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Universal Theory for Planar Deformations of an Isotropic Sandwich Beam

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등방성 샌드위치 빔의 평면 변형을 위한 통합 이론

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ABSTRACT

This work is concerned with various planar deformations of an isotropic sandwich beam, which generally consists of three layers: two stiff skin layers and one soft core layer. When one layer of the sandwich beam is modeled as a beam, the variational-asymptotic method is rigorously used to construct a zeroth-order beam model, which is similar to a generalized Timoshenko beam model capable of capturing the transverse shear deformations but still carries out the zeroth-order approximation. To analyze the planar sandwich beam, the sum of the energies of the two skin layers and one core layer is then formulated with different material and geometric properties and represented by a universal beam model in terms of the core-layer kinematics through interface displacement and stress continuity conditions. As a preliminary validation, two extreme examples are presented to demonstrate the capability and accuracy of this present approach.

Keywords : Sandwich Beam(샌드위치 빔), Variational-Asymptotic Method(변분-점근법), Timoshenko Beam Model(티모센코 빔 모델), Universal Theory(통합 이론)

1. Introduction

Due to the mismatch between constituent materials and the optimized arrangement of geometric properties, sandwich structures have demonstrated excellent multifunctionality for use as high-strength and low-weight structures, superior noise- and energy-absorption structures, and high-temperature-resistant structures[1]. Since these have rapidly increased in popularity in various engineering applications, the theoretical and numerical development of accurate and generalized models for predicting and controlling their mechanical characteristics has received consistent attention in the last several decades. In particular, according to the existing literature[2], numerous studies have done using Classical Beam Theory (CBT) without employing transverse shear deformations or the First-Order Shear Deformation

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Theory (FOSDT) with transverse shear effects to correct CBT. However, their accuracy and generalization to the analysis of sandwich beam problems are limited, and thus, researchers are reluctant to accept them with confidence[3,4]. Therefore, generalized and accurate yet simple and efficient approaches of analysis are still required.

In this paper, to investigate the planar deformations of the isotropic sandwich beam, the universal theory based on the zeroth-order Timoshenko beam model is systematically established under the Variational-Asymptotic Method (VAM)[5]. In addition, the present approach follows the previous works of isotropic sandwich plates/shells[6,7] using a similar framework.

2. Analytic Formulation of Isotropic Beam as a Layer using VAM

The isotropic sandwich beam generally consists of three isotropic layers: two stiff skin layers (faces) and a soft core layer. In this section, the asymptotic theory for the planar deformations of the isotropic beam as a layer is developed using VAM.

To this end, the undeformed state is first described on the right side of Fig. 1, where the position vector to an arbitrary point in the undeformed beam is taken to be

$$\hat{\mathrm{r}} = \mathrm{r} + y\mathrm{b}_y \tag{1}$$

where $\mathrm{r} = x\mathrm{b}_x$ is the reference line of the undeformed beam, taken for convenience as the centroid locus of a cross-section. Here, a set of unit vectors b_i $(i = x, y)$ is associated with the undeformed beam configuration, i.e. along x and y. Then, the position vector to an arbitrary point in the deformed beam can be written as

$$\hat{\mathrm{R}} = \mathrm{R} + y\mathrm{B}_y + w_x(x,y)\mathrm{B}_x + w_y(x,y)\mathrm{B}_y \tag{2}$$

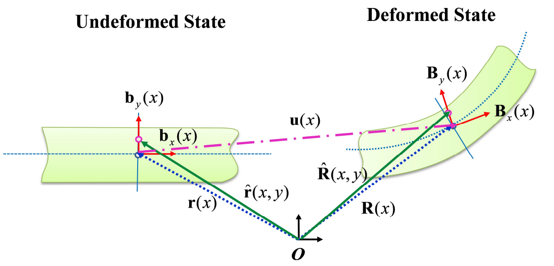

Fig. 1 Schematic of the isotropic beam deformation

where $\mathrm{R} = \mathrm{r} + u\mathrm{b}_x + v\mathrm{b}_y$, B_x is a unit vector normal to the cross-section in the deformed reference line, and B_y is normal to B_x in the same plane.

