DIRECT SUM FOR BASIC COHOMOLOGY OF CODIMENSION FOUR TAUT RIEMANNIAN FOLIATION

JIRU ZHOU

Abstract. We discuss the decomposition of degree two basic cohomology for codimension four taut Riemannian foliation according to the holonomy invariant transversal almost complex structure \( J \), and show that \( J \) is \( C^\infty \) pure and full. In addition, we obtain an estimate of the dimension of basic \( J \)-anti-invariant subgroup. These are the foliated version for the corresponding results of T. Draghici et al. [3].

1. Introduction

In order to study S. K. Donaldson’s tamed to compatible question [2], T.-J. Li and W. Zhang [7] defined two subgroups \( H^+_J(M) \), \( H^-_J(M) \) of the real degree 2 de Rham cohomology group \( H^2(M; \mathbb{R}) \) for a compact almost complex manifold \( (M, J) \). They are the sets of cohomology classes which can be represented by \( J \)-invariant and \( J \)-anti-invariant real 2-forms respectively. Later, T. Draghici, T.-J. Li and W. Zhang showed in [3] that in dimension four
\[
H^2(M; \mathbb{R}) = H^+_J(M) \oplus H^-_J(M),
\]
and they call such almost complex structure \( J \) to be \( C^\infty \)-pure and full. This is specific for four dimension, since A. Fino and A. Tomassini’s Example 3.3 in [4] gives a six dimensional almost complex manifold \( (M, J) \) with \( J \) being not \( C^\infty \)-pure, and higher dimensional non-\( C^\infty \)-pure examples can be obtained by producting it with another almost complex manifold (see Remark 2.7 in [3]). It becomes nature to ask when will the almost complex structure be \( C^\infty \)-pure and full on higher dimension. This article lays the groundwork for the case in which the higher dimensional manifold admits a codimension four taut Riemannian foliation \( F \). The main result is Theorem 4.1 which basically says a transversal almost complex structure \( J \) on a codimension four taut Riemannian foliated
manifold satisfying $\theta(V)J = 0$, $\forall V \in \Gamma TF$ is $C^\infty$-pure and full in the sense of Definition 4.

The structure of this article is the following: Section 2 are notions of transverse structures, basic forms, characteristic form and filtrations needed later. We consider the compatibility of transversal almost complex structure with a taut Riemannian metric in Section 3. After these preliminaries, basic $J$-(anti)invariant cohomology groups naturally come out and so is $C^\infty$-purenness and fullness of $J$. After some lemmas similar to those in [3], we are able to proof the decomposition of the real degree 2 de Rham cohomology group in Section 4. The last section provides bounds on the dimension of $J$-(anti)invariant cohomology groups.

2. Taut Riemannian foliation

Let's first recall some definitions and results in foliations, the below in this section is referred to [9]. Let $M$ be a closed oriented smooth manifold of dimension $n = p + q$ endowed with a codimension $q$ foliation $F$. The integrable subbundle $TF \subset TM$ is given by vectors tangent to plaques, then we further have the rank $q$ normal bundle defined as the quotient bundle $Q = TM/TF$ and the projection

$$\pi: TM \to Q$$

$$Y \mapsto \pi(Y)$$

denoted by $\overline{Y} = \pi(Y)$.

Define the $\Gamma TF$-action on $\Gamma Q$ as

$$\theta(V)s = [V,Y_s]$$

for $V \in \Gamma TF$, $s \in \Gamma Q$,

where $Y_s \in \Gamma TM$ is any choice with $\overline{Y_s} = s$. It can be checked that the definition $\theta(V)s$ is independent of the choice of $Y_s$.

Consider a Riemannian metric $g = g_{TF} \oplus g_{TF^\perp}$ on $M$ splitting $TM$ orthogonally as $TM = TF \oplus TF^\perp$, which means there is a bundle map $\sigma: Q \xrightarrow{\sim} TF^\perp \subset TM$ splitting the exact sequence

$$0 \to TF \to TM \to Q \to 0,$$

i.e., satisfying $\sigma \circ \sigma =$identity. This induces a metric on $Q$ by $g_Q = \sigma^* g_{TF^\perp}$, then the splitting map $\sigma: (Q, g_Q) \to (TF^\perp, g_{TF^\perp})$ is a metric isomorphism.

