GENERIC LIGHTLIKE SUBMANIFOLDS OF 
AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH 
a non-symmetric non-metric connection of 
type ($\ell, m$)

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Abstract. Jin [7] defined a new connection on semi-Riemannian manifolds, which is a non-symmetric and non-metric connection. He said that this connection is an ($\ell, m$)-type connection. Jin also studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an ($\ell, m$)-type connection in [7]. We study further the geometry of this subject. In this paper, we study generic lightlike submanifolds of an indefinite trans-Sasakian manifold endowed with an ($\ell, m$)-type connection.

1. Introduction

The notion of ($\ell, m$)-type connection on indefinite almost contact manifolds $\tilde{M}$ was introduced by Jin [7]. Here we quote Jin’s definition as follows:

A linear connection $\tilde{\nabla}$ on $\tilde{M}$ is called a non-symmetric non-metric connection of type ($\ell, m$), and abbreviate it to ($\ell, m$)-type connection, if there exist smooth functions $\ell$ and $m$ on $\tilde{M}$ such that $\tilde{\nabla}$ and its torsion tensor $\tilde{T}$ satisfy

\begin{align}
(\nabla X g)(\tilde{Y}, \tilde{Z}) &= -\ell(\theta(\tilde{Y})g(\tilde{X}, \tilde{Z}) + \theta(\tilde{Z})g(\tilde{X}, \tilde{Y})) \\
&\quad - m(\theta(\tilde{Y})g(J\tilde{X}, \tilde{Z}) + \theta(\tilde{Z})g(J\tilde{X}, \tilde{Y})) \\
&\quad + \ell(\theta(\tilde{Y})X - \theta(\tilde{X})\tilde{Y}) \\
&\quad + m(\theta(\tilde{Y})J\tilde{X} - \theta(\tilde{X})J\tilde{Y}),
\end{align}

where $J$ is a (1,1)-type tensor field and $\theta$ is a 1-form associated with a smooth vector field $\zeta$ by $\theta(X) = g(X, \zeta)$. We set $(\ell, m) \neq (0, 0)$ and we denote by $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ the smooth vector fields on $\tilde{M}$. Semi-symmetric non-metric connection and non-metric $\phi$-symmetric connection are important examples of this connection such that (1) $(\ell, m) = (1, 0)$ and (2) $(\ell, m) = (0, 1)$, respectively.

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Especially, in cases: (3) $(\ell, m) = (1, 0)$ in (1) and $(\ell, m) = (0, 1)$ in (2) (see [10] in details); (4) $(\ell, m) = (0, 0)$ in (1) and $(\ell, m) = (0, 1)$ in (2) and (5) $(\ell, m) = (0, 0)$ in (1) and $(\ell, m) = (1, 0)$ in (2), this connection $\nabla$ reduce to quarter-symmetric non-metric connection, quarter-symmetric metric connection and semi-symmetric metric connection, respectively.

Remark 1.1. Denote by $\bar{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(M, \bar{g})$ with respect to the metric $\bar{g}$. It is known [7] that a linear connection $\nabla$ on $M$ is an $(\ell, m)$-type connection if and only if $\nabla$ satisfies

$$\bar{\nabla}_X Y = \bar{\nabla}_X Y + \theta(Y)\{\ell X + mJX\}. \tag{3}$$

A lightlike submanifold $M$ of an indefinite almost contact manifold $\bar{M}$ is said to be generic if there exists a screen distribution $S(TM)$ on $M$ such that

$$J(S(TM)^\perp) \subset S(TM), \tag{4}$$

where $S(TM)^\perp$ is the orthogonal complement of $S(TM)$ in the tangent bundle $TM$ on $\bar{M}$, i.e., $TM = S(TM) \oplus_{\text{orth}} S(TM)^\perp$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [8] and later, studied by several geometers [3, 5, 6, 9]. Its geometry is an extension of that of lightlike hypersurfaces and half lightlike submanifolds. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

The subject of study in this paper is generic lightlike submanifolds of an indefinite trans-Sasakian manifold $M = (\bar{M}, \zeta, \theta, J, \bar{g})$ endowed with an $(\ell, m)$-type connection subject to the following two conditions that (1) the tensor field $J$ and the 1-form $\theta$, defined by (1) and (2), are identical with the indefinite trans-Sasakian structure tensor $J$ and the structure 1-form $\theta$ of $\bar{M}$, respectively, and (2) the structure vector field $\zeta$ of $\bar{M}$ is tangent to $M$.

2. $(\ell, m)$-type connections

The notion of trans-Sasakian manifold $\bar{M}$, of type $(\alpha, \beta)$, was introduced by Oubina [11]. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of trans-Sasakian manifolds such that

$$\alpha = 1, \beta = 0; \quad \alpha = 0, \beta = 1; \quad \alpha = \beta = 0,$$

respectively. If $\bar{M}$ is a semi-Riemannian manifold with a trans-Sasakian structure of type $(\alpha, \beta)$, then $\bar{M}$ is called an indefinite trans-Sasakian manifold as follows:

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an indefinite trans-Sasakian manifold if there exist (1) a structure set $\{J, \zeta, \theta, \bar{g}\}$, where $J$ is a $(1, 1)$-type tensor field, $\zeta$ is a vector field and $\theta$ is a 1-form such that

$$J^2 X = -X + \theta(X)\zeta, \quad \theta(\zeta) = 1, \quad \theta(X) = \bar{g}(X, \zeta), \tag{5}$$

$$\bar{g}(JX, JY) = \bar{g}(X, Y) - \epsilon \theta(X)\theta(Y),$$

(2) a Levi-Civita connection $\bar{\nabla}$ and two smooth functions $\alpha$ and $\beta$ such that

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \epsilon \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \epsilon \theta(Y)JX\},$$

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where \( \epsilon \) denotes \( \epsilon = 1 \) or \( \epsilon = -1 \) according as \( \zeta \) is spacelike or timelike respectively. 

\( \{ J, \zeta, \theta, \bar{g} \} \) is called an indefinite trans-Sasakian structure of type \((\alpha, \beta)\). 

In the entire discussion of this article, we shall assume that the vector field \( \zeta \) is a spacelike one, i.e., \( \epsilon = 1 \), without loss of generality.

