CLASSIFICATION OF \((k, \mu)-\)ALMOST CO-\(\mathbb{K}\)\-\(\mathbb{M}\)ANIFOLDS WITH VANISHING BACH TENSOR AND DIVERGENCE FREE COTTON TENSOR

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Abstract. The object of the present paper is to characterize Bach flat \((k, \mu)-\)almost co-\(\mathbb{K}\)\-\(\mathbb{M}\)anifolds. It is proved that a Bach flat \((k, \mu)-\)almost co-\(\mathbb{K}\)\-\(\mathbb{M}\)anifold is \(K\)-almost co-\(\mathbb{K}\)\-\(\mathbb{M}\)anifold under certain restriction on \(\mu\) and \(k\). We also characterize \((k, \mu)-\)almost co-\(\mathbb{K}\)\-\(\mathbb{M}\)anifolds with divergence free Cotton tensor.

1. Introduction

In 1921, Bach introduced a tensor [1] to study the conformal relativity in the context of conformally Einstein spaces. This tensor is known as the Bach tensor and is a symmetric \((0,2)\)-tensor \(B\) on a pseudo-Riemannian manifold \((M, g)\), defined by

\[
B(U, V) = \frac{1}{2n - 2} \sum_{i,j=1}^{2n+1} (\nabla_{e_i} \nabla_{e_j} W)(U, e_i, e_j, V) + \frac{1}{2n - 1} \sum_{i,j=1}^{2n+1} S(e_i, e_j) W(U, e_i, e_j, V),
\]

where \(\{e_i\}, i = 1, \ldots, 2n + 1\), is a local orthonormal frame on \((M, g)\), \(S\) is the Ricci tensor of type \((0,2)\), \(W\) denotes the Weyl tensor of type \((0,3)\) defined by

\[
W(U, V)Z = R(U, V)Z - \frac{1}{2n - 1} \{S(V, Z)U - S(U, Z)V\} - S(U, Z)V + g(V, Z)QU - g(U, Z)QV + \frac{r}{2n(2n - 1)} \{g(V, Z)U - g(U, Z)V\}.
\]
We recall the Cotton tensor \( C \) which is a \((0,3)\)-tensor defined by
\[
(3) \quad C(U, V)Z = \left( \nabla_U S \right)(V, Z) - \left( \nabla_V S \right)(U, Z) - \frac{1}{4n} [(Ur)g(V, Z) - (Vr)g(U, Z)].
\]
In view of (1) and (2), the Bach tensor can be expressed as [12]
\[
(4) \quad B(U, V) = \frac{1}{2n - 1} \left[ \sum_{i=1}^{2n+1} (\nabla_{e_i} C)(e_i, U, V) + \sum_{i,j = 1}^{2n+1} S(e_i, e_j)W(U, e_i, e_j, V) \right].
\]
In dimension 3, the Weyl tensor \( W \) vanishes and hence the Bach tensor reduces to
\[
(5) \quad B(U, V) = \sum_{i=1}^{3} (\nabla_{e_i} C)(e_i, U, V).
\]
If \((M, g)\) is locally conformally related to an Einstein space, \( B \) has to vanish, but there are Riemannian manifolds with \( B = 0 \), which are not conformally related to Einstein spaces [30]. By (3), it is easy to see that Bach flatness is natural generalization of Einstein and conformal flatness. For more details about Bach tensor, we refer to reader ([3, 18, 29, 30]) and references therein.

Recently, Ghosh and Sharma [19] studied Sasakian manifolds with purely transversal Bach tensor. In particular, they proved that if a Sasakian manifold admits a purely transversal Bach tensor, then \( g \) has a constant scalar curvature \( \geq 2n(2n - 1) \), with equality holds if and only if \( g \) is Einstein and the Ricci tensor \( g \) has a constant norm. Also, they studied \((k, \mu)\)-constant manifolds with divergence free Cotton tensor and vanishing Bach tensor in [20]. These works of Ghosh and Sharma ([19, 20]) turn our attention to study Bach tensor in the framework of \((k, \mu)\)-almost co-Kähler manifolds.

The paper is organized as follows: After introduction in Section 2, we give a brief description on \((k, \mu)\)-almost co-Kähler manifolds and state definition of \( K \)-almost co-Kähler manifold. In Section 3, we consider Bach flat \((k, \mu)\)-almost co-Kähler manifolds and prove that an \((k, \mu)\)-almost co-Kähler manifold with vanishing Bach tensor is a \( K \)-almost co-Kähler manifold. Finally, we characterize \((k, \mu)\)-almost co-Kähler manifolds with divergence free Cotton tensor.