Suppose $\nabla^M$ is the Levi-Civita connection induced by the Riemannian metric $g$ on $M$. For $s \in \Gamma Q$, define

$$\nabla_X s = \begin{cases} 
\pi [X, \sigma(s)] & \text{for } X \in \Gamma TF, \\
\pi (\nabla^0_s X, \sigma(s)) & \text{for } X \in \Gamma TF^\perp,
\end{cases}$$

then $\nabla$ is an adapted connection in $Q$, which means $\nabla$ restricting along $TF$ is the partial Bott connection.
Consider the $Q$-valued bilinear form on $TF$, i.e., $\alpha : TF \otimes TF \to Q$ given by

$$\alpha(U, V) = \pi \left( \nabla^M U \right)$$

for $U, V \in \Gamma TF$.

A calculation shows for $s \in \Gamma Q$,

$$\theta(Y) g_{TF}(U, V) = -2g(Y, \alpha(U, V)).$$

The Weingarten map $W(s) : TF \to TF$ is defined by

$$g_Q(\alpha(U, V), s) = g(W(s) U, V).$$

Then $\text{Tr} W \in \Gamma Q^*$, and it can be extended to a 1-form $\kappa \in \Omega^1(M)$ by setting $\kappa(V) = 0$ for $V \in \Gamma TF$, where we have used the identification $TF^\perp \cong Q$. We call $\kappa$ the mean curvature 1-form of $F$ on $(M, g)$.

Recall that a Riemannian foliation is a foliation $F$ with a holonomy invariant transversal metric $g_Q$ on $Q$, i.e.,

$$\theta(V) g_Q = 0, \forall V \in \Gamma TF.$$ 

The metric $g$ on $(M, F)$ is called bundle-like if the induced metric $g_Q$ is holonomy invariant, i.e., $\theta(V) g_Q = 0$ for all $V \in \Gamma TF$, and a Riemannian foliation $F$ is called taut if there exists a bundle-like metric for which the mean curvature 1-form $\kappa = 0$.

A differential form $\alpha \in \Omega^r(M)$ is basic, if

$$i(V) \alpha = 0, \theta(V) \alpha, \forall V \in \Gamma TF.$$

Denote by $\Omega_B^* = \Omega_B^*(F)$ the set of all basic forms, and the exterior differential $d_B = d|_{\Omega_B}$. By Cartan’s magic formula, it can be checked that $(\Omega_B^*, d_B)$ forms a sub-complex of the de Rham complex $(\Omega^*, d)$. The corresponding cohomology

$$H_B^p(F) = H_B^p(F; \mathbb{R})$$

is called the basic cohomology of $F$.

If $TF$ is oriented, the foliation $F$ with dimension $p$ is then said to be tangentially oriented. The $p$-form $\chi_F$ defined by

$$\chi_F(Y_1, \ldots, Y_p) = \det \left( g(Y_i, E_j)_{ij} \right), \forall Y_1, \ldots, Y_p \in \Gamma TM,$$

called the characteristic form of $F$, where $\{E_1, E_2, \ldots, E_p\}$ is a local oriented orthonormal frame of $TF$.

Consider the multiplicative filtration of the de Rham complex $\Omega^* = \Omega^*(M)$ as follows

$$F^q \Omega^m = \{ \alpha \in \Omega^m \mid i(V_1) \cdots i(V_{m-q+1}) \alpha = 0 \text{ for } V_1, \ldots, V_{m-q+1} \in \Gamma TF \}.$$ 

Obviously,

$$F^0 \Omega^m = \Omega^m \text{ and } F^{m+1} \Omega^m = 0.$$ 

Furthermore, for the foliation $(M, F)$, we have

$$F^{q+1} \Omega^m = 0 \quad (q = \text{codim } F).$$
3. Holonomy invariant transversal almost complex structure

If the foliation $F$ is of even codimension, and there exists almost complex structure $J$ on $Q$, i.e., an endomorphism $J : Q \to Q$ such that $J^2 = -Id_Q$, then extend $J$ onto $TM$ by setting $JX = 0$ for $X \in TF$. Such $J$ is called the transversal almost complex structure.

Lemma 3.1. For an even codimensional Riemannian foliation $(M, F)$ with a taut Riemannian metric $g = g_{TF} \oplus g_{TF}^\perp$, if there exists a transversal almost complex structure $J$ satisfying $\theta(V)J = 0$ for any $V \in \Gamma TF$ (we call such $J$ to be holonomy invariant), then the new metric $g_J$ defined by

$$g_J(X, Y) = \begin{cases} g_{TF}(X, Y) & \text{for } X, Y \in \Gamma TF \\ g_{TF}^\perp(X, Y) + g_{TF}^\perp(JX, JY) & \text{for } X, Y \in \Gamma TF^\perp \end{cases}$$

is also taut.