Let \( \nabla \) be an \((\ell, m)\)-type connection on \((M, g)\). By directed calculation from (3), (5) and the fact that \( \theta(Y) = 0 \), we obtain

\[
(\nabla_X J)\bar{Y} = \alpha \{ g(X, Y)\zeta - \theta(Y)X \} + \beta \{ g(JX, Y)\zeta - \theta(Y)JX \} - \theta(\bar{Y}) \{ \ell JX - mX + \beta \theta(X)\zeta \}.
\]

Replacing \( \bar{Y} \) by \( \zeta \) to (6) and using \( J\zeta = 0 \) and \( \theta(\nabla_X \zeta) = \ell \theta(X) \), we obtain

\[
\nabla_X \zeta = (m - \alpha)JX + (\ell + \beta)X - \beta \theta(X)\zeta.
\]

Let \((M, g)\) be an \(m\)-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold \((M, \bar{g})\) of dimension \((m + n)\). Then the radical distribution \( \text{Rad}(TM) = TM \cap TM^\perp \) on \( M \) is a subbundle of the tangent bundle \( TM \) and the normal bundle \( TM^\perp \), of rank \( r(1 \leq r \leq \min\{m, n\}) \). In case \( 1 < r < \min\{m, n\} \), we say that \( M \) is an \( r\)-lightlike submanifold \([3]\) of \( M \). In this case, there exist two complementary non-degenerate distributions \( S(TM) \) and \( S(TM^\perp) \) of \( \text{Rad}(TM) \) in \( TM \) and \( TM^\perp \), respectively, which are called the screen distribution and co-screen distribution of \( M \) such that

\[
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),
\]

where \( \oplus_{\text{orth}} \) denotes the orthogonal direct sum. Denote by \( F(M) \) the algebra of smooth functions on \( M \) and by \( \Gamma(E) \) the \( F(M) \) module of smooth sections of a vector bundle \( E \) over \( M \). Also denote by (5), the \( i \)-th equation of (5). We use the same notations for any others. Let \( X, Y, Z \) and \( W \) be the vector fields on \( M \), unless otherwise specified. We use the following range of indices:

\[
i, j, k, \ldots \in \{1, \ldots, r\}, \quad a, b, c, \ldots \in \{r + 1, \ldots, n\}.
\]

Let \( \text{tr}(TM) \) and \( \text{ltr}(TM) \) be complementary vector bundles to \( TM \) in \( TM|_M \) and \( TM^\perp \) in \( S(TM)^\perp \) respectively and let \( \{N_1, \ldots, N_r\} \) be a lightlike basis of \( \text{ltr}(TM)|_U \), where \( U \) is a coordinate neighborhood of \( M \), such that

\[
\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,
\]

where \( \{\xi_1, \ldots, \xi_r\} \) is a lightlike basis of \( \text{Rad}(TM)|_U \). In this case,

\[
TM = TM \oplus \text{tr}(TM) = \{ \text{Rad}(TM) \oplus \text{tr}(TM) \} \oplus_{\text{orth}} S(TM) = \{ \text{Rad}(TM) \oplus \text{ltr}(TM) \} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp),
\]

\[
\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, E_{r+1}, \ldots, E_n\}
\]

is a quasi-orthonormal field of frames of \( M \), where \( \{F_{r+1}, \ldots, F_m\} \) is an orthonormal basis of \( S(TM) \) and \( \{E_{r+1}, \ldots, E_n\} \) is an orthonormal basis of \( S(TM^\perp) \). Denote \( \epsilon_a = \bar{g}(E_a, E_a) \). Then \( \epsilon_a \delta_{ab} = \bar{g}(E_a, E_b) \).
Let $P$ be the projection morphism of $TM$ on $S(TM)$. Then the local Gauss-Weingarten formulae of $M$ and $S(TM)$ are given respectively by

(8) $\nabla_X Y = \nabla_X Y + \sum_{i=1}^{r} h^i(X,Y)N_i + \sum_{a=r+1}^{n} h^a_a(X,Y)E_a,$

(9) $\nabla_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ai}(X)E_a,$

(10) $\nabla_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)E_b;$

(11) $\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h^i(X,PY)\xi_i,$

(12) $\nabla_X \xi_i = -A_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j,$

where $\nabla$ and $\nabla^*$ are induced linear connections on $M$ and $S(TM)$ respectively, $h^i_i$ and $h^a_a$ are called the local second fundamental forms on $M$, $h^i_i$ are called the local second fundamental forms on $S(TM)$. $A_{N_i}, A_{E_a}$ and $A_{\xi_i}$ are called the shape operators, and $\tau_{ij}, \rho_{ai}, \lambda_{ai}$ and $\sigma_{ab}$ are 1-forms.

Let $M$ be a generic lightlike submanifold of $\bar{M}$. From (4) we show that $J(\text{Rad}(TM))$, $J(\text{ltr}(TM))$ and $J(S(TM^\bot))$ are vector subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions $H_o$ and $H$ with respect to $J$, i.e., $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = (J(\text{Rad}(TM)) \oplus J(\text{ltr}(TM))) \oplus_{\text{orth}} J(S(TM^\bot)) \oplus_{\text{orth}} H_o,$$

$$H = \text{Rad}(TM) \oplus_{\text{orth}} J(\text{Rad}(TM)) \oplus_{\text{orth}} H_o.$$

In this case, the tangent bundle $TM$ on $M$ is decomposed as follows:

(13) $\quad TM = H \oplus J(\text{ltr}(TM)) \oplus_{\text{orth}} J(S(TM^\bot)).$

Consider local null vector fields $U_i$ and $V_i$ for each $i$, local non-null unit vector fields $W_a$ for each $a$, and their 1-forms $u_i, v_i$ and $w_a$ defined by

(14) $\quad U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_a = -JE_a,$

(15) $\quad u_i(X) = g(X,V_i), \quad v_i(X) = g(X,U_i), \quad w_a(X) = \epsilon_ag(X,W_a).$

Denote by $S$ the projection morphism of $TM$ on $H$ and by $F$ the tensor field of type $(1,1)$ globally defined on $M$ by $F = J \circ S$. Then $JX$ is expressed as

(16) $\quad JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a.$
Applying $J$ to (16) and using (5)$_1$ and (14), we have

$$F^2 X = -X + \theta(X)\zeta + \sum_{i=1}^r u_i(X)U_i + \sum_{a=r+1}^n w_a(X)W_a. \tag{17}$$

