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Surfaces of Revolution of Type 1 in Galilean 3-Space

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Abstract. In this study, we classify surfaces of revolution of Type 1 in the three dimensional Galilean space $\mathbb{G}_3$ in terms of the position vector field, Gauss map, and Laplacian operator of the first and the second fundamental forms on the surface. Furthermore, we give a classification of surfaces of revolution of Type 1 generated by a non-isotropic curve satisfying the pointwise 1-type Gauss map equation.

1. Introduction

Let $\mathbf{x} : \mathbf{M} \to \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ the mean curvature and the Laplacian of $\mathbf{M}$ with respect to the Riemannian metric on $\mathbf{M}$

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induced from that of $\mathbb{E}^m$, respectively. Takahashi [15] proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta\mathbf{x} = \lambda\mathbf{x}$, that is, for which all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$, are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of the hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$ [5, 6, 18, 19].

As an extension of Takahashi's theorem, Garay [11] studied hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$\Delta\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{1.1}$$

where $\mathbf{A} \in Mat\ (m, \mathbb{R})$ is an $m \times m$ diagonal matrix, and proved that such hypersurfaces are minimal $(\mathbf{H} = 0)$ in $\mathbb{E}^m$ and are open pieces of either round hyperspheres or generalized right spherical cylinders.

Related to this, Dillen, Pas and Verstraelen [9] investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition

$$\Delta\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}, \tag{1.2}$$

where $\mathbf{A} \in Mat\ (3, \mathbb{R})$ is a 3×3 real matrix and $\mathbf{B} \in \mathbb{R}^3$ [5, 6, 18, 19].

The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with its unit normal vector field. Dillen, Pas and Verstraelen [10] studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map $\mathbf{G}$ satisfies the condition

$$\Delta\mathbf{G} = \mathbf{A}\mathbf{G}, \tag{1.3}$$

where $\mathbf{A} \in Mat\ (3, \mathbb{R})$.

In the late 1970's B.-Y. Chen introduced the notion of Euclidean immersions of finite type. Essentially these are submanifolds whose immersion into the $m-$dimensional Euclidean space $\mathbb{E}^m$ is constructed by making use of a finite number of $\mathbb{E}^m$-valued eigenfunctions of their Laplacian. The first results on this subject are collected in the book [3]. In a framework of the theory of finite type, B.-Y. Chen and P. Piccini [4] made a general study on submanifolds of Euclidean spaces with finite type Gauss maps. Several geometers also studied submanifolds of Euclidean spaces or pseudo-Euclidean spaces with finite type Gauss maps [11].

From the above definition one can see that a submanifold has 1-type Gauss map $\mathbf{G}$ if and only if $\mathbf{G}$ satisfies the equation

$$\Delta\mathbf{G} = \lambda\,(\mathbf{G} + \mathbf{C}) \tag{1.4}$$

for a constant λ and a constant vector $\mathbf{C}$, where Δ denotes the Laplace operator on a submanifold. A plane, a circular cylinder and a sphere are surfaces with a 1-type

Gauss map. Similarly, a submanifold is said to have a pointwise 1-type Gauss map if the Laplacian of its Gauss map takes the form

$$\Delta \mathbf{G} = \mathbf{F}\left(\mathbf{G} + \mathbf{C}\right) \tag{1.5}$$

for a non-zero smooth function $\mathbf{F}$ and a constant vector $\mathbf{C}$. More precisely, a pointwise 1-type Gauss map is said to be of the first kind if (1.5) is satisfied for $\mathbf{C} = 0$, and of the second kind if $\mathbf{C} \neq 0$. A helicoid, a catenoid and a right cone are the typical examples of surfaces with pointwise 1-type Gauss maps [7].

Sipus and Divjak [17] defined surfaces of revolution in the 3-dimensional pseudo-Galilean space $\mathbb{G}_3^1$ and described surfaces of revolution of constant curvature. Yoon [18, 19] characterized surfaces of revolution in $\mathbb{G}_3^1$. Dede, Ekici and Goemanse [8] defined and studied three types of surfaces of revolution in Galilean 3-space. They classified the surfaces of revolution with vanishing Gaussian curvature or vanishing mean curvature in Galilean 3-space $\mathbb{G}_3$. Choi, Kim and Yoon [6] gave the classification of surfaces of revolution generated by an isotropic curve satisfying a pointwise 1-type Gauss map equation. Choi [5] completely classified the surfaces of revolution satisfying condition (1.3). Karacan, Yoon and Bukcu [13] classified surfaces of revolution satisfying $\Delta^J \mathbf{x}_i = \lambda_i \mathbf{x}_i$, $J = 1, 2$ and $\Delta^{III} \mathbf{x}_i = \lambda_i \mathbf{x}_i$.