The displacement field is thus described in terms of beam variables $u(x)$ and $v(x)$ warping functions $w_x(x,y)$ and $w_y(x,y)$, respectively. For the dimensional reduction procedure mentioned later, the warping functions are unknown at the outset but are solved. Therefore, three constraints on the warping functions are needed to uniquely specify the deformed position vector. According to Hodges[8], these constraints are chosen in the following ways:

$$\langle \hat{\mathrm{R}} \rangle = b\langle \mathrm{R} \rangle \text{ and } \langle y\hat{\mathrm{R}} \rangle \cdot \mathrm{B}_x = 0 \tag{3}$$

where

$$\langle (\bullet) \rangle = \int_{-b/2}^{b/2} (\bullet)\, dy$$

Here, Eq. (3) implies that

$$\langle w_x \rangle = \langle w_y \rangle = \langle y w_x \rangle = 0 \tag{4}$$

The beam is assumed to be homogeneous and isotropic. Moreover, under assumption of plane stress, appropriate for a beam with narrow rectangular cross-section of width b and thickness t, twice the strain energy per unit length is given by

$$2U = \frac{Et}{1-\nu^2}\left\langle (1-\nu^2)\Gamma_{xx}^2 + \frac{1-\nu}{2}(2\Gamma_{xy})^2 + (\Gamma_{xy} + \nu\Gamma_{xx})^2 \right\rangle \tag{5}$$

with E and ν being Young's modulus and Poisson's ratio, respectively. According to the displacement field spelled out in Eq. (2), the two-dimensional(2D) strain components are

$$\begin{aligned} \Gamma_{xx} &= \varepsilon - y\kappa + w_x{}' \\ 2\Gamma_{xy} &= 2\gamma + w_{x,y} + w_y{}' \\ \Gamma_{yy} &= w_{y,y} \end{aligned} \tag{6}$$

with $(\bullet)' = \partial(\bullet)/\partial x$ and $(\bullet)_{,y} = \partial(\bullet)/\partial y$. Here, ε, κ and 2γ are the one-dimensional(1D) generalized strains measures and the 1D shearing strain measure, respectively, and they are taken as known during the dimensional reduction procedure.

Before utilizing VAM, two small parameters must be identified. First, the strain is small compared to unity. It is clear that ε, $\bar{b}\kappa$ and 2γ are $O(\bar{\varepsilon})$, where $\bar{\epsilon}$ denotes the maximum strain, and $\bar{b}$ is the maximum value taken on by y in the structure. The second small parameter is $\bar{b}/l$ where l is the wavelength of deformation along the beam, such that $\partial(\bullet)/\partial x = O(\bullet/l)$.

The VAM procedure is systematically performed as follows: All terms in Eq. (6) are first identified by two small parameters, and higher terms than $O(\bar{\epsilon})$ are negligible. Then, these resulting strain components become

$$\begin{aligned} \Gamma_{xx} &= \varepsilon - y\kappa \\ 2\Gamma_{xy} &= 2\gamma + w_{x,y} \\ \Gamma_{yy} &= w_{y,y} \end{aligned} \tag{7}$$

Second, by plugging Eq. (7) into the expression for $2U$, twice the zeroth-order approximation energy can be obtained in terms of the unknown warping functions.

$$\begin{aligned} 2U_0 = \frac{Et}{1-\nu^2} &\langle (1-\nu^2)(\varepsilon - y\kappa)^2 \\ &+ \frac{1-\nu}{2}(2\gamma + w_{x,y})^2 + [w_{y,y} + \nu(\varepsilon - y\kappa)]^2 \rangle \\ &- t\langle w_x \rangle \lambda_x - t\langle w_y \rangle \lambda_y - t\langle y w_x \rangle \lambda_{xy} \end{aligned} \tag{8}$$

where Lagrange multipliers λ_x, λ_y and λ_{xy} are used to enforce constraints on the warping functions in Eq. (4). Third, the standard procedure of the calculus of variations can be preformed in Eq. (8) to calculate the zeroth-order warping functions given by

$$\begin{aligned} w_x &= -y\left[\frac{5}{3}\left(\frac{y}{b}\right)^2 - \frac{1}{4}\right](2\gamma) \\ w_y &= -\nu\left[y\varepsilon - \frac{1}{2}\left(y^2 - \frac{b^2}{12}\right)\kappa\right] \end{aligned} \tag{9}$$

Finally, by substituting the above warping fields back into the original strain energy and discarding all terms of orders higher than $O(\bar{\varepsilon}^2)$, twice the strain energy up to the zeroth-order approximation is calculated as

$$2U_0 = tb\left[E\varepsilon^2 + \frac{b^2}{12}E\kappa^2 + \frac{1}{6}G(2\gamma)^2\right] \tag{10}$$

where $G = E/2(1+\nu)$. Note that Eq.(10) is consistent with zeroth-order Timoshenko beam theory.