3. Structure equations

Let $\bar{M}$ be an indefinite trans-Sasakian manifold with an $(\ell, m)$-type connection $\nabla$. In the following, we shall assume that $\zeta$ is tangent to $\bar{M}$. Călin [2] proved that if $\zeta$ is tangent to $\bar{M}$, then it belongs to $S(TM)$ which we assumed in this paper. Using (1), (2), (8) and (16), we see that

$$\bar{h}(X, Y) = \sum_{i=1}^r \{ h^i(X, Y)\eta_i(Z) + h^i_0(X, Z)\eta_i(Y) \}$$

$$= \ell \{ \theta(Y)g(X, Z) + \theta(Z)g(X, Y) \}$$

$$- m \{ \theta(Y)\bar{g}(JX, Z) + \theta(Z)\bar{g}(JX, Y) \}, \tag{18}$$

$$T(X, Y) = \ell \{ \theta(Y)X - \theta(X)Y \} + m \{ \theta(Y)FX - \theta(X)FY \}, \tag{19}$$

$$h^*_a(X, Y) = h^*_a(Y, X) = m \{ \theta(Y)w_a(X) - \theta(X)w_a(Y) \}, \tag{20}$$

$$h^*_a(X, Y) - h^*_a(Y, X) = m \{ \theta(Y)w_a(X) - \theta(X)w_a(Y) \}, \tag{21}$$

Theorem 3.1. Let $M$ be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with an $(\ell, m)$-type connection subject such that $\zeta$ is tangent to $M$. Then either $h^*_1$ or $h^*_a$ is symmetric if and only if $m = 0$.

Proof. (1) If $m = 0$, then $h^*_1$ are symmetric by (20). Conversely, if $h^*_1$ is symmetric, then, taking $X = \zeta$ and $Y = U_i$ to (20), we have $m = 0$.

(2) If $m = 0$, then $h^*_a$ are symmetric by (21). Conversely, if $h^*_a$ is symmetric, then, taking $X = \zeta$ and $Y = W_a$ to (21), we have $m = 0$. \[ \square \]

From the facts that $h^*_1(X, Y) = \bar{g}(\nabla_X Y, \xi_j)$ and $\epsilon_a h^*_a(X, Y) = \bar{g}(\nabla_X Y, E_a)$, we know that $h^*_1$ and $h^*_a$ are independent of the choice of $S(TM)$. Applying $\nabla_X$ to $\bar{g}(\xi_i, \xi_j) = 0, \bar{g}(\xi_i, E_a) = 0, \bar{g}(N_i, N_j) = 0, \bar{g}(N_i, E_a) = 0$ and $\bar{g}(E_a, E_b) = \epsilon_{ab}$ by turns and using (1) and (8) ~ (10), we obtain

$$h^*_1(X, \xi_j) + h^*_1(X, \xi_i) = 0, \quad h^*_a(X, \xi_i) = -\epsilon_a \lambda_{ai}(X), \tag{22}$$

$$\eta_j(A_{\xi_j} X) + \eta_i(A_{\xi_i} X) = 0, \quad \eta_i(A_{\xi_i} X) = \epsilon_{a\xi_i} \lambda_{ai}(X),$$

$$\epsilon_a \sigma_{ab} + \epsilon_a \sigma_{ba} = 0; \quad h^*_1(X, \xi_j) = 0, \quad h^*_1(\xi_j, \xi_k) = 0, \quad A^*_{\xi_j} \xi_i = 0.$$

The local second fundamental forms are related to their shape operators by

$$h^*_1(X, Y) = g(A^*_1 X, Y) + m\theta(Y)u_i(X) - \sum_{k=1}^r h^*_k(X, \xi_i)\eta_k(Y), \tag{23}$$

$$\epsilon_a h^*_a(X, Y) = g(A_{\xi_a} X, Y) + \epsilon_a m\theta(Y)w_a(X) - \sum_{k=1}^r \lambda_{ak}(X)\eta_k(Y), \tag{24}$$
Applying $\bar{\theta}$ to (28) we have

$$h^*_a(X,Y) = g(A_{\xi_a}X,Y) + \{\ell\eta(X) + mw_i(X)\} \theta(Y).$$

Replacing $Y$ by $\zeta$ to (8) and using (7), (16), (23) and (24), we have

1. $\nabla_X \zeta = (m - \alpha)FX + (\ell + \beta)X - \beta\theta(X)\zeta$,
2. $\theta(A^*_\xi_iX) = -\alpha u_i(X)$, $h^*_i(X,\zeta) = (m - \alpha)u_i(X)$,
3. $\theta(A_{\xi_a}X) = -e_a\alpha w_a(X)$, $h^*_a(X,\zeta) = (m - \alpha)w_a(X)$.

Applying $\nabla_X$ to $\dot{g}(\zeta, N_i) = 0$ and using (7), (9) and (25), we have

$$\theta(A_{\eta_i}X) = -\alpha v_i(X) + \beta\eta_i(X),$$
$$h^*_i(X,\zeta) = (\ell + \beta)\eta_i(X) + (m - \alpha)v_i(X).$$

Applying $\nabla_X$ to (14) to (16) by turns and using (6), (8) to (12), (14) to (16) and (23) to (25), we have

$$\nabla_X U_i = F(A_{\xi_i}X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a - \{\alpha\eta_i(X) + \beta v_i(X)\} \zeta,$$

$$\nabla_X V_i = F(A^*_\xi_iX) - \sum_{j=1}^r \tau_{ij}(X)V_j + \sum_{j=1}^r h^*_j(X,\zeta)U_j - \sum_{a=r+1}^n \epsilon_a\lambda_{ai}(X)W_a - \beta u_i(X)\zeta,$$

$$\nabla_X W_a = F(A_{\xi_a}X) + \sum_{i=1}^r \lambda_{ai}(X)U_i + \sum_{b=r+1}^n \sigma_{ab}(X)W_b - \epsilon_a\beta w_a(X)\zeta,$$

$$(\nabla_X F)(Y) = \sum_{i=1}^r u_i(Y)A_{\xi_i}X + \sum_{a=r+1}^n w_a(Y)A_{\xi_a}X$$

$$- \sum_{i=1}^r h^*_i(X,Y)U_i - \sum_{a=r+1}^n h^*_a(X,Y)W_a + \{\alpha g(X,Y) + \beta g(JX,Y) - \theta(X)\theta(Y)\} \zeta + (m - \alpha)\theta(Y)X - (\ell + \beta)\theta(Y)FX,$$

$$(\nabla_X u_i)(Y) = - \sum_{j=1}^r u_j(Y)\tau_{ij}(X) - \sum_{a=r+1}^n w_a(Y)\lambda_{ai}(X) - h^*_i(X,FY) - (\ell + \beta)\theta(Y)u_i(X),$$
Theorem 3.2. There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with an \((\ell, m)\)-type connection subject such that \(\zeta\) is tangent to \(M\) and \(F\) satisfies the following equation:

\[
(\nabla_X F) Y = (\nabla_Y F) X, \quad \forall X, Y \in \Gamma(TM).
\]