The main purpose of this paper is a complete classification of surfaces of revolution in the three dimensional Galilean space $\mathbb{G}_3$ in terms of the position vector field, Gauss map, pointwise 1-type Gauss map equation and Laplacian operators of the first and the second fundamental forms on the surface.

2. Preliminaries

The Galilean space G_3 is a Cayley-Klein space defined from a 3-dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure that consists of an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f. We introduce homogeneous coordinates in G_3 in such a way that the absolute plane w is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the elliptic involution by $(0 : 0 : x_2 : x_3) \rightarrow (0 : 0 : x_3 : -x_2)$. In affine coordinates defined by $(0 : x_1 : x_2 : x_3) \rightarrow (1 : x : y : z)$, distance between points $P_i = (x_i,\, y_i,\, z_i)$, $i = 1, 2$ is defined by

$$d\left(P_1, P_2\right) = \begin{cases} \left|x_2 - x_1\right|, & if \quad x_1 \neq x_2 \\ \sqrt{\left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}, & if \quad x_1 = x_2. \end{cases} \tag{2.1}$$

The group of motions of G_3 is a six-parameter group given (in affine coordinates) by

$$\overline{x} = a + x,\ \overline{y} = b + cx + y\cos\theta + z\sin\theta,\ \overline{z} = d + ex - y\sin\theta + z\cos\theta.$$

A C^r-surface S, $r \geq 1$, immersed in the Galilean space, $\mathbf{x} : U \rightarrow S$, $U \subset \mathbb{R}^2$, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, has the following first fundamental form

$$\mathbf{I} = \left(g_1 du + g_2 dv\right)^2 + \epsilon\left(h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2\right),$$

where the symbols $g_i = x_i$, $h_{ij} = \widetilde{\mathbf{x}}_i . \widetilde{\mathbf{x}}_j$ stand for derivatives of the first coordinate function $x(u,v)$ with respect to u, v and for the Euclidean scalar product of the projections $\widetilde{\mathbf{x}}_k$ of vectors $\mathbf{x}_k$ onto the yz-plane, respectively. Furthermore,

$$\epsilon = \begin{cases} 0, & \text{if direction } du : dv \text{ is non-isotropic,} \\ 1, & \text{if direction } du : dv \text{ is isotropic.} \end{cases}$$

In every point of a surface there exists a unique isotropic direction defined by $g_1 du + g_2 dv = 0$. In that direction, the arc length is measured by

$$ds^2 = h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2 = \frac{h_{11} g_2^2 - 2h_{12} g_1 g_2 + h_{22} g_1^2}{g_1^2} = \frac{W^2}{g_1^2} dv^2,$$

where $g_1 \neq 0$.

A surface is called admissible if it has no Euclidean tangent planes. Therefore, for an admissible surface either $g_1 \neq 0$ or $g_2 \neq 0$ holds. An admissible surface can always locally be expressed as

$$z = f(u,v).$$

The Gaussian $\mathbf{K}$ and mean curvature $\mathbf{H}$ are C^{r-2} functions, $r \geq 1$, defined by

$$\mathbf{K} = \frac{LN - M^2}{W^2}, \; \mathbf{H} = \frac{g_2^2 L - 2g_1 g_2 M + g_1^2 N}{2W^2},$$

where

$$L_{ij} = \frac{x_1 \mathbf{x}_{ij} - x_{ij} \mathbf{x}_1}{x_1} . \mathbf{G}, \;\; x_1 = g_1 \neq 0.$$

We will use L_{ij}, $i, j = 1, 2$, for L, M, N if more convenient. The vector $\mathbf{G}$ defines a normal vector to a surface

$$\mathbf{G} = \frac{1}{W} \left(0, -x_2 z_1 + x_1 z_2, x_2 y_1 - x_1 y_2 \right),$$

where $W^2 = \left(x_2 \mathbf{x}_1 - x_1 \mathbf{x}_2 \right)^2$ [8, 14].