Proof. Since $\theta(V)J = 0$ for any $V \in \Gamma TF$ and $g$ is bundle-like,

$$\theta(V)g_J Q(s, s') = (\theta(V)g_Q)(s, s') + (\theta(V)g_Q)(Js, Js') = 0,$$

i.e., $g_J$ is also bundle-like.

For the tautness part, let $e_1, \ldots, e_n$ be an orthonormal basis of $T_xM$ such that $e_1, \ldots, e_p \in TF_x$ and $e_{p+1}, \ldots, e_n \in TF_x^\perp$. Then by (1), we have the mean curvature 1-form $\kappa$ for $g$,

$$\kappa(s)_x = \text{Tr} W(s)_x$$

\begin{align*}
= & \sum_{i=1}^p g(W(s)e_i, e_i) \\
= & \sum_{i=1}^p g_Q(\alpha(e_i, e_i), s) \\
= & -\frac{1}{2} \sum_{i=1}^p (\theta(s)g_{TF})(e_i, e_i),
\end{align*}

which shows that $\kappa$ is independent of $g_Q$.

We denote by $\kappa_J$ the mean curvature 1-form with respect to $g_J$. Since $g$ is taut, $\kappa$ vanishes, and so is $\kappa_J$, i.e., $g_J$ is also taut.

In the sequel, we still denote this $g_J$ by $g$, and call the transversal almost Hermitian structure $(\mathcal{F}, g, J, F)$ is called a transversal almost Hermitian structure.
4. $C^\infty$-pure and full

For an even codimensional Riemannian foliation $\mathcal{F}$ on $M$ endowed with a transversal almost complex structure $J$ satisfying $\theta(V)J = 0$, $\forall \, V \in T\mathcal{F}$, denote by $\Lambda^2_B$ the bundle of real basic 2-forms. Since $\theta(V)J = 0$, $\forall \, V \in T\mathcal{F}$, we have a well-defined action of $J$ on $\Lambda^2_B$ by:

$$J : \Lambda^2_B \to \Lambda^2_B$$

$$\alpha(\cdot, \cdot) \mapsto \alpha(J\cdot, J\cdot).$$

Then by the formula:

$$\alpha(\cdot, \cdot) = \frac{\alpha(\cdot, \cdot) + \alpha(J\cdot, J\cdot)}{2} + \frac{\alpha(\cdot, \cdot) - \alpha(J\cdot, J\cdot)}{2},$$

we get a splitting

$$\Lambda^2_B = \Lambda^+_J \oplus \Lambda^-_J,$$

where $\Lambda^+_J$ is the bundle of $J$-invariant basic 2-forms, and $\Lambda^-_J$ is the bundle of $J$-anti-invariant basic 2-forms.

Let $\Omega^2_B$ be the space of basic 2-forms on $M$, $\Omega^+_J (\Omega^-_J)$ the space of $J$-invariant ($J$-anti-invariant) basic 2-forms.

**Definition.** Let $Z^2_B$ be the space of basic closed 2-forms on $M$, and let $Z^+_J = Z^2_B \cap \Omega^+_J$. Define

$$H^+_J(\mathcal{F}) = \{ \alpha \in H^2_B(\mathcal{F}; \mathbb{R}) | \exists \alpha \in Z^+_J \text{ such that } [\alpha] = \alpha \},$$

and the dimension of $H^+_J(\mathcal{F})$ are denoted by $h^+_J$ respectively.

It is obvious that

$$H^+_J(\mathcal{F}) + H^-_J(\mathcal{F}) \subseteq H^2_B(\mathcal{F}; \mathbb{R}).$$

**Definition.** $J$ is said to be $C^\infty$-pure if $H^+_J(\mathcal{F}) \cap H^-_J(\mathcal{F}) = 0$, and is said to be $C^\infty$-full if $H^+_J(\mathcal{F}) + H^-_J(\mathcal{F}) = H^2_B(\mathcal{F}; \mathbb{R})$. $J$ is $C^\infty$-pure and full if $H^+_J(\mathcal{F}) \oplus H^-_J(\mathcal{F}) = H^2_B(\mathcal{F}; \mathbb{R})$.

The main result is the following:

**Theorem 4.1.** Given a codimension four taut Riemannian foliation $\mathcal{F}$ on a closed smooth manifold $M$, if $J$ is a transversal almost complex structure satisfying $\theta(V)J = 0$ for any $V \in T\mathcal{F}$, then $J$ is $C^\infty$-pure and full.

**Remark 4.2.** The condition that $\theta(V)J = 0$ for any $V \in T\mathcal{F}$ seems to be necessary. One of the reason is we need this condition to guarantee $J$ preserves basic 2-forms. The other is that for a taut metric, we can easily construct a $J$ compatible taut metric and the corresponding transversal fundamental 2-form will be a basic form.