Proof. Let \((\nabla_X F) Y = (\nabla_Y F) X\). Using (20), (21) and (34), we obtain

\[
(36) \quad (\nabla_X v_i)(Y) = \sum_{j=1}^{r} v_j(Y) \tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y) \rho_{ia}(X)
\]

\[+ \sum_{j=r+1}^{r} u_j(Y) \eta_i(A_{N_j} X) - g(A_{N_i} X, FY) \]

\[+ (m - \alpha) \theta(Y) \eta_i(X) - (\ell + \beta) \theta(Y) v_i(X).\]

Proof. Let \((\nabla_X F) Y = (\nabla_Y F) X\). Using (20), (21) and (34), we obtain

\[
(37) \quad \sum_{i=1}^{r} \{u_i(Y) A_{N_i} X - u_i(X) A_{N_i} Y\}
\]

\[+ \sum_{a=r+1}^{n} \{w_a(Y) A_{E_a} X - w_a(X) A_{E_a} Y\} - 2\beta \bar{g}(X, JY) \zeta
\]

\[+ \{\theta(X) u_i(Y) - \theta(Y) u_i(X)\} U_i + \{\theta(X) w_a(Y) - \theta(Y) w_a(X)\} W_a
\]

\[+ (m - \alpha) \{\theta(Y) X - \theta(X) Y\} - (\ell + \beta) \{\theta(Y) FX - \theta(X) FY\} = 0.
\]

Taking the scalar product with \(\zeta\) and using (28)1 and (29)1, we have

\[
\alpha \sum_{i=1}^{r} \{u_i(Y) v_i(X) - u_i(X) v_i(Y)\}
\]

\[= \beta \sum_{i=1}^{r} \{u_i(Y) \eta_i(X) - u_i(X) \eta_i(Y)\} - 2\beta \bar{g}(X, JY).
\]

Taking \(X = V_j, Y = U_j\) and \(X = \xi_j, Y = U_j\) to this equation by turns, we obtain \(\alpha = 0\) and \(\beta = 0\), respectively. Taking \(X = \xi_j\) to (37), we have

\[
\theta(X) \{m \xi_i + \ell V_i\} + \sum_{j=1}^{r} u_j(X) A_{N_j} \xi_i + \sum_{a=r+1}^{n} w_a(X) A_{E_a} \xi_i = 0.
\]

Taking \(X = \zeta\) to this, we have \(m \xi_i + \ell V_i = 0\). It follows that \(\ell = m = 0\). It is a contradiction to \((\ell, m) \neq (0, 0)\). Thus we have our theorem. □

Corollary 3.3. There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with an \((\ell, m)\)-type connection subject such that \(\zeta\) is tangent to \(M\) and \(F\) is parallel with respect to the connection \(\nabla\).

Theorem 3.4. Let \(M\) be a generic lightlike submanifold of an indefinite trans-Sasakian manifold \(M\) with an \((\ell, m)\)-type connection such that \(\zeta\) is tangent to \(M\). If \(U_i\)'s or \(V_i\)'s are parallel with respect to \(\nabla\), then \(\tau_{ij} = 0\) and \(\alpha = \beta = 0\), i.e., \(M\) is an indefinite cosymplectic manifold.
Proof. (1) If $U_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta$, $V_i$, $W_a$, $U_j$ and $N_j$ to (31) such that $\nabla_X U_i = 0$ respectively, we get

$$\alpha = \beta = 0, \quad \tau_{ij} = 0, \quad \rho_{ia} = 0, \quad \eta_j(A_{N_j} X) = 0, \quad h^i_j(X, U_j) = 0.$$  \hfill (38)

As $\alpha = \beta = 0$, $M$ is an indefinite cosymplectic manifold.

(2) If $V_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta$, $U_j$, $V_j$, $W_\alpha$ and $N_j$ to (32) with $\nabla_X V_i = 0$ respectively, we get

$$\beta = 0, \quad \tau_{ij} = 0, \quad h^i_j(X, \zeta) = 0, \quad \lambda_{ai} = 0, \quad h^i_j(X, U_j) = 0.$$  \hfill (39)

As $h^i_j(X, U_j) = 0$, we get $h^i_j(\zeta, U_j) = 0$. Taking $X = U_j$ and $Y = \zeta$ to (20), we get $h^i_j(U_j, \zeta) = m \delta_{ij}$. On the other hand, replacing $X$ by $U$ to (27)2, we have $h^i_j(U_j, \zeta) = (m - \alpha) \delta_{ij}$. It follows that $\alpha = 0$. Since $\alpha = \beta = 0$, $M$ is an indefinite cosymplectic manifold. \qed

4. Indefinite generalized Sasakian space forms

Definition. An indefinite trans-Sasakian manifold $M$ is said to be an indefinite generalized Sasakian space form and denote it by $M(f_1, f_2, f_3)$ if there exist three smooth functions $f_1$, $f_2$ and $f_3$ on $M$ such that

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = f_1(\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y})$$

$$+ f_2(\tilde{g}(\tilde{X}, J\tilde{Z})J\tilde{Y} - \tilde{g}(\tilde{Y}, J\tilde{Z})J\tilde{X} + 2\tilde{g}(\tilde{X}, J\tilde{Y})J\tilde{Z})$$

$$+ f_3(\theta(\tilde{X})\theta(\tilde{Z})\tilde{Y} - \theta(\tilde{Y})\theta(\tilde{Z})\tilde{X})$$

$$+ \tilde{g}(\tilde{X}, \tilde{Z})\theta(\tilde{Y})\zeta - \tilde{g}(\tilde{Y}, \tilde{Z})\theta(\tilde{X})\zeta,$$

where $\tilde{R}$ is the curvature tensor of the Levi-Civita connection $\nabla$.

The notion of generalized Sasakian space form was introduced by Alegre et al. \cite{Alegre}, while the indefinite generalized Sasakian space forms were introduced by Jin \cite{Jin}. Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \frac{c-3}{4}, \quad f_2 = f_3 = \frac{c-1}{4}; \quad f_1 = \frac{c-3}{4}, \quad f_2 = f_3 = \frac{c+1}{4}; \quad f_1 = f_2 = f_3 = \frac{c}{4}$$

respectively, where $c$ is a constant J-sectional curvature of each space forms.