It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operators $\Delta^{\mathbf{I}}$ and $\Delta^{\mathbf{II}}$ of the first and the second fundamental forms on $\mathbf{M}$ are defined by

$$(2.3) \qquad \Delta^{\mathbf{I}} \mathbf{x} = -\frac{1}{\sqrt{EG - F^2}} \left[\frac{\partial}{\partial u} \left(\frac{G \mathbf{x}_u - F \mathbf{x}_v}{\sqrt{EG - F^2}} \right) - \frac{\partial}{\partial v} \left(\frac{F \mathbf{x}_u - E \mathbf{x}_v}{\sqrt{EG - F^2}} \right) \right],$$

$$(2.4) \qquad \Delta^{\mathbf{II}} \mathbf{x} = -\frac{1}{\sqrt{LN - M^2}} \left[\frac{\partial}{\partial u} \left(\frac{N \mathbf{x}_u - M \mathbf{x}_v}{\sqrt{LN - M^2}} \right) - \frac{\partial}{\partial v} \left(\frac{M \mathbf{x}_u - L \mathbf{x}_v}{\sqrt{LN - M^2}} \right) \right]$$

[1, 2, 12, 13, 16].

3. Surfaces of Revolution in G_3

In the Galilean space $\mathbb{G}_3$ there are two types of rotations: Euclidean rotations given by the normal form

$$\overline{x} = x,\ \overline{y} = y\cos v + z\sin v,\ \overline{z} = -y\sin v + z\cos v \tag{3.1}$$

and isotropic rotations with the normal form

$$\overline{x} = x + ct,\ \overline{y} = y + xt + c\frac{t^2}{2},\ \overline{z} = z. \tag{3.2}$$

Then the surface of revolution of Type 1 can be written as

$$\mathbf{x}(u,v) = (f(u), g(u)\cos v, -g(u)\sin v)\,. \tag{3.3}$$

Suppose that α is parametrized by arc-length. In this case, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u,v) = (u, g(u)\cos v, -g(u)\sin v)\,. \tag{3.4}$$

Next, we consider the isotropic rotations. By rotating the isotropic curve $\alpha(u) = (0, f(u), g(u))$ about the $z-$axis by isotropic rotation (3.2), we obtain the parametrization of the surface of revolution of Type 2 as

$$\mathbf{x}(u,v) = \left(cv, f(u) + \frac{cv^2}{2}, g(u)\right), \tag{3.5}$$

where f and g are smooth functions and $c \neq 0 \in \mathbb{R}$ [12].

Finally, we assume, again without loss of generality, that the profile curve $\alpha(u) = (f(u), g(u), 0)$ lies in the isotropic $xy-$ plane and is parameterized by

$$\mathbf{x}(u,v) = \left(f(u) + cv, g(u), vf(u) + \frac{cv^2}{2}\right), \tag{3.6}$$

where f and g are smooth functions and $c \neq 0 \in \mathbb{R}$. The surface (3.6) is called the surface of revolution of Type 3 [12].

4. Surfaces of Revolution of Type 1 Satisfying $\Delta^{\mathbf{I}}\mathbf{x} = \mathbf{A}\mathbf{x}$

In this section, we classify surface of revolution of Type 1 given in $\mathbb{G}_3$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{I}}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$ and

$$\Delta^{\mathbf{I}}\mathbf{x}_i = \left(\Delta^{\mathbf{I}}\mathbf{x}_1, \Delta^{\mathbf{I}}\mathbf{x}_2, \Delta^{\mathbf{I}}\mathbf{x}_3\right), \tag{4.2}$$

where

$$\mathbf{x}_1 = u,\ \mathbf{x}_2 = g(u)\cos v,\ \mathbf{x}_3 = -g(u)\sin v. \tag{4.3}$$

For this surface of revolution, the coefficients of the first and second fundamental forms are

$$\begin{aligned} &g_1 = 1, g_2 = 0, h_{11} = g'^2(u),\ h_{12} = 0,\ h_{22} = g^2(u), \\ &L_{11} = L = -g''(u),\quad L_{22} = \ N = g(u),\ L_{12} = M = 0, \\ &E = 1,\quad F = 0,\quad G = g^2(u), \end{aligned} \tag{4.4}$$

respectively. The Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are

$$\mathbf{K} = -\frac{g''(u)}{g(u)},\quad \mathbf{H} = \frac{1}{2g(u)}. \tag{4.5}$$

