CLEANNESS OF SKEW GENERALIZED POWER SERIES RINGS

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Abstract. A skew generalized power series ring \( R[[S, \omega]] \) consists of all functions from a strictly ordered monoid \( S \) to a ring \( R \) whose support contains neither infinite descending chains nor infinite antichains, with pointwise addition, and with multiplication given by convolution twisted by an action \( \omega \) of the monoid \( S \) on the ring \( R \). Special cases of the skew generalized power series ring construction are skew polynomial rings, skew Laurent polynomial rings, skew power series rings, skew Laurent series rings, skew monoid rings, skew group rings, skew Mal’cev-Neumann series rings, the “untwisted” versions of all of these, and generalized power series rings. In this paper we obtain some necessary conditions on \( R, S \) and \( \omega \) such that the skew generalized power series ring \( R[[S, \omega]] \) is (uniquely) clean. As particular cases of our general results we obtain new theorems on skew Mal’cev-Neumann series rings, skew Laurent series rings, and generalized power series rings.

1. Introduction

Given a ring \( R \), a strictly ordered monoid \((S, \leq)\) and a monoid homomorphism \( \omega : S \to \text{End}(R) \), one can construct the skew generalized power series ring \( R[[S, \omega]] \) (see Section 2 for details). Skew generalized power series rings are a common generalization of skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew Laurent series rings, skew monoid rings, skew group rings, skew Mal’cev-Neumann series rings, and of course the “untwisted” versions of all of these. Hence any result on skew generalized power series rings has its counterpart for each of these particular ring extensions, and these counterparts follow immediately from a single proof. This property makes skew generalized power series rings a useful tool for unifying results on the ring extensions listed above; such an approach was applied, e.g., in [18], [19], [20], [22], [25], [31], [32], [33], [34], [36], [39] and [42].

An element \( a \) of a ring \( R \) is called (uniquely) clean if it can be expressed (uniquely) as the sum of an idempotent and a unit in \( R \). The ring \( R \) is called a

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(uniquely) clean ring if every element of $R$ is (uniquely) clean. It follows that every local ring is clean. More generally, Camillo and Yu [1, Theorem 9] showed that a ring is semiperfect if and only if it is clean and contains no infinite family of orthogonal idempotents. In addition, they showed that every unit-regular ring is clean [1, Theorem 5]. Clean rings were first studied by Nicholson [27] in connection with exchange rings and lifting of idempotents.

A ring $R$ is Boolean in case every element in $R$ is idempotent. Clearly, the class of uniquely clean rings is a natural generalization of that of Boolean rings. In fact, a ring $R$ is uniquely clean if and only if $R/J(R)$ is Boolean and idempotents lift uniquely modulo $J(R)$ (cf. [29, Theorem 20]), where $J(R)$ is the Jacobson radical of $R$. Studies of (uniquely) clean of some ring extensions was considered in many papers (see [1], [5], [9], [27], [28], [29], [38], and [46]).

Because of the importance of (uniquely) clean rings in general theory of rings, it is natural to ask under what conditions on a ring $R$, a strictly ordered monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \to \text{End}(R)$, the skew generalized power series ring $R[[S, \omega]]$ is (uniquely) clean. In this paper we obtain some necessary conditions on $R$, $S$ and $\omega$ such that the skew generalized power series ring $R[[S, \omega]]$ is (uniquely) clean.

The paper is organized as follows. In Section 2, we recall the skew generalized power series ring construction and show that (skew) polynomial rings, (skew) Laurent polynomial rings, (skew) power series rings, (skew) Laurent series rings, (skew) monoid rings and the Mal’cev-Neumann construction are special cases of the construction. In Section 3, we study when the skew generalized power series ring $R[[S, \omega]]$ is (uniquely) clean. In particular, it is proved that, under suitable conditions, for a 2-primal ring $R$, a strictly ordered monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \to \text{End}(R)$, the skew generalized power series ring $R[[S, \omega]]$ is clean if and only if $R$ is semiregular with $J(R)$ nil, where $(S, \leq)$ is a totally ordered group or $(S, \leq)$ is an abelian torsion-free group such that $\leq$ is subtotal and $\omega : S \to \text{Aut}(R)$ a group homomorphism (see Theorem 3.7). As a consequence of the main result of this paper (Corollary 3.9), we obtain some characterizations a skew Laurent series ring $R[[x, x^{-1}; \alpha]]$ to be clean. In particular, we will show that, $R[[x, x^{-1}; \alpha]]$ is clean if and only if $R$ is semiregular with $J(R)$ nil, where $R$ is an $(\alpha, \delta)$-compatible 2-primal ring and it is either right Goldie, or has right Krull dimension, or is a ring with ACC on both right and left annihilators. The results were motivated by [46, Theorem 2.5] of Zhou and Ziembowski. Finally, we prove that $R[[S, \omega]]$ is uniquely clean if and only if $R$ is uniquely clean and $\omega_s$ is idempotent-stabilizing for all $s \in S$, where $(S, \leq)$ is a positively strictly ordered monoid, and $\omega : S \to \text{End}(R)$ a monoid homomorphism (Theorem 3.12). As an application, we provide (apparently) new examples of (uniquely) clean rings.