Denote by $\tilde{R}$ the curvature tensors of the non-metric $\phi$-symmetric connection $\nabla$ on $M$. By directed calculations from (2), (3) and (5), we see that

$$(41) \quad \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}$$

$$+ (\nabla_X \theta)(\tilde{Z})\{(\tilde{X} + mJ\tilde{Y}) - (\nabla_{\tilde{Y}} \theta)(\tilde{Z})\{\tilde{X} + mJ\tilde{Y}\}$$

$$+ \theta(\tilde{Z})((\tilde{X} \tilde{X})\tilde{Y} - (\tilde{Y} \tilde{X})\tilde{X} + (\tilde{X} m)J\tilde{Y} - (\tilde{Y} m)J\tilde{X}$$

$$- m\alpha[\theta(\tilde{Y})\tilde{X} - \theta(\tilde{X})\tilde{Y}] - m\beta[\theta(\tilde{Y})J\tilde{X} - \theta(\tilde{X})J\tilde{Y}]$$

$$- 2m\delta(\tilde{X}, J\tilde{Y})\zeta).$$

Taking the scalar product with $\xi$ and $N_i$ to (41) by turns and, then denote by $R$ and $R^*$ the curvature tensors of the induced linear connections $\nabla$ and
\( \nabla^* \) on \( M \) and \( S(TM) \) respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for \( M \) and \( S(TM) \), respectively:

\[
\bar{R}(X,Y)Z = R(X,Y)Z \\
+ \sum_{i=1}^{r} \{ h^i(X,Y) A_{\alpha_i} Y - h^i(Y,Z) A_{\alpha_i} X \} \\
+ \sum_{a=r+1}^{n} \{ h^a(X,Z) A_{\alpha_a} Y - h^a(Y,Z) A_{\alpha_a} X \} \\
+ \sum_{i=1}^{r} \left\{ (\nabla_X h^i)(Y,Z) - (\nabla_Y h^i)(X,Z) \right\} \\
+ \sum_{j=1}^{r} \left[ \tau_{ji}(X) h^j(Y,Z) - \tau_{ji}(Y) h^j(X,Z) \right] \\
+ \sum_{a=r+1}^{n} \left\{ \lambda_{ai}(X) h^a(Y,Z) - \lambda_{ai}(Y) h^a(X,Z) \right\} \\
- \ell[\theta(X) h^a(Y,Z) - \theta(Y) h^a(X,Z)] \\
- m[\theta(X) h^a(FY,Z) - \theta(Y) h^a(FX,Z)] \} N_i \]

\[
+ \sum_{a=r+1}^{n} \left\{ (\nabla_X h^a)(Y,Z) - (\nabla_Y h^a)(X,Z) \right\} \\
+ \sum_{i=1}^{r} \left\{ \rho_{ia}(X) h^i(Y,Z) - \rho_{ia}(Y) h^i(X,Z) \right\} \\
+ \sum_{a=r+1}^{n} \left\{ \sigma_{ba}(X) h^a(Y,Z) - \sigma_{ba}(Y) h^a(X,Z) \right\} \\
- \ell[\theta(X) h^a(Y,Z) - \theta(Y) h^a(X,Z)] \\
- m[\theta(X) h^a(FY,Z) - \theta(Y) h^a(FX,Z)] \right\} E_a, \\
\]

(43) \[
R(X,Y)PZ = R^*(X,Y)PZ \\
+ \sum_{i=1}^{r} \{ h^*_i(X,PZ) A_{\xi_i} Y - h^*_i(Y,PZ) A_{\xi_i} X \} \\
+ \sum_{i=1}^{r} \left\{ (\nabla_X h^*_i)(Y,PZ) - (\nabla_Y h^*_i)(X,PZ) \right\} \\
+ \sum_{k=1}^{r} \left[ \tau_{ik}(Y) h^*_k(X,PZ) - \tau_{ik}(X) h^*_k(Y,PZ) \right] \\
- \ell[\theta(X) h^*_i(Y,PZ) - \theta(Y) h^*_i(FX,PZ)] \\
- m[\theta(X) h^*_i(FY,PZ) - \theta(Y) h^*_i(FX,PZ)] \right\} \xi_i. 
\]
substituting (42) and (40) and using (22) and (43), we get

\[
(44) \quad (\nabla_X h^j_i)(Y, Z) - (\nabla_Y h^j_i)(X, Z) \\
+ \sum_{j=1}^3 \{\tau_{ji}(X)h^j_i(Y, Z) - \tau_{ji}(Y)h^j_i(X, Z)\} \\
+ \sum_{a=r+1}^n \{\lambda_{ai}(X)h^a_i(Y, Z) - \lambda_{ai}(Y)h^a_i(X, Z)\} \\
- \ell\{\theta(X)h^j_i(Y, Z) - \theta(Y)h^j_i(X, Z)\} \\
- m\{\theta(X)h^j_i(FY, Z) - \theta(Y)h^j_i(FX, Z)\} \\
- m\{(\nabla_X \theta)(Z)u_i(Y) - (\nabla_Y \theta)(Z)u_i(X)\} \\
- \theta(Z)\{[Xm + m\beta\theta(X)]u_i(Y) - [Ym + m\beta\theta(Y)]u_i(X)\} \\
= f_2\{u_i(Y)\bar{g}(X, JZ) - u_i(X)\bar{g}(Y, JZ) + 2u_i(Z)\bar{g}(X, JY)\},
\]

\[
(45) \quad (\nabla_X h^*_i)(Y, PZ) - (\nabla_Y h^*_i)(X, PZ) \\
- \sum_{j=1}^r \{\tau_{ij}(X)h^*_j(Y, PZ) - \tau_{ij}(Y)h^*_j(X, PZ)\} \\
- \sum_{a=r+1}^n \epsilon_a\{\rho_{ai}(X)h^a_i(Y, PZ) - \rho_{ai}(Y)h^a_i(X, PZ)\} \\
+ \sum_{j=1}^r \{h^j_i(X, PZ)\eta_i(A_{X_j} Y) - h^j_i(Y, PZ)\eta_i(A_{X_j} X)\} \\
- \ell\{\theta(X)h^*_i(Y, PZ) - \theta(Y)h^*_i(X, PZ)\} \\
- m\{\theta(X)h^*_i(FY, PZ) - \theta(Y)h^*_i(FX, PZ)\} \\
- (\nabla_X \theta)(PZ)\{\epsilon_i(Y) + m\nu_i(Y)\} + (\nabla_Y \theta)(PZ)\{\epsilon_i(X) + m\nu_i(X)\} \\
- \theta(PZ)\{[X\ell + m\alpha\theta(X)]\eta_i(Y) - [Y\ell + m\alpha\theta(Y)]\eta_i(X)\} \\
+ [Xm + m\beta\theta(X)]u_i(Y) - [Ym + m\beta\theta(Y)]u_i(X)\} \\
= f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\
+ f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY)\} \\
+ f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\} \theta(PZ).
\]