Corollary 4.1. *There are no minimal surfaces of revolution (3.4).*

Corollary 4.2. *The profile curve of surface of revolution of Type 1 of constant Gaussian curvature in $\mathbb{G}_3$ is as follows:*

(1) *If $\mathbf{K} = \frac{1}{a^2}$, then the general solution of the differential equation (4.5) is*

$$g(u) = c_1 \cos\frac{u}{a} + c_2 \sin\frac{u}{a},$$

where $c_1, c_2, a \in \mathbb{R}$.

(2) *If $\mathbf{K} = 0$, then the general solution of the differential equation (4.5) is*

$$g(u) = c_1 u + c_2,$$

where $c_1, c_2 \in \mathbb{R}$.

The Laplacian operator on $\mathbf{M}$ with the help of (2.3), (4.3) and (4.4) turns out to be

$$\boldsymbol{\Delta}^{\mathbf{I}}\mathbf{x} = \left(-\frac{g'}{g}, \frac{\cos v\left(1 - g'^2 - gg''\right)}{g}, -\frac{\sin v\left(1 - g'^2 - gg''\right)}{g}\right).$$

Suppose that $\mathbf{M}$ satisfies (4.1). Then from (4.2) and (4.3), we have

$$\left.\begin{aligned} a_{11}u + a_{12}g(u)\cos v - a_{13}g(u)\sin v &= -\frac{g'}{g} \\ a_{21}u + a_{22}g(u)\cos v - a_{23}g(u)\sin v &= \frac{\cos v\left(1 - g'^2 - gg''\right)}{g} \\ a_{31}u + a_{32}g(u)\cos v - a_{33}g(u)\sin v &= -\frac{\sin v\left(1 - g'^2 - gg''\right)}{g} \end{aligned}\right\} \tag{4.6}$$

Since the functions $\cos v, \sin v$ and the constant function are linearly independent, by (4.6) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = \lambda, a_{22} = a_{33} = \mu$. Consequently the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{4.7}$$

and (4.6) can be rewritten as

$$\lambda u = -\frac{g'}{g}, \tag{4.8}$$

$$\mu g(u) \cos v = \frac{\cos v \left(1 - g'^2 - gg''\right)}{g}, \tag{4.9}$$

$$\mu g(u) \sin v = \frac{\sin v \left(1 - g'^2 - gg''\right)}{g}. \tag{4.10}$$

From (4.8), (4.9) and (4.10), we obtain

$$\begin{aligned} \lambda u &= -\frac{g'}{g} \text{ or } \ g = -\frac{g'}{\lambda u}, \ \lambda \neq 0. \\ \mu g(u) &= \frac{\left(1 - g'^2 - gg''\right)}{g}. \end{aligned} \tag{4.11}$$

Combining the first and the second equation of (4.11), we obtain

$$\left(\lambda u + \frac{\mu}{\lambda u}\right) g'(u) - \frac{\lambda u}{g'(u)} - g''(u) = 0. \tag{4.12}$$

If we solve ordinary differential equation (4.12) with Mathematica, we get

$$g(u) = c_1 \pm \int_1^u e^{\frac{\lambda x^2}{2}} \sqrt{\left(c_2 x^{\frac{2\mu}{\lambda}} + (\lambda x^2)^{\frac{\mu}{\lambda}}\, Gamma\left[1 - \frac{\mu}{\lambda}, \lambda x^2\right]\right)} dx, \tag{4.13}$$

where $\lambda \neq 0$, $\mu \neq 0$, $c_i \in \mathbb{R}$. The solution (4.13) does not satisfy (4.8) and (4.9). Let $\lambda \neq 0, \mu = 0$, from (4.12), we obtain

$$(\lambda u)\, g'(u) - \frac{\lambda u}{g'(u)} - g''(u) = 0. \tag{4.14}$$

Its general solution is

$$g(u) = c_1 \pm \int_1^u \sqrt{1 + e^{2c_2 + \lambda x^2}} dx, \tag{4.15}$$

where $c_1, c_2 \in \mathbb{R}$. The solution (4.15) does not satisfy (4.8) and (4.9). Since $\lambda \neq 0$, there is no harmonic surface of revolution given by (3.4) in the three dimensional Galilean space $\mathbb{G}_3$.