3. Universal Theory of Isotropic Sandwich Beams

As depicted in Fig. 2, the sandwich beam theory requires an expression for the strain energy, which is made of two stiff skin layers and a soft core layer, per unit length in terms of each 1D generalized strain and transverse shear strain measure. Therefore, this strain energy of the isotropic sandwich beam under the same beam thickness can be easily calculated by obtaining the final result in the previous section:

$$\begin{aligned} 2U_{sd} = tb^+ &\left[E^+(\varepsilon^+)^2 + \frac{b^{+2}}{12}E^+(\kappa^+)^2 + \frac{5}{6}G^+(2\gamma^+)^2\right] \\ + tb &\left[E(\varepsilon)^2 + \frac{b^2}{12}E(\kappa)^2 + \frac{5}{6}G(2\gamma)^2\right] \\ + tb^- &\left[E^-(\varepsilon^-)^2 + \frac{b^{-2}}{12}E^-(\kappa^-)^2 + \frac{5}{6}G^-(2\gamma^-)^2\right] \end{aligned} \tag{11}$$

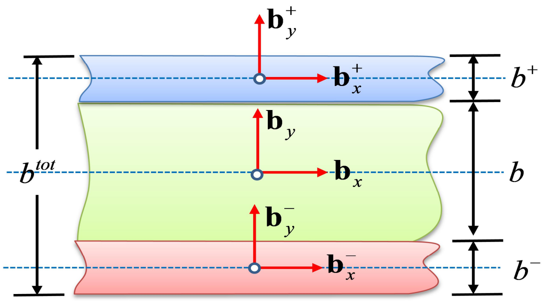

Fig. 2 Configuration of the isotropic sandwich beam problem

Here, the first, second, and third expressions on the left-hand side of Eq. (11) denote the strain energy for the top skin layer (+), core-layer, and bottom skin layer (−), respectively. Moreover, the corresponding generalized strain and shearing strain measures are denoted by ϵ^+, κ^+, $2\gamma^+$, ϵ, κ, 2γ and ϵ^-, κ^-, $2\gamma^-$, respectively.

In addition, similar to Berdichevsky's methodology[6,7], six continuity conditions at the interfaces of the layers must be introduced into Eq.(11) to develop a universal beam model in terms of ϵ, κ and 2γ. When the 2D displacement field is kinematically defined as $\mathrm{U}=\hat{\mathrm{R}}-\hat{\mathrm{r}}$ in Fig. 1, the 2D displacement continuity can be specified by the following two conditions at the face-core interfaces:

$$\begin{aligned} \mathrm{U}^+(x,-b^+/2) &= \mathrm{U}(x,b/2) \\ \mathrm{U}^-(x,b^-/2) &= \mathrm{U}(x,-b/2) \end{aligned} \qquad (12)$$

Alternatively, by using $\mathrm{r}^+=\mathrm{r}+(b/2)[1+(b^+/b)]\mathrm{b}_y^+$ and $\mathrm{r}^-=\mathrm{r}-(b/2)[1+(b^-/b)]\mathrm{b}_y^-$ with $\mathrm{b}_y^+=\mathrm{b}_y^-=\mathrm{b}_y$, as shown in Fig. 2, Eq.(12) can be replaced with the following continuity conditions for deformed position vectors at the interfaces of the core layer with two skin-layers

$$\begin{aligned} \hat{\mathrm{R}}^+(x,-b^+/2) &= \hat{\mathrm{R}}(x,b/2) \\ \hat{\mathrm{R}}^-(x,b^-/2) &= \hat{\mathrm{R}}(x,-b/2) \end{aligned} \qquad (13)$$

On the other hand, four of the continuity conditions can be obtained by a proper definition of the 2D transverse shear and normal stress fields such that

$$\begin{aligned} \sigma_{xy}^+(x,-b^+/2) &= \sigma_{xy}(x,b/2) \\ \sigma_{yy}^+(x,-b^+/2) &= \sigma_{yy}(x,b/2) \\ \sigma_{xy}^-(x,b^-/2) &= \sigma_{xy}(x,-b/2) \\ \sigma_{yy}^-(x,b^-/2) &= \sigma_{yy}(x,-b/2) \end{aligned} \qquad (14)$$

where and $\sigma_{xy}=G(2\Gamma_{xy})$ and $\sigma_{yy}=\frac{E\nu}{1-\nu^2}(\Gamma_{yy}+\nu\Gamma_{xx})$.

