

3.Contact CR-submanifolds of Sasaki-like Almost Contact Manifolds with B-metric

In this section, we define and study contact CR-submanifolds of Sasaki-like almost contact manifolds with B-metric and investigate their integrability conditions.

Definition 3.1. A submanifold (M, g) of a $(2n+1)$-dimensional Sasaki-like almost contact manifold with B-metric $\bar{M}$ with structure tensors (φ, ζ, η) is called a *contact CR-submanifold* if there exists a differentiable distribution $D : x \longrightarrow D_x \subseteqq T_xM$ and the complementary orthogonal distribution $D^\perp : x \longrightarrow D_x^\perp \subset T_xM$ on M which satisfies the following conditions:

(i) $\zeta \in D$,

(ii) $\varphi D_x \subset T_xM$ for each $x \in M$,

(iii) $\varphi D_x^\perp \subseteqq T_xM^\perp$ for each point $x \in M$.

The complementary orthogonal distribution of $\varphi D^\perp$ in $TM^\perp$ is denoted by π. The tangent bundle $T\bar{M}$ of $\bar{M}$ has the following decomposition

$$T\bar{M} = TM \perp TM^\perp = TM \perp \varphi D^\perp \perp \pi = D \perp D^\perp \perp \varphi D^\perp \perp \pi.$$

Let E and G be the projection morphisms of TM on the distributions D and $D^\perp$ respectively, then for any $X \in TM$, we can write

$$X = EX + GX, \tag{3.1}$$

where $EX \in D$ and $GX \in D^\perp$. Applying φ to (3.1), we get

$$\varphi X = HX + KX,$$

where $\varphi EX = HX \in D$ and $\varphi GX = KX \in \varphi D^\perp$ are the tangential and the normal components of φX, respectively.

Similarly for any $V \in TM^\perp$, we have

$$\varphi V = tV + fV,$$

where tV and fV are the tangential and the normal parts of φV, respectively.

Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections of $\bar{g}$ and g on $\bar{M}$ and M, respectively. Then the Gauss and Weingarten formulae for $\bar{M}$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3.2}$$

$$\bar{\nabla}_X V = -A_V X + D_X V, \tag{3.3}$$

for any vector fields $X, Y \in TM$ and $V \in TM^\perp$, where h is the second fundamental form of M, A_V is the shape operator of M with respect to V and D is the normal connection on $TM^\perp$ which is a metric linear connection. Since $\zeta \in TM$, we have, for any vector field $X \in TM$,

$$\bar{\nabla}_X \zeta = \varphi X = \nabla_X \zeta + h(X, \zeta), \tag{3.4}$$

and on equating the components of φX, we get

$$\nabla_X \zeta = HX, \quad KX = h(X, \zeta). \tag{3.5}$$

Let E_1 and E_2 be the projection morphisms of $TM^\perp$ on $\varphi D^\perp$ and π, respectively. Then (3.2) and (3.3) can be written as

$$\bar{\nabla}_X Y = \nabla_X Y + h^1(X,Y) + h^2(X,Y), \quad \bar{\nabla}_X V = -A_V X + D^1_X V + D^2_X V, \tag{3.6}$$

where

$$h^1(X,Y) = E_1(h(X,Y)), \qquad h^2(X,Y) = E_2(h(X,Y)),$$
$$D^1_X V = E_1(D_X V), \qquad D^2_X V = E_2(D_X V).$$

It should be noted that D^1 and D^2 are not linear connections on $TM^\perp$ but are Otsuki connections with respect to the vector bundle morphisms E_1 and E_2 respectively. Thus the above equation (3.6) reduces to

$$\bar{\nabla}_X Y = \nabla_X Y + h^1(X,Y) + h^2(X,Y), \tag{3.7}$$
$$\bar{\nabla}_X N = -A_N X + \nabla^1_X N + D^2(X,N), \tag{3.8}$$
$$\bar{\nabla}_X W = -A_W X + D^1(X,W) + \nabla^2_X W, \tag{3.9}$$

where $\nabla^1_X N = D^1_X N$ and $\nabla^2_X W = D^2_X W$ are metric connections on $\varphi D^\perp$ and π, respectively and $D^1(X,W) = D^1_X W$ and $D^2(X,N) = D^2_X N$ are $\mathbf{F}(\bar{M})$ bilinear mappings. Making use of equations (3.7) to (3.9), we have

$$\bar{g}(h^1(X,Y),N) = \bar{g}(Y, A_N X), \tag{3.10}$$
$$\bar{g}(h^2(X,Y),W) = \bar{g}(Y, A_W X), \tag{3.11}$$
$$\bar{g}(D^2(X,N),W) = -\bar{g}(D^1(X,W),N). \tag{3.12}$$