2. Preliminaries

Let a \((2n + 1)\)-dimensional smooth manifold \( M^{2n+1} \) \((n \geq 1)\) admits a tensor field \( \phi \) of type \((1,1)\), a vector field \( \xi \) and a 1-form \( \eta \) such that
\[
(6) \quad \phi^2 + I = \eta \otimes \xi, \quad \eta(\xi) = 1,
\]
where \( I \) denotes the identity transformation. Then the structure \( \langle \phi, \xi, \eta \rangle \) is called an almost contact structure of \( M^{2n+1} \). The manifold \( M^{2n+1} \) equipped with structure \( \langle \phi, \xi, \eta \rangle \) is known as an almost contact manifold ([5, 6]). From (6), we can easily prove that
\[
(7) \quad \phi \xi = 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad \text{rank}(\phi) = 2n.
\]
A $(2n+1)$-dimensional almost contact metric manifold is said to be normal if the induced almost complex structure $J$ defined on $M^{2n+1} \times \mathbb{R}$ by

$$J(U, f \frac{\partial}{\partial t}) = (\phi U - f\xi, \eta(U) \frac{\partial}{\partial t}),$$

where $U$ is any vector field on $M^{2n+1}$, $f$ is a smooth function defined on $M^{2n+1} \times \mathbb{R}$ and $t$ is the coordinate of $\mathbb{R}$, is normal, that is, $J$ is a complex structure. The normality of an almost contact structure $(\phi, \xi, \eta)$ is expressed as

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $d$ denotes the exterior derivative and $[\phi, \phi]$ represents the Nijenhuis tensor of $\phi$ defined as

$$[\phi, \phi](U, V) = \phi^2[U, V] + [\phi U, \phi V] - \phi[\phi U, V] - \phi[U, \phi V]$$

for any vector fields $U$ and $V$ on $M^{2n+1}$.

If the associated Riemannian metric $g$ of an almost contact manifold $M^{2n+1}$ satisfies

$$g(\phi U, \phi V) + \eta(U)\eta(V) = g(U, V)$$

for all vector fields $U$ and $V$ on $M^{2n+1}$, then an almost contact manifold endowed with $g$ is known as almost contact metric manifold. In consequence of (6), (7) and (8), we have

$$g(\phi U, V) + g(U, \phi V) = 0 \quad \text{and} \quad \eta(U) = g(\xi, U)$$

for all vector fields $U$ and $V$ on $M^{2n+1}$.

A $(2n+1)$-dimensional almost contact metric manifold is said to be a contact metric manifold if

$$d\eta(U, V) = \Phi(U, V) = g(U, \phi V)$$

for all vector fields $U$ and $V$ on $M^{2n+1}$, where $\Phi$ denotes the fundamental 2-form on an almost contact metric manifold.

According to Cappelletti-Montano et al. [10], an almost co-K"{a}hler manifold is an almost contact metric manifold with closed contact 1-form $\eta$ and fundamental 2-form $\Phi$. If the associated almost contact structures of an almost co-K"{a}hler manifold $M^{2n+1}$ is normal, that is, $\nabla \phi = 0$, (or equivalently $\nabla \Phi = 0$), then $M^{2n+1}$ is called co-K"{a}hler manifold [22]. The simplest example of (almost) co-K"{a}hler manifold is the Riemannian product of a real line or a circle and a (almost) co-K"{a}hler manifold. However, there do exist some examples of (almost) co-K"{a}hler manifolds which are not globally the product of a real line or a circle and a (almost) co-K"{a}hler manifold ([13,23–27], [31–35]).

The notion of an almost cosymplectic manifold was introduced by Goldberg and Yano in 1969 [21]. The simplest examples of such manifolds are those being the products of almost K"{a}hlerian manifolds and the real line $\mathbb{R}$ or the circle $S^1$. In particular, cosymplectic manifolds in the sense of Blair [4] are of this type. However, the class of almost cosymplectic manifolds is much more wider. In early literature, (almost) co-K"{a}hler manifolds were usually referred to as (almost) cosymplectic manifolds. For example, in 1967, Blair and Goldberg in [7] obtained that the Betti numbers of any compact cosymplectic manifold are non-zero.
If we set $h = \frac{1}{2}L\phi$ and $h' = h \circ \phi$ on an almost co-Kähler manifold, then we recall the following results proved by Dacko et al. ([14–16]), Endo [17] and Olszak ([24,26]):