5. Surfaces of Revolution of Type 1 Satisfying $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{Ax}$

In this section, we classify surfaces of revolution of Type 1 with non-degenerate second fundamental form in $\mathbb{G}_3$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x} = \mathbf{Ax}, \tag{5.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$. By a straightforward computation, the Laplacian $\boldsymbol{\Delta}^{\mathbf{II}}$ of the second fundamental form $\mathbf{II}$ on $\mathbf{M}$ is expressible as

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \begin{pmatrix} \left(\frac{g'(u)g''(u) - g(u)g'''(u)}{2g(u)g''^2(u)}\right) \\ \cos v\left(\frac{g'^2(u)g''(u) + 4g(u)g''^2(u) - g(u)g'(u)g'''(u)}{2g(u)g''^2(u)}\right) \\ -\sin v\left(\frac{g'^2(u)g''(u) + 4g(u)g''^2(u) - g(u)g'(u)g'''(u)}{2g(u)g''^2(u)}\right) \end{pmatrix}. \tag{5.2}$$

Suppose that $\mathbf{M}$ satisfies (5.1). Then from (2.4) and (4.4), we have

$$\begin{aligned} a_{11}u + a_{12}g(u)\cos v - a_{13}g(u)\sin v &= \frac{1}{2g''}\left(\frac{g'}{g} - \frac{g'''}{g''}\right), \\ a_{21}u + a_{22}g(u)\cos v - a_{23}g(u)\sin v &= \cos v\left(\frac{1}{2g''}\left(\frac{g'}{g} - \frac{g'''}{g''}\right) + 2\right), \\ a_{31}u + a_{32}g(u)\cos v - a_{33}g(u)\sin v &= -\sin v\left(\frac{1}{2g''}\left(\frac{g'}{g} - \frac{g'''}{g''}\right) + 2\right). \end{aligned} \tag{5.3}$$

Since the functions $\cos v$, $\sin v$ and the constant function are linearly independent, by (5.3) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = \lambda$, $a_{22} = a_{33} = \mu$. Consequently matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{5.4}$$

Then the system (5.3) reduces now to the equations

$$\begin{aligned} 2\lambda u g''^2 &= g'g'' - gg''', \\ -2\mu g^2 g''^2 - 4\mu g g''^2 &= g'\left(g'g'' - gg'''\right). \end{aligned} \tag{5.5}$$

Combining the first and the second equation of (5.5), we get

$$2 - \mu g + \lambda u g' = 0, \tag{5.6}$$

where $g \neq 0$ and $g'' \neq 0$. Its general solution is given by

$$g(u) = \frac{2}{\mu} + c_1 u^{\frac{\mu}{\lambda}}, \tag{5.7}$$

where $c_1 \in \mathbb{R}$. The solution (5.7) does not satisfies (5.5). If $\lambda \neq 0$, $\mu = 0$, then we have

$$g(u) = c_1 - \frac{2 \ln u}{\lambda}. \tag{5.8}$$

The solution (5.8) does not satisfies (5.5). Let $\lambda = 0, \mu = 0$, from (5.6), we have a contradiction. Consequently, we have:

Theorem 5.1. *Let* $\mathbf{M}$ *be a non-isotropic surface of revolution of Type 1 with non-degenerate second fundamental form given by (3.4) in the three dimensional Galilean space* $\mathbb{G}_3$. *There is no the surface* $\mathbf{M}$ *satisfying the condition* $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{A}\mathbf{x}$, $\mathbf{A} \in Mat(3, R)$.

6. Surfaces of Revolution of Type 1 Satisfying $\Delta^{\mathbf{I}}\mathbf{G} = \mathbf{AG}$

In this section, we classify surfaces of revolution of Type1 in $\mathbb{G}_3$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{I}}\mathbf{G} = \mathbf{AG}, \tag{6.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$.