Lemma 3.1. *Let (M,g) be a contact CR-submanifold of an indefinite Sasaki-like almost contact manifold $(\bar{M},\varphi,\zeta,\eta,\bar{g})$ with B-metric. Then we have*

$$(\nabla_X H)Y = A_{KY}X + th^1(X,Y) - g(X,Y)\zeta - \eta(Y)X + 2\eta(Y)\eta(X)\zeta, \tag{3.13}$$
$$(\nabla^1_X K)Y = -h^1(X,HY), \tag{3.14}$$
$$D^2(X,KY) = fh^2(X,Y) - h^2(X,HY), \tag{3.15}$$

where

$$(\nabla_X H)Y = \nabla_X HY - H\nabla_X Y,$$
$$(\nabla^1_X K)Y = \nabla^1_X KY - K\nabla_X Y, \tag{3.16}$$

for any $X,Y \in TM$.

Proof. Since $\bar{M}$ is a Sasaki-like almost contact manifold with B-metric,

$$\bar{\nabla}_X \varphi Y = \varphi(\bar{\nabla}_X Y) + (\bar{\nabla}_X \varphi)Y,$$

for any $X, Y \in TM$, and using (3.7) and (3.8), we obtain

$$\begin{aligned} &\nabla_X HY + h^1(X, HY) + h^2(X, HY) - A_{KY}X + \nabla^1_X KY + D^2(X, KY) \\ &= H\nabla_X Y + K\nabla_X Y + th^1(X, Y) + fh^2(X, Y) \\ &\quad - g(X, Y)\zeta - \eta(Y)X + 2\eta(Y)\eta(X)\zeta. \end{aligned} \tag{3.17}$$

Considering the tangential, $(\varphi D^{\perp})$, and π components, respectively, of the above equation, the lemma follows. □

Lemma 3.2. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B-metric. Then we have*

$$(\nabla^1_X t)N = -HA_N X + A_{fN} X, \tag{3.18}$$

$$h^1(X, tN) = -FA_N X - \nabla^1_X fN, \tag{3.19}$$

$$h^2(X, tN) = -(D^2_X f), N, \tag{3.20}$$

$$-A_{fW} X = HA_W X + tD^1(X, W), \tag{3.21}$$

$$D^1(X, fW) = KA_W X, \tag{3.22}$$

$$\nabla^2_X fW = f\nabla^2_X W, \tag{3.23}$$

for any $X, Y \in TM$, $N \in \varphi D^{\perp}$ and $W \in \pi$.

Proof. Let $N \in \varphi D^{\perp}$ then we have $\bar{\nabla}_X tN = \bar{\varphi}\bar{\nabla}_X N$. By using (3.7) and (3.8), we obtain

$$\begin{aligned} &\nabla_X tN + h^1(X, tN) + h^2(X, tN) \\ &= \bar{\varphi}(-A_N X + \nabla^1_X N + D^2(X, N)) \\ &= -HA_N X - KA_N X + t\nabla^1_X N + fD^2(X, N). \end{aligned}$$

Comparing the tangential components, we get $(\nabla^1_X t)N = -HA_N X + A_{fN} X$, where $(\nabla^1_X t)N = \nabla_X tN - t\nabla^1_X N$ and comparing $\varphi D^{\perp}$ and π components, the relations (3.19) and (3.20) follows respectively.

Next, let $W \in \pi$ and making use of (3.9) we have

$$\begin{aligned} -A_{fW} X + \nabla^2_X fW + D^1(X, fW) &= \bar{\varphi}(-A_W X + \nabla^2_X W + D^1(X, W)) \\ &= -HA_W X - KA_W X + f\nabla^2_X W + tD^1(X, W). \end{aligned}$$

Comparing the tangential, $\varphi D^{\perp}$ and π components of above equation the relations (3.21), (3.22) and (3.23) follows respectively. Thus the proof is completed. □

Theorem 3.3. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B-metric. Then $\nabla^1 K = 0$ if and only if $\nabla^1 t = 0$.*

Proof. Let $\nabla^1 K = 0$. Using (3.14), we have $h^1(X, HY) = 0$. Now taking the inner product with respect to $N \in \varphi D^{\perp}$ and using relation (3.10), we get

$$\bar{g}(HY, A_N X) = \bar{g}(h^1(X, HY), N) = 0,$$

which implies $A_N X \in D^{\perp}$, that is, $HA_N X = 0$. Next, by making use of relation (3.18), we have $\nabla^1 t = 0$. The converse follows similarly. □

Lemma 3.4. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B-metric. Then we have*

$$A_{\varphi W} Z = A_{\varphi Z} W, \tag{3.24}$$

and

$$\nabla^1_Z \varphi W - \nabla^1_W \varphi Z \in \varphi D^{\perp}, \tag{3.25}$$

for any $Z, W \in D^{\perp}$.