$$h\xi = h'\xi = 0, \quad trh = trh' = 0, \quad h\phi + \phi h = 0,$$

where $tr$ stand for trace, $L$ represents the Lie derivative operator and $R$ is the Riemannian curvature tensor of the Riemannian connection $\nabla$ defined as

$$R(U,V)Z = \nabla_U(\nabla_V Z) - \nabla_V(\nabla_U Z) - \nabla_{[U,V]}Z$$

for all vector fields $U$, $V$ and $Z$ on $M^{2n+1}$. Notice that the vector fields $h$ and $h'$ are symmetric operators with respect to the Riemannian metric $g$. In an almost co-Kähler manifold, the 1-form $\eta$ is closed, that is,

$$(\nabla_U \eta)(V) - (\nabla_V \eta)(U) = 0 \iff g(h\phi U, V) - g(h\phi V, U) = 0$$

for all vector fields $U$ and $V$ on $M^{2n+1}$. In almost co-Kähler manifolds

$$\nabla_U \xi = h'U \iff (\nabla_U \eta)(V) = g(h\phi U, V)$$

for all vector fields $U$ and $V$ on $M^{2n+1}$.

The Lie derivative of Riemannian metric $g$ along the Reeb vector $\xi$ and (12) infer that the relation $(L_\xi g)(U,V) = 2g(h'U, V)$ holds on an almost co-Kähler manifold. This reflects that the Reeb vector field $\xi$ of $M^{2n+1}$ is Killing if and only if the tensor field $h$ vanishes on $M^{2n+1}$. Thus we have the following:

**Definition.** An almost co-Kähler manifold is said to be a $K$-almost co-Kähler manifold if the Reeb vector field $\xi$ is Killing.

If the distribution $\mathcal{D}$ on an almost co-Kähler manifold $M^{2n+1}$ is defined by $\mathcal{D} = ker\eta$. Then by using (6)-(8) and relation $d\Phi = 0$, one can define an almost $K$-Kähler structure $(g\mathcal{D}, \phi\mathcal{D})$ on $\mathcal{D}$. In 1987, Olszak [26] proved that an associated almost $K$-Kähler structure is integrable if and only if

$$(\nabla_U \phi)(V) = g(hU, V)\xi - \eta(V)hU$$

for all vector fields $U$ and $V$ on $M^{2n+1}$. This reflects that an almost co-Kähler manifold is co-Kähler if and only if it is $K$-almost co-Kähler and the associated almost $K$-Kähler structure is integrable. It is noticed that a 3-dimensional almost co-Kähler manifold is co-Kähler if and only if it is $K$-almost co-Kähler.

Blair et al. [8] introduced the notion of $(k, \mu)$-contact metric manifolds, where $k$ and $\mu$ are real numbers. The full classification of these manifolds was given by Boeckx [9]. By a $(k, \mu)$-almost co-Kähler manifold we mean an almost co-Kähler manifold such that the Reeb vector field $\xi$ belongs to the $(k, \mu)$-nullity distribution, that is,

$$R(U,V)\xi = k[\eta(V)U - \eta(U)V] + \mu[\eta(V)hU - \eta(U)hV]$$

for any vector fields $U, V$ in $\chi(M)$ and $k, \mu \in \mathbb{R}$. In this paper, a $(k, \mu)$-almost co-Kähler manifold with $k < 0$ will be called a proper $(k, \mu)$-almost co-Kähler manifold or a non-coKähler $(k, \mu)$-almost co-Kähler manifold. This was first
introduced by Endo [17] and was generalized to \((k, \mu, v)\)-spaces by Dacko and Olszak in [16] (see also Carriazo and Martin-Monina [11] and [28]).

It is well known that an almost co-Kähler \((k, \mu)\)-manifold of dimension \((2n + 1), n > 1\) is a K-almost co-Kahler manifold if and only if \(k = 0\).

A \((k, \mu)\)-almost co-Kähler manifold satisfies the curvature properties [28]

\[
h^2 U = k \phi^2 U, \quad k \leq 0.
\]

Balkan et al. proved the following lemmas [2].

**Lemma 2.1.** If the almost cosymplectic manifold \(M^{2n+1}\) with \(\xi\) belonging to the \((k, \mu)\)-nullity distribution, then the following relations hold:

\[
\begin{align*}
(\nabla_U h)V - (\nabla_V h)U &= k(\eta(V)\phi U - \eta(U)\phi V + 2g(\phi U, V)\xi) + \mu(\eta(V)\phi hU - \eta(U)\phi hV), \\
R(\xi, U)V &= k[g(U, V)\xi - \eta(V)U] + \mu[g(hU, V)\xi - \eta(V)hU], \\
S(U, \xi) &= 2nk\eta(U),
\end{align*}
\]

where \(U\) and \(V\) are vector fields on \(M^{2n+1}\), \(k, \mu \in \mathbb{R}\) and \(S\) is the Ricci tensor of \(M^{2n+1}\) [8].