Theorem 6.1. *Let* $\mathbf{M}$ *be a surface of revolution given by (3.4) in the three dimensional Galilean space* $\mathbb{G}_3$. *Then* $\mathbf{M}$ *satisfies (6.1) if and only if it is an open part of a cylinder.*

Proof. Let $\mathbf{M}$ be a surface of revolution generated by a unit speed nonisotropic curve in $\mathbb{G}_3$. Then $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u, v) = (u, g(u) \cos v, -g(u) \sin v) . \tag{6.2}$$

where g is a positive function. The Gauss map $\mathbf{G}$ of $\mathbf{M}$ is obtained by

$$\mathbf{G} = (0, -\cos v, \sin v) . \tag{6.3}$$

Suppose that $\mathbf{M}$ satisfies (6.1). Then from (4.4) and (5.3) we get the system of differential equations

$$\begin{aligned} &- a_{12} \cos v + a_{13} \sin v = 0, \\ &- a_{22} \cos v + a_{23} \sin v = -\frac{\cos v}{g^2}, \\ &- a_{32} \cos v + a_{33} \sin v = \frac{\sin v}{g^2}. \end{aligned} \tag{6.4}$$

In order to prove the theorem we have to solve (6.4). From (6.4) we easily deduce that

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{32} = 0,\ a_{22} = a_{33},\ a_{22} = a_{33} = \frac{1}{g^2(u)} \tag{6.5}$$

and

$$\Delta^{\mathbf{I}}\mathbf{G} = \frac{1}{g^2(u)}\mathbf{G}. \tag{6.6}$$

From this $g(u)$ is a constant function. Consequently, $\mathbf{M}$ is an open part of a cylinder. It can be easily shown that the converse assertion is also true. □

Theorem 6.2. *There is no surfaces of revolution of Type 1 generated by a non-isotropic curve in $\mathbb{G}_3$ with harmonic Gauss map.*

Proof. Let $\mathbf{M}$ be a surface of revolution of Type 1 defined by (3.4) in $\mathbb{G}_3$. If $\mathbf{M}$ has harmonic Gauss map, that is, $\mathbf{M}$ satisfies $\boldsymbol{\Delta}^{\mathbf{I}}\mathbf{G} = 0$, then $g^{-2}(u)\mathbf{G} = 0$. It is impossible because $g(u)$ is a positive function and $\mathbf{G}$ is the unit normal vector field of $\mathbf{M}$. □

Theorem 6.3. *Let $\mathbf{M}$ be a surface of revolution of Type 1 generated by a non-isotropic curve in the three dimensional Galilean space $\mathbb{G}_3$. Then $\mathbf{M}$ has point wise 1-type Gauss map of the first kind.*

Proof. Let $\mathbf{M}$ be a surface of revolution of Type 1 generated by a non-isotropic curve in $\mathbb{G}_3$. Suppose that $\mathbf{M}$ has pointwise 1-type Gauss map. Combining (1.5) and (6.6), one gets $\mathbf{F}(u) = g^{-2}(u)$ and $\mathbf{C} = 0$. Thus the Gauss map $\mathbf{G}$ of $\mathbf{M}$ is of pointwise 1-type of the first kind. □

Theorem 6.4. *There is no surface of revolution of Type 1 generated by a non-isotropic curve in $\mathbb{G}_3$ with pointwise 1-type Gauss map of the second kind.*

Proof. Let $\mathbf{M}$ be a surface of revolution of Type 1 defined by (3.4) in $\mathbb{G}_3$. By Theorem 6.3, $\mathbf{M}$ has only pointwise 1-type Gauss map of the first kind. Thus, the theorem is proved. □

Remark 6.5. We consider a surface defined by

$$\mathbf{x}(u,v) = \left(u, \left(a^2u + b^2\right)\cos v, -\left(a^2u + b^2\right)\sin v\right), \tag{6.7}$$

where $a, b \in \mathbb{R}$ and $u > -\frac{b^2}{a^2}$. The surface is a cone satisfying the $\left(a^2x + b^2\right)^2 = y^2 + z^2$. From (6.6) the Laplacian $\Delta^{\mathbf{I}}\mathbf{G}$ of the Gauss map $\mathbf{G}$ of the surface is obtained by

$$\Delta^{\mathbf{I}}\mathbf{G} = \frac{1}{\left(a^2u + b^2\right)^2}\mathbf{G}. \tag{6.8}$$

Thus, a cone in $\mathbb{G}_3$ has pointwise 1-type Gauss map of the first kind.