Proof. Since $\bar{M}$ is a Sasaki-like almost contact manifold with B-metric, we have, for any $Z, W \in D^{\perp}$, $\nabla_Z \varphi W = \varphi \nabla_Z W$. Taking inner products with $U \in \pi$, we obtain

$$\bar{g}(\varphi U, \bar{\nabla}_Z W) = \bar{g}(U, \nabla^1_Z \varphi W),$$

which implies

$$\bar{g}(U, \nabla^1_Z \varphi W - \nabla^1_W \varphi Z) = g(A_{\varphi Z} W, U) - g(A_{\varphi W} Z, U) = 0.$$

Thus, the proof is completed. □

Lemma 3.5. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B-metric. Then the anti-invariant distribution $D^{\perp}$ is integrable.*

Proof. For any Z, W on $D^{\perp}$, making use of relation (3.24) yields

$$\varphi[Z, W] = \varphi(\nabla_Z W - \nabla_W Z) = (\nabla^1_Z \varphi W - \nabla^1_W \varphi Z) + (D^2(Z, \varphi W) - D^2(W, \varphi Z)).$$

By virtue of relation (3.15), we have

$$D^2(Z, \varphi W) = f h^2(Z, W)$$

and

$$\varphi[Z, W] = \nabla^1_Z \varphi W - \nabla^1_W \varphi Z$$

thus the assertion follows using relation (3.25). □

Lemma 3.6. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B-metric. Then the distribution D is integrable if and only if $h^1(X, \varphi Y) = h^1(Y, \varphi X)$.*

Proof. For any X, Y on D and taking into consideration the relations (3.14) and (3.15) infers that $K \nabla_X Y = h^1(X, \varphi Y)$. Further for any Z on $D^{\perp}$ yields $\bar{g}(K \nabla_X Y, \varphi Z) = \bar{g}(h^1(X, \varphi Y), \varphi Z)$ which implies $\bar{g}(\nabla_X Y, Z) = \bar{g}(h^1(X, \varphi Y), \varphi Z)$. It follows that $g([X, Y], Z) = \bar{g}(h^1(X, \varphi Y), \varphi Z) - \bar{g}(h^1(Y, \varphi X), \varphi Z)$. Thus, making use of the non-degenerate property of $\varphi(D^{\perp})$, the result follows. □

Like a contact CR-submanifold of a Sasakian manifold, a contact CR-submanifold of a Sasaki-like almost contact manifold with B-metric is known as *contact CR-product* if it is locally a product of M^1 and M^2, where M^1 and M^2 are the leaf of the distribution $D \oplus \zeta$ and $D^\perp$ respectively.

Theorem 3.7. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B-metric. Then (M, g) is a contact CR-product if and only if*

$$A_{\varphi X} Z = \eta(X) Z, \tag{3.26}$$

$\forall X \in D^\perp$ and $Z \in D$.

Proof. Proof is similar with that of Theorem 6.1 of [8]. □

Theorem 3.8. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold $(\bar{M}, \varphi, \zeta, \eta, \bar{g})$ with B-metric. Then the following assertions are equivalent :*

(i) *D^1 is a metric Otsuki connection on $TM^\perp$.*

(ii) *$D^1(X, W) = 0$ for any $X \in TM$ and $W \in \pi$.*

(iii) *$D^2(X, N) = 0$ for any $X \in TM$ and $N \in \varphi D^\perp$.*

(iv) *D^2 is a metric Otsuki connection on $TM^\perp$.*

Proof. Since ∇^1 and ∇^2 are metric connections on $\varphi(TM)$ and π, respectively and making use of relation (3.12) we have

$$\bar{g}(D^1(X, N), N') = -\bar{g}(D^2(X, N'), N) = 0,$$
$$\bar{g}(D^1(X, W), W') = -\bar{g}(D^2(X, W'), W) = 0,$$