**Theorem 4.1.** Let $M$ be a generic lightlike submanifold of an indefinite generalized Sasakian space form $M(f_1, f_2, f_3)$ with an $(\ell, m)$-type connection such that $\zeta$ is tangent to $M$. Then the functions $\alpha$, $\beta$, $f_1$, $f_2$ and $f_3$ satisfy

1. $\alpha$ is a constant on $M$,
2. $\alpha\beta = 0$, and
3. $f_1 - f_2 = \alpha^2 - \beta^2$ and $f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$. 
Proof. Applying $\nabla_X$ to $\theta(U_i) = 0$ and $\theta(V_j) = 0$ by turns and using (8), (31), (32) and the facts that $g(FX, \zeta) = 0$ and $\zeta \in \Gamma(S(TM))$, we get

$$\nabla_X \theta(U_i) = \alpha \eta_i(X) + \beta \nu_i(X), \quad \nabla_X \theta(V_i) = \beta \nu_i(X).$$

Applying $\nabla_X$ to (30)$_1$: $h^j_j(Y, U_i) = h^*_j(Y, V_j)$ and using (5), (16), (23), (25), (27)$_2$, (29)$_2$, (30)$_1, 2, 4$, (31) and (32), we obtain

$$(\nabla_X h^*_j)(Y, U_i) = (\nabla_X h^*_j)(Y, V_j)$$

$$\quad - \sum_{k=1}^r \{\tau_{kj}(X)h^*_k(Y, U_i) + \tau_{jk}(X)h^*_k(Y, V_j)\}$$

$$\quad - \sum_{a=r+1}^n \{\lambda_{aj}(X)h^*_a(Y, U_i) + \epsilon_{a}(X)h^*_a(Y, V_j)\}$$

$$\quad + \sum_{k=1}^r \{h^*_k(Y, U_i)h^*_k(X, \xi_j) + h^*_k(X, U_i)h^*_k(Y, \xi_j)\}$$

$$\quad - g(A^*_k X, F(A_{N_k} Y)) - g(A^*_k Y, F(A_{N_k} X))$$

$$\quad - \sum_{k=1}^r h^*_k(Y, V_j)\eta_k(A_{N_k} Y)$$

$$\quad + \beta(m - \alpha)\{u_j(Y)\nu_i(X) - u_j(X)\nu_i(Y)\}$$

$$\quad + \alpha(m - \alpha)u_j(Y)\eta_i(X) - \beta(\ell + \beta)u_j(X)\eta_i(Y).$$

Substituting this and (29) into the modified equation (44) which is change $i$ with $j$ and $Z$ with $U_i$, and using (22)$_3$, (30)$_3$ and (46)$_1$, we have

$$(\nabla_X h^*_j)(Y, V_j) - (\nabla_Y h^*_j)(X, V_j)$$

$$\quad - \sum_{k=1}^r \{\tau_{jk}(X)h^*_k(Y, V_j) - \tau_{ik}(Y)h^*_k(X, V_j)\}$$

$$\quad - \sum_{a=r+1}^n \{\epsilon_{a}(X)h^*_a(Y, V_j) - \lambda_{aj}(X)h^*_a(Y, V_j)\}$$

$$\quad + \sum_{k=1}^r \{h^*_k(Y, V_j)\eta_k(A_{N_k} Y) - h^*_k(Y, V_j)\eta_k(A_{N_k} X)\}$$

$$\quad - \theta(X)h^*_j(FY, V_j) + \theta(Y)h^*_j(FX, V_j)$$

$$\quad + \beta(m - 2\alpha)\{u_j(Y)\nu_i(X) - u_j(X)\nu_i(Y)\}$$

$$\quad + (\ell \beta - \alpha^2 + \beta^2)\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\}$$

$$= f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} + 2\delta_{ij}g(X, JY).$$

Comparing this with (45) such that $PZ = V_j$ and using (46)$_2$, we obtain

$$\{f_1 - f_2 - \alpha^2 + \beta^2\}\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\}$$

$$= 2\alpha \beta\{u_j(Y)\nu_i(X) - u_j(X)\nu_i(Y)\}.$$
Taking $Y = U_j$, $X = \xi_i$ and $Y = U_j$, $X = V_i$ to this by turns, we have
$$f_1 - f_2 = \alpha^2 - \beta^2, \quad \alpha \beta = 0.$$  

Applying $\nabla_X$ to $\theta(\zeta) = 1$ and using (7) and the fact: $\theta \circ J = 0$, we get
$$\nabla_X \theta(\zeta) = -\ell \theta(X).$$

Applying $\nabla_X$ to $\eta_i(Y) = \tilde{g}(Y, N_i)$ and using (1) and (9), we have
$$\nabla_X \eta_i(Y) = -g(A_{x_i} X, Y) + \sum_{j=1}^r \tau_{ij}(X) \eta_j(Y)$$
$$- \{\ell \eta_i(X) + m \eta_i(X)\} \theta(Y).$$

Applying $\nabla_X$ to $h^*_i(Y, \zeta) = (\ell + \beta) \eta_i(Y) + (m - \alpha) \eta_i(Y)$ and using (25), (26), (36), (48) and the fact that $\alpha \beta = 0$, we get
$$\nabla_X h^*_i(Y, \zeta) = X(\ell + \beta) \eta_i(Y) + X(m - \alpha) \eta_i(Y)$$
$$+ \{\ell + \beta\} \{g(A_{x_i} X, Y) - g(A_{x_i} Y, Y) - \sum_{j=1}^r \tau_{ij}(X) \eta_j(Y)$$
$$+ \beta \theta(X) \eta_i(Y) - \ell \theta(X) \eta_i(Y) + \theta(X) \eta_i(Y)\}$$
$$+ \{m [\theta(Y) \eta_i(X) + \theta(X) \eta_i(Y)]$$
$$- \{m \eta_i(X) \theta(Y) \eta_i(Y)\} + (m - \alpha) \{\eta_i(X) \eta_i(Y) - g(A_{x_i} Y, FX)$$
$$+ \sum_{j=1}^r \eta_j(Y) \tau_{ij}(X) + \sum_{a=r+1}^n \epsilon_a w_a(Y) \rho_a(X)$$
$$+ \sum_{j=1}^r \eta_j(Y) \eta_i(A_{x_i} X) + (m - \alpha) \theta(Y) \eta_i(X)$$
$$+ \beta \theta(X) \eta_i(Y) - (\ell + \beta) \theta(Y) \eta_i(Y)\}.$$