7. Surfaces of Revolution of Type 1 Satisfying $\Delta^{\mathbf{II}}\mathbf{G} = \mathbf{AG}$

In this section, we classify surfaces of revolution of Type 1 in $\mathbb{G}_3$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{G}= \mathbf{AG}, \tag{7.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$.

Theorem 7.1. *There is no non-isotropic surfaces of revolution of Type 1 given by (4.2) satisfying (7.1) in the three dimensional Galilean space* $\mathbb{G}_3$.

Proof. Let $\mathbf{M}$ be a surface of revolution generated by a unit speed nonisotropic curve in $\mathbb{G}_3$. Suppose that $\mathbf{M}$ satisfies (7.1). Then from (4.4) and (5.3) we get the system of equations

$$\begin{aligned} &- a_{12}\cos v + a_{13}\sin v = 0,\\ &- a_{22}\cos v + a_{23}\sin v = -\frac{\cos v}{g},\\ &- a_{32}\cos v + a_{33}\sin v = \frac{\sin v}{g}. \end{aligned} \tag{7.2}$$

In order to prove the theorem we have to solve (7.2). From (7.2) we easily deduce that

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{32} = 0,\ a_{22} = a_{33},\ a_{22} = \frac{1}{g(u)}$$

and

$$\Delta^{\mathbf{II}}\ \mathbf{G} = \frac{1}{g(u)}\ \mathbf{G}\ . \tag{7.3}$$

From this $g(u)$ is a constant function. For the nondegeneracy of the second fundamental form of $\mathbf{M}$, we assume that g'' is nonvanishing everywhere. If a non-isotropic surface of revolution of Type 1 satisfies (7.1), then the function g is constant. It is a contradiction. □

Theorem 7.2. *There is no surfaces of revolution of Type 1 generated by a non-isotropic curve in* $\mathbb{G}_3$ *with harmonic Gauss map.*

Proof. Let $\mathbf{M}$ be a surface of revolution of Type 1 defined by (4.1) in $\mathbb{G}_3$. If $\mathbf{M}$ has harmonic Gauss map, that is, $\mathbf{M}$ satisfies $\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{G}=0$, then $g^{-1}(u)\ \mathbf{G} = 0$. It is impossible because $g(u)$ is a positive function and $\mathbf{G}$ is the unit normal vector field of $\mathbf{M}$. □

Theorem 7.3. *Let* $\mathbf{M}$ *be a surface of revolution of Type 1 generated by a non-isotropic curve in the three dimensional Galilean space* $\mathbb{G}_3$*. Then* $\mathbf{M}$ *has point wise 1-type Gauss map of the first kind.*

Proof. Let $\mathbf{M}$ be a surface of revolution of Type 1 generated by a non-isotropic curve in $\mathbb{G}_3$. Suppose that **M** has pointwise 1-type Gauss map. Combining (1.5) and (7.3), one gets $\mathbf{F}(u) = g^{-1}(u)$ and $\mathbf{C} = 0$. Thus the Gauss map **G** of **M** is of pointwise 1-type of the first kind. □

Theorem 7.4. *There is no surface of revolution of Type 1 generated by a non-isotropic curve in* $\mathbb{G}_3$ *with pointwise 1-type Gauss map of the second kind.*

Proof. Let **M** be a surface of revolution of Type 1 defined by (4.1) in $\mathbb{G}_3$. By Theorem 7.3, **M** has only pointwise 1-type Gauss map of the first kind. Thus, the theorem is proved. □

Remark 7.5. We consider a surface defined by

$$\mathbf{x}(u,v) = \left(u, \left(a^2u + b^2\right)\cos v, -\left(a^2u + b^2\right)\sin v\right), \tag{7.4}$$

where $a, b \in \mathbb{R}$ and $u > -\frac{b^2}{a^2}$. The surface is a cone satisfying the $\left(a^2x + b^2\right)^2 = y^2 + z^2$. From (7.3) the Laplacian $\Delta^{\mathbf{II}}\mathbf{G}$ of the Gauss map **G** of the surface is obtained by

$$\Delta^{\mathbf{II}}\ \mathbf{G} = \frac{1}{(a^2u + b^2)}\ \mathbf{G}. \tag{7.5}$$

Thus, a cone in $\mathbb{G}_3$ has pointwise 1-type Gauss map of the first kind.

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