thus, the assertions (i), (ii), (iii) and (iv) are equivalent. □

4. Lightlike Submanifold of a Sasaki-like Almost Contact Manifold with B-metric

Consider an m-dimensional submanifold (M, g) immersed in a real $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant index q such that $m, n \geq 1$, $1 \leq q \leq m + n - 1$ and let g be the induced metric of $\bar{g}$ on M. Then M is called a lightlike submanifold of $\bar{M}$ if $\bar{g}$ is a degenerate metric on the tangent bundle TM of M. For a degenerate metric g on M, $TM^\perp$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^\perp$ are no longer complementary but degenerate orthogonal subspaces of $T\bar{M}$. So, there exists a subspace called radical or null subspace, that is,

$$Rad(T_xM) = T_xM \cap T_xM^\perp.$$

Further, if the mapping $Rad(TM) : x \in M \longrightarrow RadT_xM$, defines a smooth distribution of rank $r > 0$ on M then the submanifold M is called an r-lightlike submanifold of $\bar{M}$and $Rad(TM)$ is known as the radical distribution on M. A semi-Riemannian complementary distribution $S(TM)$ of $Rad(TM)$ in TM is a *screen distribution*. We have

$$TM = Rad(TM) \perp S(TM)$$

and that $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}).$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}).$$

For a quasi-orthonormal fields of frames of $\bar{M}$ along M, we have the following.

Theorem 4.1.([4]) *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $ltr(TM)\mid_{\mathbf{U}}$ consisting of smooth section $\{N_a\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathbf{U}}$, where $\mathbf{U}$ is a coordinate neighbourhood of M, such that*

$$\bar{g}(N_a, \xi_b) = \delta_{ab}, \quad \bar{g}(N_a, N_b) = 0, \quad \text{for any} \quad a, b \in \{1, 2, .., r\},$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $Rad(TM)$.

Let $(M, \tilde{g})$ be an r-lightlike submanifold of a Sasaki-like almost contact manifold with B metric$(\bar{M}, \varphi, \bar{g}, \bar{\tilde{g}})$. Let $\bar{\tilde{\nabla}}$ be the Levi-Civita connection of the metric $\bar{\tilde{g}}$ on $\bar{M}$ and $\tilde{\nabla}$ be the induced connection on M then Gauss and Weingarten formulae are given by

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \tilde{h}(X, Y), \quad \bar{\tilde{\nabla}}_X V = -\tilde{A}_V X + \nabla^t_X V,$$

for arbitrary $X, Y \in TM$ and $V \in tr(TM)$, where $\{\tilde{\nabla}_X Y, \tilde{A}_V X\}$ and $\{\tilde{h}(X, Y), \nabla^t_X V\}$ belong to TM and $tr(TM)$, respectively and $\tilde{\nabla}$ and ∇^t are linear connections on TM and $tr(TM)$, respectively. Moreover, $\tilde{\nabla}$ is torsion-free linear connection, $\tilde{h}$ is a $tr(TM)$-valued symmetric $\mathcal{F}(M)$-bilinear form on TM and $\tilde{A}$ is a TM-valued $\mathcal{F}(M)$-bilinear form on $tr(TM) \times (TM)$. In general, $\tilde{\nabla}$ and ∇^t are not metric connections. Let L and S be the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\tilde{\perp}})$, respectively then

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + h^l(X, Y) + h^s(X, Y), \tag{4.1}$$

$$\bar{\tilde{\nabla}}_X V = -\tilde{A}_V X + D^l_X V + D^s_X V. \tag{4.2}$$

where

$$D^l_X V = L(\nabla^t_X V); \quad D^s_X V = S(\nabla^t_X V).$$

Besides D^l and D^s do not define linear connections on $tr(TM)$ but they are Otsuki connections on $tr(TM)$ with respect to L and S, respectively. Therefore (4.1) and (4.2) become