**Lemma 2.2.** Let \(M\) be an almost cosymplectic manifold with \(\xi\) belonging to the \((k, \mu)\)-nullity distribution. For any vector field \(U\), the Ricci operator \(Q\) is given by

\[
\begin{align*}
(i) &\quad QU = \mu hU + 2nk\eta(U)\xi \\
(ii) &\quad r = 2nk.
\end{align*}
\]

Also putting \(U = \xi\) in Lemma 2.2(i), we get

\[
Q\xi = 2nk\xi.
\]

**3. Bach flat \((k, \mu)\)-almost co-Kähler manifold**

Lemma 2.2(i) implies

\[
S(V, Z) = \mu g(hV, Z) + 2nk\eta(V)\eta(Z).
\]

Differentiating (17) along \(U\) and using (10) and (12), we have

\[
(\nabla_U S)(V, Z) = \mu g((\nabla_U h)V, Z) - 2nk g(V, \phi hU)\eta(Z) - 2nk g(Z, \phi hU)\eta(V).
\]

In \((k, \mu)\)-almost co-Kähler manifold \(r\) is constant, then (3) implies

\[
\]

Using (18), (9), (10) and Lemma 2.1(i) in the above equation yields

\[
C(U, V)Z = k\mu\{g(\phi U, Z)\eta(V) - g(\phi V, Z)\eta(U) + 2g(\phi U, V)\eta(Z)\} - (\mu^2 - 2nk)\{g(\phi hV, Z)\eta(U) - g(\phi hU, Z)\eta(V)\}.
\]

Substituting \(Z = \xi\) in the above equation provides

\[
C(U, V)\xi = 2\mu g(\phi U, V).
\]
Differentiating $C(U, V)\xi$ along $X$ and using (10) and (12), we get

\[(\nabla_X C)(U, V)\xi = 2k\mu g((\nabla_X \phi)U, V) - k\mu \{g(\phi V, \phi hX)\eta(U) - g(\phi U, \phi hX)\eta(V)\} - (\mu^2 - 2nk)\{g(\phi hU, \phi hX)\eta(V) - g(\phi hV, \phi hX)\eta(U)\}.
\]

Using (8), (10) and (13) in the above equation yields

\[(\nabla_X C)(U, V)\xi = 2k\mu \{g(hX, U)\eta(V) - g(hX, V)\eta(U)\} - k\mu \{g(V, hX)\eta(U) - g(U, hX)\eta(V)\} - (\mu^2 - k)\{g(hU, hX)\eta(V) - g(hV, hX)\eta(U)\}.
\]

Putting $X = U = e_i$ in (22), where \(\{e_i\}\) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$ ($1 \leq i \leq n$), we get

\[(\nabla_{e_i} C)(e_i, V)\xi = 2k\mu \{g(he_i, e_i)\eta(V) - g(he_i, V)\eta(e_i)\} - k\mu \{g(V, he_i)\eta(e_i) - g(e_i, he_i)\eta(V)\} - (\mu^2 - 2nk)\{g(he_i, he_i)\eta(V) - g(hV, he_i)\eta(e_i)\}.
\]

Using (6), (10) and (15) in the above equation infers

\[(\nabla_{e_i} C)(e_i, V)\xi = -k(\mu^2 - 2nk)(1 - n)\eta(V).
\]

Substituting $\xi$ for $Z$ in (2) and operating by the Ricci operator $Q$, we get

\[QW(U, V)\xi = R(U, V)\xi Q - \frac{1}{2n - 1}\{S(V, \xi)QU - S(U, \xi)QV + g(V, \xi)Q^2U - g(U, \xi)Q^2V\} + \frac{r}{2n(2n - 1)}\{g(V, \xi)QU - g(U, \xi)QV\}.
\]

Substituting $e_i$ for $V$ in the above equation, taking inner product with $e_i$, summation over $i$, and using (6), (9), (14), Lemma 2.1(iii) and (16), we get

\[
\sum_{i=1}^{2n+1} g(QW(U, e_i)\xi, e_i) = -4n^2k^2\eta(U) - \frac{1}{2n - 1}\{4n^2k^2 - 2nk r + 4n^2k^2 - |Q|^2\}\eta(U) + \frac{r}{2n(2n - 1)}\{2nk - r\}\eta(U).
\]