Following Yu[9], the relationship between two sets of deformed base vectors (B_i^+ and B_i^-) for two skin layers and B_i for the core layer can uniquely be specified by a direction cosine matrix expressed in therms of the transverse shear measure of the core layer under the small strain assumption, such as

$$\begin{aligned} &\mathrm{B}_x^+=\mathrm{B}_x+2\gamma\mathrm{B}_y \ \text{ and } \ \mathrm{B}_y^+=-2\gamma\mathrm{B}_x+\mathrm{B}_y \\ &\mathrm{B}_x^-=\mathrm{B}_x+2\gamma\mathrm{B}_y \ \text{ and } \ \mathrm{B}_y^-=-2\gamma\mathrm{B}_x+\mathrm{B}_y \end{aligned}$$
(15)

Therefore, from Eqs. (15) and (12), one can derive the following kinematic identity relationships between the two skin layers and one core layer

$$\begin{aligned} &\varepsilon^+=\varepsilon-\frac{b}{2}\left[1+\left(\frac{b^+}{b}\right)\right]\kappa-\frac{b}{12}\left[1-6\left(\frac{b^+}{b}\right)\right](2\gamma') \\ &\kappa^+=\kappa-2\gamma' \ \text{ and } \ 2\gamma^+=0 \\ &\varepsilon^-=\varepsilon+\frac{b}{2}\left[1+\left(\frac{b^-}{b}\right)\right]\kappa+\frac{b}{12}\left[1-6\left(\frac{b^-}{b}\right)\right](2\gamma') \\ &\kappa^-=\kappa-2\gamma' \ \text{ and } \ 2\gamma^-=0 \end{aligned} \qquad (16)$$

Here, the four interface conditions of 2D transverse stress continuity can be automatically satisfied due to Eqs. (7), (9), and (13).

Finally, by plugging Eq. (16) into Eq. (11), we find the strain energy of the isotropic sandwich beam:

$$\begin{aligned} 2U_{sd} &= A_{\text{eff}}\varepsilon^2+2B_{\text{eff}}\varepsilon\kappa+D_{\text{eff}}\kappa^2+G_{\text{eff}}(2\gamma)^2 \\ &\quad +2E_{\text{eff}}\varepsilon(2\gamma')+2F_{\text{eff}}\kappa(2\gamma')+H_{\text{eff}}(2\gamma')^2 \end{aligned} \qquad (17)$$

where

$$
\begin{aligned}
A_{\text{eff}} &= t(b^+E^+ + bE + b^-E^-) \\
B_{\text{eff}} &= t\left\{-\frac{b^+b}{2}\left[1+\left(\frac{b^+}{b}\right)\right]E^+ + \frac{b^-b}{2}\left[1+\left(\frac{b^-}{b}\right)\right]E^-\right\} \\
D_{\text{eff}} &= t\left\{\frac{b^+b^2}{2}\left[1+\left(\frac{b^+}{b}\right)\right]^2E^+ + \frac{b^-b^2}{2}\left[1+\left(\frac{b^-}{b}\right)\right]^2E^- \right. \\
&\quad \left. + \frac{{b^+}^3}{12}E^+ + \frac{b^3}{12}E + \frac{{b^-}^3}{12}E^-\right\} \\
G_{\text{eff}} &= \frac{5}{6}tbG \\
E_{\text{eff}} &= t\left\{-\frac{b^+b}{12}\left[1-6\left(\frac{b^+}{b}\right)\right]E^+ + \frac{b^-b}{12}\left[1-6\left(\frac{b^-}{b}\right)\right]E^-\right\} \\
F_{\text{eff}} &= t\left\{\frac{b^+b^2}{24}\left[1+\left(\frac{b^+}{b}\right)\right]\left[1-6\left(\frac{b^+}{b}\right)\right]E^+ \right. \\
&\quad \left. + \frac{b^-b^2}{24}\left[1+\left(\frac{b^-}{b}\right)\right]\left[1-6\left(\frac{b^-}{b}\right)\right]E^- - \frac{{b^+}^3}{12}E^+ - \frac{{b^-}^3}{12}E^-\right\} \\
H_{\text{eff}} &= t\left\{\frac{b^+b^2}{144}\left[1-6\left(\frac{b^+}{b}\right)\right]^2E^+ + \frac{b^-b^2}{144}\left[1-6\left(\frac{b^-}{b}\right)\right]^2E^- \right. \\
&\quad \left. + \frac{{b^+}^3}{12}E^+ + \frac{{b^-}^3}{12}E^-\right\}
\end{aligned}
\tag{18}
$$

4. Validation Examples

To validate the present approach, two numerical examples are presented as a preliminary validation. In particular, we consider two extreme cases of the isotropic sandwich beam with two different face-to-core stiffness ratios (FCSR), $E^+/E = E^-/E = 10^1$ and $E^+/E = E^-/E = 10^5$, and the same Poisson's ratio, $\nu = 0.3$. The thicknesses of the two skin layers and the core layer are $b^+ = 0.2$, $b^- = 0.1$, and $b = 0.7$, respectively. The total thickness of the beam is defined as $b_{\text{tot}} = b^+ + b + b^- = 1$, and the length of the beam is $L = 10b_{\text{tot}}$. A unit width is assumed.