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + h^l(X,Y) + h^s(X,Y), \tag{4.3}$$

$$\bar{\tilde{\nabla}}_X N = -\tilde{A}_N X + \nabla^l_X N + D^s(X,N), \tag{4.4}$$

$$\bar{\tilde{\nabla}}_X W = -\tilde{A}_W X + D^l(X,W) + \nabla^s_X W, \tag{4.5}$$

where ∇^l and ∇^s are defined by $\nabla^l_X N = D^l_X N$ and $\nabla^s_X W = D^s_X W$ are metric linear connections on $ltr(TM)$ and $S(TM^{\tilde{\perp}})$, respectively. D^l and D^s are defined by $D^l(X,W) = D^l_X W$ and $D^s(X,N) = D^s_X N$ are $\mathcal{F}(M)$-bilinear mappings. Using (4.3) $--$(4.5) and taking into account that $\bar{\tilde{\nabla}}$ is a metric connection, we obtain

$$\bar{\tilde{g}}(h^s(X,Y),W) + \bar{\tilde{g}}(Y, D^l(X,W)) = \bar{\tilde{g}}(\tilde{A}_W X, Y),$$

$$\bar{\tilde{g}}(D^s(X,N),W) = \bar{\tilde{g}}(\tilde{A}_W X, N),$$

$$\bar{\tilde{g}}(\tilde{A}_N X, N') + \bar{\tilde{g}}(\tilde{A}_{N'} X, N) = 0,$$

$$\bar{\tilde{g}}(\tilde{\nabla}_X Y, N) + \bar{\tilde{g}}(Y, \nabla^l_X N) = \bar{\tilde{g}}(\tilde{A}_N X, Y).$$

$$\bar{\tilde{g}}(h^l(X,Y),\xi) + \bar{\tilde{g}}(Y, h^l(X,\xi)) + \bar{\tilde{g}}(Y, \tilde{\nabla}_X \xi)) = 0. \tag{4.6}$$

Let P' be the projection morphism of TM on $S(TM)$ then new induced geometric objects on the screen distribution $S(TM)$ are given as below.

$$\nabla_X P'Y = \nabla^*_X P'Y + h^*(X, P'Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{4.7}$$

for any $X, Y \in TM$ and $\xi \in Rad(TM)$, where $\{\nabla^*_X P'Y, A^*_\xi X\}$ and $\{h^*(X, P'Y), \nabla^{*t}_X \xi\}$ belong to $S(TM)$ and $Rad(TM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$, respectively. h^* and A^* are $Rad(TM)$-valued and $S(TM)$-valued bilinear forms and they are called as the second fundamental forms of distributions $S(TM)$ and $Rad(TM)$, respectively, for any $X, Y \in TM$, $\xi \in Rad(TM)$ and $N \in ltr(TM)$.

From the geometry of Riemannian submanifolds, it is known that on a non degenerate submanifold the induced connection ∇ is a metric connection. But unfortunately, this is not true incase of a lightlike submanifold. Indeed, considering $\bar{\nabla}$ a metric connection, we have

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y),$$

for any $X, Y, Z \in TM$.

In [11], Sahin defined radical transversal lightlike submanifolds for an indefinite Sasakian manifold. In this paper, we define radical transversal lightlike submanifold of a Sasaki-like almost contact manifold with B-metric as follows.

Definition 4.2. Let $(M, \tilde{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \bar{g}, \bar{\bar{g}})$. Then M is a *radical transversal lightlike submanifold of* $\bar{M}$ if

$$\varphi(RadTM) = ltr(TM), \tag{4.8}$$

$$\varphi(S(TM)) = S(TM). \tag{4.9}$$

It is important to note that for an indefinite Sasakian manifold [11] there do not exist any 1-lightlike radical transversal lightlike submanifolds. But for a Sasaki-like almost contact manifold with B-metric there exists an 1-lightlike radical transversal lighlike submanifold.

Lemma 4.3. *There exists an* 1*-lightlike radical transversal lighlike submanifold* $(M, \tilde{g})$ *of a Sasaki-like almost contact manifold with* B*-metric* $(\bar{M}, \varphi, \bar{g}, \bar{\bar{g}})$.

Proof. Let us suppose that $(M, \tilde{g})$ be an 1-lightlike radical transversal lighlike submanifold of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \bar{g}, \bar{\bar{g}})$. Then we have, $Rad(TM) = span\zeta$ and $ltr(TM) = spanN$. Using the definition of Sasaki-like almost contact manifold, we have,

$$\bar{\bar{g}}(\varphi\zeta, \zeta) = -\bar{\bar{g}}(\varphi^2\zeta, \varphi\zeta) + \eta(\varphi\zeta)\eta(\zeta) = \bar{\bar{g}}(\zeta, \varphi\zeta) \neq 0.$$

Let $N \in \Gamma(ltr(TM))$, then using(4.8), we have $\bar{\bar{g}}(\varphi\zeta, \zeta) = \bar{\bar{g}}(N, \zeta) = 1$, Thus, we conclude that there exists an 1-lightlike radical transversal lighlike submanifold of a Sasaki-like almost contact manifold with B-metric. □

Theorem 4.4. *Let* $(M, \tilde{g})$ *be a radical lightlike submanifold of a Sasaki-like almost contact manifold with* B*-metric* $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\bar{g}})$*. Then the induced metric connection* $\tilde{\nabla}$ *on* M *is a metric connection if and only if* $A_{\varphi Y}X$ *has no component in* $S(TM)$ *for* $X \in TM$ *and* $Y \in Rad(TM)$.