Substituting this and (29) into (45) with $PZ = \zeta$ and using (47), we get
$$\{X \beta + (f_1 - f_3 - \alpha^2 + \beta^2) \theta(X)\} \eta_i(Y)$$
$$- \{Y \beta + (f_1 - f_3 - \alpha^2 + \beta^2) \theta(Y)\} \eta_i(X)$$
$$= (X \alpha) \eta_i(Y) - (Y \alpha) \eta_i(X).$$

Taking $X = \zeta$, $Y = \xi_i$ and $X = U_j$, $Y = V_i$ to this by turns, we have
$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta \beta, \quad U_j \alpha = 0.$$  

Applying $\nabla_X$ to $h^*_i(Y, \zeta) = (m - \alpha) \eta_i(Y)$ and using (26) and (35), we get
$$\nabla_X h^*_i(Y, \zeta) = X(m - \alpha) \eta_i(Y) - (\ell + \beta) h^*_i(Y, X)$$
$$- (m - \alpha) \{\sum_{j=1}^r \eta_j(Y) \tau_{ij}(X) + \sum_{a=r+1}^n \epsilon_a w_a(Y) \lambda_a(X)$$
$$+ h^*_i(X, FY) + h^*_i(Y, FX) + \ell \theta(Y) \eta_i(X)}$$
$$+ h^*_i(Y, \zeta).$$
Substituting this into (44) with $Z = \zeta$ and using (20) and (47), we have

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Taking $Y = U_i$ to this result and using the fact that $U_i\alpha = 0$, we have $X\alpha = 0$. Therefore $\alpha$ is a constant. This completes the proof of the theorem. 

**Definition.** (1) A screen distribution $S(TM)$ is said to be totally umbilical [3] in $M$ if there exist smooth functions $\gamma_i$ on a neighborhood $U$ such that

$$h^*_i(X, PY) = \gamma_i g(X, PY).$$

In case $\gamma_i = 0$, we say that $S(TM)$ is totally geodesic in $M$.

(2) An $r$-lightlike submanifold $M$ is said to be screen conformal [4] if there exist non-vanishing smooth functions $\varphi_i$ on $U$ such that

$$(49) \quad h^*_i(X, PY) = \varphi_i h^*_i(X, PY).$$

**Theorem 4.2.** Let $M$ be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ with an $(\ell, m)$-type connection subject such that $\zeta$ is tangent to $M$. If

1. $S(TM)$ is totally umbilical, or
2. $M$ is screen conformal,

then $\tilde{M}(f_1, f_2, f_3)$ is an indefinite $\beta$-Kenmotsu manifold with a semi-symmetric non-metric connection such that

$$\alpha = m = 0, \quad \beta = -\ell \neq 0, \quad f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = -\zeta\beta.$$

**Proof.** (1) If $S(TM)$ is totally umbilical, then (29)$_2$ is reduced to

$$\gamma_i \theta(X) = (\ell + \beta) \eta_i(X) + (m - \alpha) v_i(X).$$

Taking $X = \zeta$, $X = \xi_i$ and $X = V_i$ to this equation by turns, we have

$$(50) \quad \gamma_i = 0, \quad \ell = -\beta, \quad m = \alpha,$$

respectively. As $\gamma_i = 0$, we obtain $h^*_i = 0$. Thus, from (30)$_{1, 2}$, we have

$$(51) \quad h^*_i(X, U_i) = 0, \quad h^*_i(X, U_i) = 0.$$

Replacing $Y$ by $U_j$ to (20) and using (50)$_1$ and the result: $m = \alpha$, we get

$$h^*_i(U_j, X) = \alpha \theta(X) \delta_{ij}.$$ 

Taking $X = \zeta$ to this and using (27)$_2$ such that $m = \alpha$, we have $\alpha = 0$.

As $\alpha = m = 0$ and $\beta = -\ell \neq 0$, $\tilde{M}$ is an indefinite $\beta$-Kenmotsu manifold with a semi-symmetric non-metric connection and $f_1 + \beta^2 = f_2$ by Theorem 4.1. Taking $PZ = U_j$ to (45) and using (46)$_1, (50)$ and (51), we have

$$f_2 \{ [v_j(Y) \eta_i(X) - v_i(X) \eta_j(Y)] + [v_i(Y) \eta_j(X) - v_j(X) \eta_i(Y)] \} = 0.$$

Taking $X = \xi_i$ and $Y = U_j$, we get $f_2 = 0$. Thus $f_1 = -\beta^2$ and $f_3 = -\zeta\beta.$
(2) If $M$ is screen conformal, then, from $(28)_2, (29)_2$ and $(49)$, we have
\[(l + \beta)\eta_i(X) + (m - \alpha)v_i(X) = \varphi_i(m - \alpha)u_i(X)\].

Taking $X = \xi_i$ and $X = V_i$ to this equation by turns, we see that $l = -\beta$ and $m = \alpha$, respectively. As $\alpha\beta = 0$, it follows that
\[(52) \quad \ell m = \ell\alpha = m\beta = 0, \quad \ell\beta = -\beta^2, \quad m\alpha = \alpha^2.\]

Denote by $\mu_i$ the $r$-th vector fields on $S(TM)$ such that $\mu_i = U_i - \varphi_iV_i$. Using $(30)_{1,2,3,4}$ and $(49)$, we see that
\[(53) \quad h_{ij}^\beta(X, \mu_i) = 0, \quad h_{ij}^\alpha(X, \mu_i) = 0, \quad g(\mu_i, \mu_j) = -(\varphi_j + \varphi_i)\delta_{ij}, \quad J\mu_i = N_i - \varphi_i\xi_i.\]

Applying $\nabla_X$ to $\mu_i = U_i - \varphi_iV_i$ and then, taking the scalar product with $\xi$ to the resulting equation and using $(31)$ and $(32)$, we obtain
\[g(\nabla_X\mu_i, \xi) = -\{\alpha\eta_i(X) + \beta v_i(X) - \varphi_i\beta u_i(X)\}.\]

Applying $\nabla_X$ to $\theta(\mu_i) = 0$ and using $(8)$ and the last equation, we get
\[(54) \quad (\nabla_X\theta)(\mu_i) = \alpha\eta_i(X) + \beta v_i(X) - \varphi_i\beta u_i(X).\]

Applying $\nabla_Y$ to $(49)$, we have
\[(\nabla_Xh_{ij}^\beta)(Y, PZ) = (X\varphi_i)h_{ij}^\beta(Y, PZ) + \varphi_i(\nabla_Xh_{ij}^\beta)(Y, PZ).\]