Putting $U = V = e_i$ in (17) infer that

\[|Q| = 2nk.
\]

Using (27) and Lemma 2.2(ii) in (26) provides

\[
\sum_{i=1}^{2n+1} g(QW(U, e_i)\xi, e_i) = -4n^2k^2\eta(U).
\]
Now we notice that the last term of the Bach tensor in (4) can be written as
\[ g(Qe_i, e_j)g(W(U, e_i)e_j, V) = -g(W(U, e_i)V, Qe_i) = -g(QW(U, e_i)V, e_i). \]
Using (29) in (4), we have
\[ B(U, V) = \frac{1}{2n-1} \left[ \sum (\nabla e_i)C(e_i, U, V) - \sum g(QW(U, e_i)V, e_i) \right]. \]
Substituting \( \xi \) for \( V \) in the above equation, using the hypothesis \( B(U, \xi) = 0 \) and with the help of (24) and (28) yields
\[ k\{\mu^2(1-n) - 2nk(1 + n)\} \eta(U) = 0. \]
Taking \( U = \xi \) in the above equation, we get
\[ k\{\mu^2(1-n) - 2nk(1 + n)\} = 0. \]
This implies either \( k = 0 \) or \( \mu^2(1-n) - 2nk(1 + n) = 0. \)

When \( k = 0 \) we say from definition the manifold is \( K \)-almost co-Kähler manifold, provided \( \mu^2(1-n) - 2nk(1 + n) \neq 0. \)

Thus, in view of the above result, we can state the following theorem.

**Theorem 3.1.** A Bach flat \((k, \mu)\)-almost co-Kähler manifold is \(K\)-almost co-Kähler manifold, provided \( \mu^2(1-n) - 2nk(1 + n) \neq 0. \)

Now we consider 3-dimensional Bach flat \((k, \mu)\)-almost co-Kähler manifolds. Then from (32), we get either \( k = 0 \) or \( 4k = 0. \)

From these two cases, we get \( k = 0. \)

This implies that the manifold is \(K\)-almost co-Kähler manifold.

It is known that any 3-dimensional almost co-Kähler manifold is co-Kähler manifold if and only if it is \(K\)-almost co-Kähler manifold [24].

Thus we conclude that:

**Corollary 3.2.** A 3-dimensional Bach flat \((k, \mu)\)-almost co-Kähler manifold is co-Kähler manifold.

**4. \((k, \mu)\)-almost co-Kähler manifolds satisfying \( divC = 0 \)**

By hypothesis \( divC = 0 \) and this implies that
\[ (\nabla_U S)(V, Z) - (\nabla_V S)(U, Z) = \frac{1}{4n}[dr(U)g(V, Z) - dr(V)g(U, Z)]. \]
In an almost co-Kähler manifold \( r \) is constant. Hence the above equation implies
\[ (\nabla_U S)(V, Z) - (\nabla_V S)(U, Z) = 0. \]
Using (18), (9) and (10) in the above equation yields
\[ \mu[g((\nabla_U h)U - (\nabla_V h)V, Z)] - 2nk \mu(\phi h U, Z) \eta(V) + 2nk \mu(\phi h V, Z) \eta(U) = 0. \]
Using Lemma 2.1(i) in (35), we get
\[ k \mu \{ g(\phi U, Z) \eta(V) - g(\phi V, Z) \eta(U) + 2g(\phi U, V) \eta(Z) \} \]
(36) \[-(\mu^2 - 2nk)\{g(\phi V, Z)\eta(U) - g(\phi U, Z)\eta(V)\} = 0.\]

Substituting \(\xi\) for \(Z\) in the above equation, we get
(37) \[2k\mu g(\phi U, V) = 0.\]

This implies either \(k = 0\) or \(\mu = 0\) or, both are zero.

If \(\mu = 0\), then the manifold is \(N(k)\)-almost co-Kähler manifold.

Again, if \(k = 0\), then the manifold is \(K\)-almost co-Kähler manifold and if \(\mu\) and \(k\) both are zero, then \(R(U, V)\xi = 0\).

Dacko [14] proved that in an almost co-Kähler manifold \(R(U, V)\xi = 0\) if and only if the manifold is locally a product of an open interval and an almost Kähler manifold. Hence we can state the following theorem.

**Theorem 4.1.** In a \((k, \mu)\)-almost co-Kähler manifold with divergence free Cotton tensor, one of the following cases occur:

(i) the manifold is \(K\)-almost co-Kähler manifold,
(ii) the manifold is \(N(k)\)-almost co-Kähler manifold,
(iii) the manifold is locally a product of an open interval and an almost Kähler manifold.

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