Proof. We know that the necessary and sufficient condition for an induced connection to be a metric connection is that for any $X \in TM$ and $Y \in Rad(TM)$, $\nabla_X Y \in Rad(TM)$. Suppose $\tilde{\nabla}$ is a metric connection on M then for any $Z \in S(TM)$ and making use of (4.1), we have

$$\bar{\bar{g}}(\bar{\bar{\nabla}}_X Y, Z) = 0.$$

By making use of (2.2) yields

$$-\bar{\bar{g}}(\varphi\bar{\bar{\nabla}}_X Y, \varphi Z) + \eta(\bar{\bar{\nabla}}_X Y)\eta(Z) = 0$$

$$\bar{\bar{g}}((\bar{\bar{\nabla}}_X\varphi)Y - \bar{\bar{\nabla}}_X\varphi Y), \varphi Z) = 0$$

Next to it, using (4.4), the proof is completed. □

5. Relations between Contact CR-submanifold and Radical Transversal Lightlike Submanifold of a Sasaki-like Almost Contact Manifold with B-metric

Since there are two B-metrics on an indefinite almost contact manifold $\bar{M}$ with B-metric, there are two corresponding induced metrics on the submanifold M of $\bar{M}$. Hence M is either non-degenerate (degenerate) with respect to both the induced metrics or degenerate with respect to one and non-degenerate with respect to other. In [9], Nakova studied the case when the submanifold (M, g) is non-degenerate and $(M, \tilde{g})$ is a degenerate submanifold of $\bar{M}$. In particular, Nakova proved the following theorem for an almost complex manifold with a Norden metric.

Theorem 5.1.([9]) *Let $(\bar{M}, \bar{J}, \bar{g}, \bar{\tilde{g}})$ be a $2n$-dimensional almost complex manifold with a Norden metric and M be an m-dimensional submanifold of $\bar{M}$. The submanifold (M, g) is a CR-submanifold with an r-dimensional totally real distribution $D^{\perp}$ if and only if $(M, \tilde{g})$ is an r-lightlike radical transversal lightlike submanifold of $\bar{M}$.*

Thus, in this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ has the following decomposition.

$$T\bar{M} = S(TM)\tilde{\perp}S(TM^{\tilde{\perp}})\tilde{\perp}(Rad(TM)\tilde{\oplus}ltr(TM)),$$

where $S(TM) = D$, $RadTM = D^{\perp}$, $S(TM^{\tilde{\perp}}) = (\varphi D^{\perp})^{\perp}$ and $ltrTM = \varphi D^{\perp}$. For a Sasaki-like almost contact manifold with B-metric, we have

$$\begin{aligned}
\bar{\tilde{\nabla}}_X Y &= \bar{\nabla}_X Y, & \tilde{\nabla}_X Y &= \nabla_X Y,\\
h^l(X,Y) &= h^1(X,Y), & h^s(X,Y) &= h^2(X,Y),\\
\tilde{A}_N X &= A_N X, & \tilde{A}_W X &= A_W X,\\
\nabla^l_X N &= \nabla^1_X N, & \nabla^s_X W &= \nabla^2_X W\\
D^l(X,W) &= D^1(X,W), & D^s(X,N) &= D^2(X,N),
\end{aligned} \tag{5.1}$$

We have verify the above result for a Sasaki-like almost contact manifold with B-metric and the proof is same as above theorem if we consider that $\zeta \in S(TM)$ [3].