Substituting this and $(49)$ into $(45)$ and using $(44)$, we have
\[
\begin{align*}
\sum_{j=1}^n &\left\{(X\varphi_i)\delta_{ij} - \varphi_i\tau_{ij}(X) - \varphi_j\tau_{ij}(X) - \eta_i(A_{N_j}X)\right\}h_{ij}^\beta(Y, PZ) \\
- &\sum_{j=1}^n \left\{(Y\varphi_i)\delta_{ij} - \varphi_i\tau_{ij}(Y) - \varphi_j\tau_{ij}(Y) - \eta_i(A_{N_j}Y)\right\}h_{ij}^\beta(X, PZ) \\
- &\sum_{a=r+1}^n \left\{\epsilon_a\rho_a(X) + \varphi_i\lambda_{a}(X)\right\}h_{ij}^\alpha(Y, PZ) \\
+ &\sum_{a=r+1}^n \left\{\epsilon_a\rho_a(Y) + \varphi_i\lambda_{a}(Y)\right\}h_{ij}^\alpha(X, PZ) \\
- &\left\{\nabla_X\theta\right\}(PZ)\left\{\ell\eta_i(Y) + m\nu_i(Y) - \varphi\nu_i\nu_i(Y)\right\} \\
+ &\left\{\nabla_Y\theta\right\}(PZ)\left\{\ell\eta_i(X) + m\nu_i(X) - \varphi\nu_i\nu_i(X)\right\} \\
- &\theta(PZ)\left\{[X\ell + \alpha^2\theta(X)]\eta_i(Y) - [Y\ell + \alpha^2\theta(Y)]\eta_i(X)\right\} \\
+ &\left\{(Xm)g(\mu_i, Y) - (Ym)g(\mu_i, X)\right\} \\
= &\left\{f_1(g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y))\right\} \\
+ &\left\{f_2(g(\mu_i, Y)g(X, JPZ) - g(\mu_i, X)g(Y, JPZ) + 2g(\mu_i, PZ)g(X, JY))\right\} \\
+ &\left\{f_3(\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X))\right\}\theta(PZ).
\end{align*}
\]
Replacing $PZ$ by $\mu_j$ to this and using (46) and (52) $\sim$ (54), we obtain

\begin{equation}
(f_1 + \beta^2)(v_j(Y)\eta_k(X) - v_j(X)\eta_k(Y))
- \varphi_j(f_1 + \beta^2)(u_j(Y)\eta_k(X) - u_j(X)\eta_k(Y))
+ (f_2 + \alpha^2)v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)
- \varphi_i(f_2 + \alpha^2)\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y)\}
= 2f_2\delta_{ij}(\varphi_j + \varphi_i)\tilde{g}(X)j(Y).
\end{equation}

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we obtain

\[f_1 + f_2 = -(\alpha^2 + \beta^2).\]

From this result and Theorem 5.1, we see that $\alpha = 0$. As $\alpha = m = 0$ and $\beta = -\ell \neq 0$, $M(f_1, f_2, f_3)$ is an indefinite $\beta$-Kenmotsu manifold with a semi-symmetric non-metric connection. Taking $X = \xi_i$ and $Y = V_j$ to the modified equation (55) which is change $j$ with $i$, we obtain $f_2\varphi_i = 0$. As all $\varphi_i$ are non-vanishing functions, we get $f_2 = 0$. Thus $f_1 = -\beta^2$ and $f_3 = -\xi_\beta$. \hfill \Box

**Theorem 4.3.** Let $M$ be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an $(l, m)$-type connection such that $\zeta$ is tangent to $M$. If $U_i$’s or $V_i$’s are parallel with respect to $\nabla$, then $\bar{M}(f_1, f_2, f_3)$ is a flat manifold with an indefinite cosymplectic structure;

\[\alpha = \beta = 0, \quad f_1 = f_2 = f_3 = 0.\]

**Proof.** (1) If $U_i$’s are parallel with respect to the connection $\nabla$, then we have the equations of (38). As $\alpha = \beta = 0$, we get $f_1 = f_2 = f_3$ by Theorem 4.1. Applying $\nabla_Y$ to (38)$_5$ and using the fact that $\nabla_Y U_j = 0$, we obtain

\[(\nabla_X h_i^k)(Y, U_j) = 0.\]

Substituting this equation and (38) into (45) with $PZ = U_j$, we have

\[f_1(v_j(Y)\eta_k(X) - v_j(X)\eta_k(Y)) + f_2(v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)) = 0.\]

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we get $f_1 + f_2 = 0$. Thus we see that $f_1 = f_2 = f_3 = 0$ and $M$ is flat.

(2) If $V_i$’s are parallel with respect to the connection $\nabla$, then we have the equations in (39). As $\alpha = \beta = 0$, $f_1 = f_2 = f_3$ by Theorem 4.1. Taking $Y = \xi_j$ and $Y = U_j$ to (20) by turns and using (39)$_{3,5}$, we have

\[h_i^j(\xi_j, X) = 0, \quad h_i^j(U_j, X) = m\theta(X)\delta_{ij}.\]

Using these two equations and (30), we see that

\begin{align}
\tag{56}
h_i^j(\xi_i, V_j) &= 0, \quad h_i^j(\xi_j, V_j) = \epsilon_a h_i^j(\xi_i, W_a) = 0, \\
h_k^i(U_j, V_j) &= 0, \quad h_k^i(U_j, V_j) = \epsilon_a h_k^i(U_j, W_a) = 0.
\end{align}

From (30)$_1$ and (39)$_5$ and using the fact that $\nabla_X V_i = 0$, we have

\[h_i^j(Y, V_j) = 0.\]
Applying \( \nabla_X \) to this equation and using the fact that \( \nabla_X \)\( h^a_i(Y, V_j) = 0 \), we have
\[
(\nabla_X h^a_i)(Y, V_j) = 0.
\]
Substituting the last two equations into (45) such that \( PZ = V_j \), we get
\[
\sum_{a=r+1}^{n} e_{a} \{ \rho_{ia}(Y)h^a_i(X, V_j) - \rho_{ia}(X)h^a_i(Y, V_j) \} \\
+ \sum_{k=1}^{r} \{ h^k_i(X, V_j)\eta_l(\mathcal{A}_{\kappa k} Y) - h^k_j(Y, V_j)\eta_l(\mathcal{A}_{\kappa k} X) \} \\
= f_1(u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)) + 2f_2\delta_{ij}\bar{g}(X, JY).
\]
Taking \( X = \xi_i \) and \( Y = U_j \) to this equation and using (56), we obtain
\[
f_1 + 2f_2 = 0.
\]
It follows that \( f_1 = f_2 = f_3 = 0 \) and \( \bar{M} \) is flat. \( \square \)

References


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