**Remark 4.3.** For a K-contact manifold $(M, \xi, \eta, \phi, g)$, we have proved that $\phi$ is $C^\infty$-pure and full $[10]$. For the characteristic foliation $\mathcal{F}_\xi$, $g$ is taut and $\theta(\xi)\phi = 0$, so this can be considered as a special case of Theorem 4.1.
In order to prove Theorem 4.1, we do some preparation. Let $g$ be a bundle-like metric inducing $g_Q$ on $Q$. Define the Hodge star operator:

$$\tilde{\alpha} = (-1)^{p(q-r)} (\alpha \wedge \chi_F).$$

The relation between $\tilde{\alpha}$ and the Hodge star operator $\ast$ with respect to $g$ is [9]

$$\ast \alpha = \tilde{\alpha} \wedge \chi_F,$$

where $\chi_F$ is the characteristic $p$-form of $F$ defined in Section 2.

The scalar product in $\Omega^r_B(F)$ is defined by

$$\langle \alpha, \beta \rangle_B = \int_M \alpha \wedge \tilde{\beta} \wedge \chi_F,$$

which is just the restriction of the usual scalar product on $\Omega^r(M)$ to the sub-space $\Omega^r_B(F)$ [9].

Define the formal adjoint $\delta_B : \Omega^r_B(F) \rightarrow \Omega^{r-1}_B(F)$ of $d_B : \Omega^{r-1}_B(F) \rightarrow \Omega^r_B(F)$ by

$$\langle d_B \alpha, \beta \rangle_B = \langle \alpha, \delta_B \beta \rangle_B.$$

It was shown in [6,9] that, on $\Omega^r_B(F)$

$$\delta_B = (-1)^{q(r+1)+1} \tilde{\alpha} (d_B - \kappa \wedge \tau).$$

Define the basic Laplacian

$$\Delta_B = d_B \delta_B + \delta_B d_B,$$

then set

$$\mathcal{H}_B^r(F) = \{ \text{the harmonic basic } r\text{-forms } \omega \mid \Delta_B \omega = 0 \}.$$
2-form \( \alpha \). Furthermore, we have \( \Delta_B \bar{\pi} = \bar{\pi} \Delta_B \) (note that if \( \kappa \neq 0 \), \( \Delta_B \) and \( \bar{\pi} \) do not commute). Hence,

\[
H^2(\mathcal{F}, \mathbb{R}) = H^2_B(\mathcal{F}) = H^+_B(\mathcal{F}) \oplus H^-_B(\mathcal{F}),
\]

and we denote the dimension of \( H^2_B(\mathcal{F}) \), \( H^+_B(\mathcal{F}) \), \( H^-_B(\mathcal{F}) \) by \( b^+_B \), \( b^-_B \), \( b^-_B \) respectively.

For a codimension four transversal almost Hermitian manifold \((M, F, J, g, F)\), we have the following relation

\[
\Lambda^+ + J = R F \oplus \Lambda^- + g Q,
\]

\[
\Lambda^+ \cap \Lambda^- + g Q = R F,
\]

Hence, similar to [3], we have the following two lemmas:

**Lemma 4.6.** If \( \alpha \in \Omega^+_B \) and \( \alpha = \alpha_h + d\theta + \delta\Psi \) is its basic Hodge decomposition, then \( (d\theta)_B^+ = (\delta\Psi)_B^+ \) and \( (d\theta)_B^- = -(\delta\Psi)_B^- \). In particular, the basic 2-form

\[
\alpha - 2(d\theta)_B^+ = \alpha_h
\]

is harmonic and the 2-form

\[
\alpha + 2(d\theta)_B^- = \alpha_h + 2d\theta
\]

is closed.

**Lemma 4.7.** Let \((M^{p+4}, \mathcal{F}, g, J, F)\) be a closed codimension four taut transversal almost Hermitian manifold. Then \( Z^-_J \subset H^+_g \), and \( Z^-_J \subset H^-_J \) is bijective. Furthermore, \( H^-_J = Z^-_J = H^+_g F^+ \).

With the above preparation, we can present the proof of the main result.