Theorem 5.2. *Let (M, g) be a submanifold of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\tilde{g}})$. Then the submanifold (M, g) is a contact CR-submanifold with an r-dimensional totally real distribution if and only if $(M, \tilde{g})$ is an r-lightlike contact radical transversal lightlike submanifold of $\bar{M}$.*

Theorem 5.3. *Let $(M, \tilde{g})$ be a radical transversal lightlike submanifold of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\tilde{g}})$. Then the radical distribution of $(M, \tilde{g})$ is integrable.*

Proof. Let $\xi_1, \xi_2 \in \Gamma(Rad(TM))$ and $X \in \Gamma(S(TM))$ then we have

$$\begin{aligned}\bar{\bar{g}}(\bar{\bar{\nabla}}_{\xi_1}\xi_2, X) &= -\bar{g}(\varphi\xi_2, \bar{\nabla}_{\xi_1}X) \\ &= \bar{g}(\bar{\nabla}_{\xi_1}\varphi\xi_2, X) \\ &= -\bar{g}(A_{\varphi\xi_2}\xi_1, X).\end{aligned}$$

Hence

$$\bar{\bar{g}}(\bar{\bar{\nabla}}_{\xi_1}\xi_2 - \bar{\bar{\nabla}}_{\xi_2}\xi_1, X) = \bar{g}(A_{\varphi\xi_1}\xi_2 - A_{\varphi\xi_2}\xi_1, X).$$

Since $\xi_1, \xi_2 \in \Gamma(D^\perp) = \Gamma(Rad(TM))$, using (3.24) we obtain $\bar{\bar{g}}([\xi_1, \xi_2], X) = 0$ which implies that $[\xi_1, \xi_2] \in \Gamma(Rad(TM))$. Hence the result follows. □

Theorem 5.4. *Let $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\bar{g}})$ be a Sasaki-like almost contact manifold with B-metric and (M, g) be a contact CR submanifold of $\bar{M}$. Then the invariant distribution D of contact CR-submanifold (M, g) is integrable if and only if the screen distribution $S(TM)$ of radical transversal lightlike submanifold $(M, \tilde{g})$ is integrable.*

Proof. Let $X, Y \in S(TM)$ and $N \in ltr(TM)$ then

$$\bar{\bar{g}}(\bar{\bar{\nabla}}_X Y, N) = \bar{g}(\varphi\bar{\nabla}_X Y, N) = \bar{g}(h^1(X, \varphi Y), N)$$

and hence $\bar{\bar{g}}([X, Y], N) = \bar{g}(h^1(X, \varphi Y) - h^1(Y, \varphi X), N)$. Thus the assertion follows using Lemma 3.6. □

Theorem 5.5. *Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\bar{g}})$. Then the induced connection $\tilde{\nabla}$ on the radical transversal lightlike submanifold $(M, \tilde{g})$ is a metric connection if and only if (M, g) is a contact CR-product.*

Proof. Let (M, g) be a contact CR-submanifold of a Sasaki-like almost contact manifold with B-metric then using (3.7), (3.8) and (3.18), we have

$$\begin{aligned}-A_{\varphi Z}X + \nabla^1_X\varphi Z + D^2(X, \varphi Z) &= -g(X, Z)\zeta - \eta(X)Z + 2\eta(Z)\eta(X)\zeta \\ &= \varphi(\nabla_X Z + h^1(X, Z) + h^2(X, Z))\end{aligned}$$

Comparing the tangential parts, we have

$$\varphi\tilde{\nabla}_X Z = A_{\varphi Z}X - \eta(X)Z.$$

Taking inner product with $Y \in Rad(TM)$ and using (2.3) we have

$$\begin{aligned}\bar{\bar{g}}(\varphi\tilde{\nabla}_X Z, Y) &= \bar{\bar{g}}(A_{\varphi Z}X - \eta(X)Z, Y) \\ &= \bar{g}(A_{\varphi Z}X - \eta(X)Z, \varphi Y) + \eta(A_{\varphi Z}X - \eta(X)Z)\eta(Y) \\ &= 0.\end{aligned}$$

Using Theorem 3.7. we have $\tilde{\nabla}_X Z \in Rad(TM)$ and hence, the induced connection $\tilde{\nabla}$ is a metric connection on $(M, \tilde{g})$. □

Theorem 5.6. *Let (M, g) be a contact CR-submanifold of a of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\bar{g}})$. If K is parallel, that is, $\nabla^1 K = 0$ then the screen distribution $S(TM)$ of radical transversal lightlike submanifold $(M, \tilde{g})$ is a parallel distribution with respect to $\tilde{\nabla}$ and h^* vanishes identically on lightlike submanifold $(M, \tilde{g})$.*

Proof. Let K be parallel then using (3.14), we get $h^1(X, HY) = 0$ and using this in (3.10), we obtain $g(\varphi Y, A_N X) = 0$, for any $X, Y \in TM$. Further using (2.2) and (4.4), we obtain $\tilde{\bar{g}}(Y, \tilde{A_N} X) = 0$, implies $\tilde{A_N}$ is $\Gamma(Rad(TM))$-valued operator. Thus from Theorem 2.6 on page 162 of [4], the assertion follows. □