For the purpose of comparing this solution with the available exact 3D solution[3], the cylindrical bending problem of a geometrically linear formulation, static analysis is carried out. For the sake of saving space, axial and transverse shear stress components(σ_{xx} and σ_{xy}) are only plotted at points where their maximum values occur ($x = L/2$).

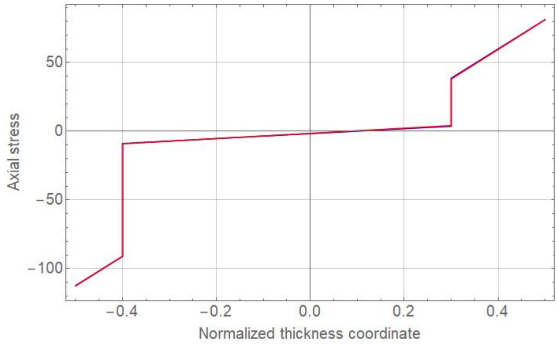

Fig. 3 Distribution of σ_{xx} vs. the through-thickness coordinate for $E^+/E = 10^1$

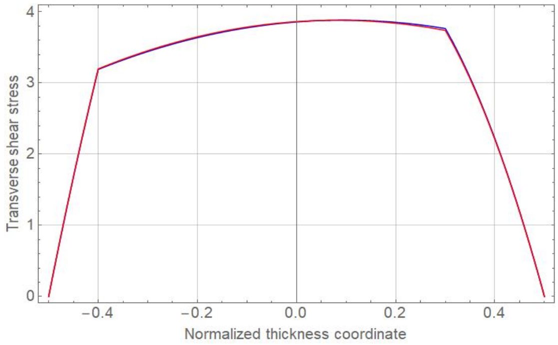

Fig. 4 Distribution of σ_{xy} vs. the through-thickness coordinate for $E^+/E = 10^1$

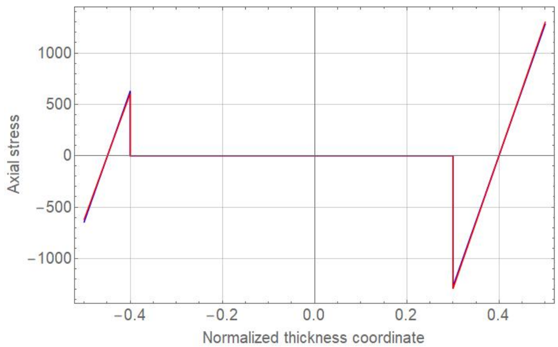

Fig. 5 Distribution of σ_{xx} vs. the through-thickness coordinate for $E^+/E = 10^5$

From the plotted results in Figs. 3 and 4 for FCSR = 10^1 and Figs. 5 and 6 for FCSR = 10^5, it

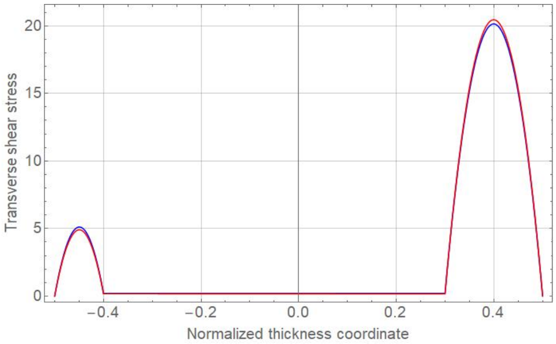

Fig. 6 Distribution of σ_{xy} vs. the through-thickness coordinate for $E^{+}/E=10^{5}$

is clear that the results obtained from the present approach (blue line) have excellent agreement with the exact 3D solution (red line) for all the stress components through the thickness direction. Therefore, this clearly proves that the present approach can be used to model isotropic sandwich beams confidently and obtain accurate results, even in extreme cases of FCSR.

5. Conclusion

The present work represents a new contribution, as there is currently no published work on using VAM for the universal modeling of a sandwich beam, each layer of which consists of isotropic material. The following conclusions can be drawn:

1. The universal theory based on the zeroth-order Timoshenko beam model for investigating the planar deformations of an isotropic sandwich beam is systematically established.
2. The close agreement between exact 3D solutions and those of the present approach demonstrates its capability and accuracy to predict the mechanical behavior of isotropic sandwich beams.

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