**Proof of Theorem 4.1.** Let \( g \) be the \( J \)-compatible metric, and \( F \) be the basic 2-form. If \( a \in H^+_J(\mathcal{F}) \cap H^-_J(\mathcal{F}) \), let \( a' \in Z^+_J \), \( a'' \in Z^-_J \) be the representative for \( a \). Then see page 39 in [9],

\[
d\chi_F + \kappa \wedge \chi_F = \varphi_0 \in F^2\Omega^{p+1}.
\]

Hence, on a codimension four foliation \((M, \mathcal{F})\), for basic 1-form \( \gamma \) and basic 2-form \( \alpha'' \), \( \gamma \wedge \alpha'' \wedge \phi_0 \in F^3\Omega^{p+1} = 0 \) vanishes. Therefore, by integration by parts, we have

\[
0 = \int_M \alpha' \wedge \alpha'' \wedge \chi_F
\]

\[
= \int_M (\alpha'' + d_B \gamma) \wedge \alpha'' \wedge \chi_F
\]

\[
= \int_M \alpha'' \wedge \alpha'' \wedge \chi_F + \int_M d_B \gamma \wedge \alpha'' \wedge \chi_F
\]

\[
= \int_M \alpha'' \wedge \bar{\pi} \alpha'' \wedge \chi_F + \int_M \gamma \wedge d_B \alpha'' \wedge \chi_F + \int_M \gamma \wedge \alpha'' \wedge d\chi_F
\]
\[
\int_M |\alpha''|^2 g \, \text{dvol} + \int_M \gamma \wedge \alpha'' \wedge (\phi_0 - \kappa \wedge \chi) \\
= \int_M |\alpha''|^2 g \, \text{dvol}.
\]

Hence, \(\alpha'' = 0\), i.e., \(a = 0\), that’s to say \(H^+_J (\mathcal{F}) \cap H^-_J (\mathcal{F}) = 0\).

The proof of fullness part is technically almost the same as the proof of Theorem 2.3 in [3]. \(\square\)

D. Domínguez’s remarkable theorem [1] says that for a Riemannian foliation \(\mathcal{F}\) on a closed manifold, there always exists a bundle-like metric for \(\mathcal{F}\) such that the mean curvature form \(\kappa\) is a basic 1-form. F. Kamber and Ph. Tondeur shows \(\kappa\) should be closed [5]. Furthermore, if \([\kappa] \in H^1_b (\mathcal{F})\) is trivial, then by a suitable conformal change to \(g_{TF}\), the bundle-like metric \(g\) can be modified to be a taut metric [5]. Since we have an injective map

\[H^1_b (\mathcal{F}) \to H^1 (M),\]

closed and simply connected Riemannian foliation is always taut [9]. Hence, we have the following corollary:

**Corollary 4.8.** For a codimension four Riemannian foliation \(\mathcal{F}\) on a closed and simply connected smooth manifold \(M\), if \(J\) is a transversal almost complex structure satisfying \(\theta(V)J = 0\) for any \(V \in \Gamma_{TF}\), then \(J\) is \(C^\infty\)-pure and full.

### 5. Bounds on \(h^\pm_J\)

Under the condition of Theorem 4.1 and by (3), we have

\[h^+_J + h^-_J = b^+_2 = b^+_B + b^-_B.\]

Furthermore, by relations (4), the following inequalities holds:

\[h^+_J \geq b^-_B, \quad h^-_J \leq b^+_B.\]

This can be strengthened as follows:

**Lemma 5.1.** Let \((M, \mathcal{F}, g, J, F)\) be a closed codimension four almost Hermitian taut Riemannian foliation. Assume that the harmonic part \(F_h\) of the transversal Hodge decomposition of \(F\) is not identically zero. Then

\[h^+_J \geq b^-_B + 1, \quad h^-_J \leq b^+_B - 1.\]

**Proof.** Let \(F = F_h + d\theta + \delta \Psi\) be the transversal Hodge decomposition of \(F\), then \(F + 2(d\theta)\) is a closed \(J\)-invariant basic 2-form, and \([F_h + 2d\theta] \in H^+_B \cap H^-_B\) is nontrivial since \(F_h\) is not identically zero. \(\square\)

A more specific case is when \(F\) is closed, i.e., the manifold \(M\) in Lemma 5.1 is transversal almost Kähler, we let \(\omega = F\).

**Theorem 5.2.** If \((M, \mathcal{F}, g, J, \omega)\) is taut transversal almost Kähler of codimension four, then

\[h^+_J \geq b^-_B + 1, \quad h^-_J \leq b^+_B - 1.\]
Proof. Since $g$ is taut, $\tau \Delta = \Delta \tau$. Hence, $d\omega = 0$ and $\omega \in \Omega^+_g$ induces that $\delta_B \omega = 0$, i.e., $\omega$ is basic harmonic itself. $\square$

References


Jiuru Zhou
School of Mathematical Sciences
Yangzhou University
Yangzhou 225009, P. R. China
Email address: zhoujiuru@yzu.edu.cn