Theorem 5.7. *Let (M, g) be a contact CR-submanifold of of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\bar{g}})$. If $(\nabla^1_X t)N = 0$ then the screen distribution $S(TM)$ of radical transversal lightlike submanifold $(M, \tilde{g})$ is a parallel distribution with respect to $\tilde{\nabla}$ and h^* vanishes identically on lightlike submanifold $(M, \tilde{g})$.*

Proof. Using the hypothesis in (3.18), we get $HA_N X = 0$ implies $A_N X \in D^{\perp}$, for any $X \in TM$. Therefore for any $Y \in D$,

$$g(HY, A_N X) = g(\varphi Y, A_N X) = 0.$$

The rest of the proof is similar to that of above Theorem. □

Theorem 5.8. *Let (M, g) be a contact CR-submanifold of of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\bar{g}})$. then the radical distribution $Rad(TM)$ of radical transversal lightlike submanifold $(M, \tilde{g})$ is integrable and the shape operator A^*_ξ of the screen distribution of $(M, \tilde{g})$ vanishes identically on $Rad(TM)$, for any $\xi \in Rad(TM)$.*

Proof. We know that for a contact CR-submanifold of of a Sasaki-like almost contact manifold with B-metric $(\bar{M}, \varphi, \eta, \zeta, \bar{g}, \bar{\bar{g}})$, the totally real distribution $D^{\perp}$ is always integrable. Therefore for any $X \in D$ and $Z, Z' \in D^{\perp}$, we have

$$\begin{aligned} 0 = \bar{g}([Z, Z'], \varphi X) &= \bar{g}(\bar{\nabla}_Z \varphi Z', X) - \bar{g}(\bar{\nabla}_{Z'} \varphi Z, X) \\ &= -\bar{g}(\varphi Z', h^1(X, Z)) + \bar{g}(\varphi Z, h^1(X, Z')). \end{aligned}$$

Since $X \in D$, we obtain, using (2.2),

$$\bar{\bar{g}}(Z', h^l(HX, Z)) - \bar{\bar{g}}(Z, h^l(HX, Z')) = 0. \tag{5.2}$$

Now replace ξ by Z and Y by Z' in (3.19), where $Z, Z' \in \Gamma(D^{\perp}) = Rad(TM)$, we get

$$\bar{\bar{g}}(Z', h^l(HX, Z)) + \bar{\bar{g}}(Z, h^l(HX, Z')) = 0. \tag{5.3}$$

Adding (5.2) and (5.3), we obtain $\bar{\bar{g}}(Z', h^l(HX, Z)) = 0$ consequently $h^l(HX, Z) = 0$. Thus by virtue of Theorem 2.7 on page 162 of [4], the assertion follows. □

References

[1] A. Bejancu, *CR-submanifolds of a Kaehler manifold I*, Proc. Amer. Math. Soc., **69**(1978), 135–142.

[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhauser, Boston 2002.

[3] C. Calin, *Contributions to geometry of CR-submanifold*, Thesis, University of Iasi, Romania, 1998.

[4] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and its Applications **364**, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[5] K. L. Duggal and B. Sahin, *Lightlike submanifolds of indefinite Sasakian manifolds*, Int. J. Math. Math. Sci., (2007), Art. ID 57585, 21 pp.

[6] G. Ganchev, V. Mihova and K. Gribachev, *Almost contact manifolds with B-metric*, Math. Balkanica (N.S.), **7(3–4)**(1993), 261–276.

[7] S. Ivanov, H. Manev and M. Manev *Sasaki-like almost contact complex Riemannian manifolds*, J. Geom. Phys., **107**(2016), 136–148.

[8] K. Matsumoto, *On contact CR-submanifolds of Sasakian manifolds*, Internat. J. Math. Math. Sci., **6(2)**(1983), 313–326.

[9] G. Nakova, *Some lightlike submanifolds of almost complex manifolds with Norden metric*, J. Geom., **103**(2012), 293–312.

[10] K. Yano and M. Kon, *Differential geometry of CR submanifolds*, Geom. Dedicata, **10**(1981), 369–391.

[11] C. Yildirim and B. Sahin, *Transversal lightlike submanifolds of indefinite Sasakian manifolds*, Turkish J. Math., **34**(2010), 561–583.