Throughout this paper all monoids and rings are with identity element that is inherited by submonoids and subrings and preserved under homomorphisms, but neither monoids nor rings are assumed to be commutative. We will denote by $\text{End}(R)$ the monoid of ring endomorphisms of $R$, and by $\text{Aut}(R)$ the group of ring automorphisms of $R$. If $S$ is a monoid or a ring, then the group of
invertible elements of $S$ is denoted by $U(S)$. When we consider an ordering relation $\leq$ on a set $S$, then the word “order” means a partial ordering unless otherwise stated. The order $\leq$ is total (respectively trivial) if any two different elements of $S$ are comparable (respectively incomparable) with respect to $\leq$. We will use the symbol $1$ to denote the identity elements of the monoid $S$, the ring $R$, and the ring $R[[S,\omega]]$, as well as the trivial monoid homomorphism $1 : S \to \text{End}(R)$ that sends every element of $S$ to the identity endomorphism. Also we use $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ for the integers, positive integers, rational numbers and the field of real numbers, respectively. For a ring $R$, we denote by $J(R)$ the Jacobson radical of $R$. The prime radical of a ring $R$ and the set of all nilpotent elements in $R$ are denoted by $P(R)$ and $\text{nil}(R)$, respectively.

2. Preliminaries

A partially ordered set $(S, \leq)$ is called artinian if every strictly decreasing sequence of elements of $S$ is finite, and $(S, \leq)$ is called narrow if every subset of pairwise order-incomparable elements of $S$ is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of $S$ has at least one but only a finite number of minimal elements. An ordered monoid is a pair $(S, \leq)$ consisting of a monoid $S$ and an order $\leq$ on $S$ such that for all $a, b, c \in S$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$. An ordered monoid $(S, \leq)$ is said to be strictly ordered if for all $a, b, c \in S$, $a < b$ implies $ca < cb$ and $ac < bc$.

For a strictly ordered monoid $S$ and a ring $R$, Ribenboim [42] defined the ring of generalized power series $R[[S]]$ consisting of all maps from $S$ to $R$ whose support is artinian and narrow with the pointwise addition and the convolution multiplication. This construction provided interesting examples of rings (e.g., Elliott and Ribenboim, [4]; Ribenboim, [40], [41]) and it was extensively studied by many authors.

In [23], Mazurek and Ziembowski, introduced a “twisted” version of the Ribenboim construction and studied when it produces a von Neumann regular ring. Now we recall the construction of the skew generalized power series ring introduced in [23]. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to \text{End}(R)$ a monoid homomorphism. For $s \in S$, let $\omega_s$ denote the image of $s$ under $\omega$, that is $\omega_s = \omega(s)$. Let $A$ be the set of all functions $f : S \to R$ such that the support $\text{supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(x, y) \in \text{supp}(f) \times \text{supp}(g) : s = xy\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in A$ as follows:

$$fg(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\omega_u(g(v)),$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, $A$ becomes a ring, called the ring of skew generalized power series with coefficients in $R$ and exponents in $S$ (one can
think of a map $f : S \to R$ as a formal series $\sum_{s \in S} r_s s$, where $r_s = f(s) \in R$ and denoted either by $R[[S, \omega]]$ or by $R[[S, \omega]]$ (see [18] and [23]).

To each $r \in R$ and $s \in S$, we associate elements $c_r, e_s \in R[[S, \omega]]$ defined by

$$c_r(x) = \begin{cases} r & x = 1 \\ x \in S \setminus \{1\} \end{cases}, \quad e_s(x) = \begin{cases} 1 & x = s \\ 0 & x \in S \setminus \{s\} \end{cases}.$$ 

It is clear that $r \mapsto c_r$ is a ring embedding of $R$ into $R[[S, \omega]]$ and $s \mapsto e_s$ is a monoid embedding of $S$ into the multiplicative monoid of the ring $R[[S, \omega]]$, and $e_r c_r = c_{\omega(r)} e_r$.

Below we quote from [19], how the classical constructions mentioned in Section 1 can be viewed as special cases of the skew generalized power series ring construction.

Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then for the additive monoid $S = \mathbb{N} \cup \{0\}$ of nonnegative integers, the map $\omega : S \to \text{End}(R)$ given by

$$\omega(n) = \sigma^n \text{ for any } n \in S,$$

is a monoid homomorphism. If furthermore $\sigma$ is an automorphism of $R$, then (2.1) defines also a monoid homomorphism $\omega : S \to \text{Aut}(R)$ for $S = \mathbb{Z}$, the additive monoid of integers. We can consider two different orders on each of the monoids $\mathbb{N} \cup \{0\}$ and $\mathbb{Z}$: the trivial order and the natural linear order. In both cases these monoids are strictly ordered, and thus in each of the cases we can construct the skew generalized power series ring $R[[S, \omega]]$. As a result, we obtain the following extensions of the ring $R$:

1. If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the trivial order, then the ring $R[[S, \omega]]$ is isomorphic to the skew polynomial ring $R[x, \sigma]$.
2. If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the natural linear order, then $R[[S, \omega]]$ is isomorphic to the skew power series ring $R[[x; \sigma]]$.
3. If $S = \mathbb{Z}$ and $\leq$ is the trivial order, and $\sigma$ is an automorphism of $R$, then $R[[S, \omega]]$ is isomorphic to the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$.
4. If $S = \mathbb{Z}$ and $\leq$ is the natural linear order, and $\sigma$ is an automorphism of $R$, then $R[[S, \omega]]$ is isomorphic to the skew Laurent series ring $R[[x, x^{-1}; \sigma]]$.

By applying the above points (1)-(4) to the case where $\sigma$ is the identity map of $R$, we can see that also the following ring extensions are special cases of the skew generalized power series ring construction: the ring of polynomials $R[x]$, the ring of power series $R[[x]]$, the ring of Laurent polynomials $R[x, x^{-1}]$, and the ring of Laurent series $R[[x, x^{-1}]]$.

Furthermore, any monoid $S$ is a strictly ordered monoid with respect to the trivial order on $S$. Hence if $R$ is a ring, $S$ is a monoid and $\omega : S \to \text{End}(R)$ is a monoid homomorphism, then we can impose the trivial order on $S$ and construct the skew generalized power series ring $R[[S, \omega]]$, which in this case will be denoted by $R[[S, \omega]]$. It is clear that the ring $R[[S, \omega]]$ is isomorphic to the classical skew monoid ring built from $R$ and $S$ using the action $\omega$ of $S$ on
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R. If ω is trivial, we write \( R[S] \) instead of \( R[[S,ω]] \). Obviously the ring \( R[S] \) is isomorphic to the ordinary monoid ring of \( S \) over \( R \).

Also, the construction of skew generalized power series rings generalizes another classical ring constructions such as the Mal’cev-Neumann Laurent series rings ((\( S, ≤ \)) a totally ordered group and trivial ω; see [3, p. 528]), the Mal’cev-Neumann construction of twisted Laurent series rings ((\( S, ≤ \)) a totally ordered group; see [10, p. 242]), and generalized power series rings \( R[[S]] \) (trivial ω; see [42, Section 4]), twisted generalized power series rings (see [14, Section 2], [23]).

Recall that a monoid \( S \) is said to be torsion-free if for any \( n \in \mathbb{N} \) and \( s,t \in S \), \( s^n = t^n \) implies \( s = t \). By use the terminology of [23], an order \( ≤ \) on a monoid \( S \) is said to be subtotal (see [42]) if for any \( s,t \in S \) there exists \( n \in \mathbb{N} \) such that \( s^n ≤ t^n \) and \( t^n ≤ s^n \). Clearly, every total order \( ≤ \) is subtotal. Furthermore, Example 3.8 in [25] show that in general a subtotal order need not be total. If \((S,·,≤)\) is an abelian ordered group, then the order \( ≤ \) is subtotal if and only if for every \( s \in S \) there exists \( n \in \mathbb{N} \) such that \( s^n ≥ 1 \) or \( s^n ≤ 1 \).

It is easy to see that if \((S,·,≤)\) is an ordered torsion-free commutative monoid such that \( ≤ \) is subtotal, then the binary relation \( ≤ \) on \( S \) defined by

\[
s ≤ t \quad \text{if and only if} \quad s^n ≤ t^n \quad \text{for some} \quad n \in \mathbb{N}
\]

is a total order on \( S \) and \((S,·,≤)\) is an ordered monoid. The order \( ≤ \) will be called the total order associated with \( ≤ \). Clearly, \( s ≤ t \) implies \( s ≤ t \) for any \( s,t \in S \), and thus by [23, Proposition 1.1], if a subset \( T \) of \( S \) is artinian and narrow with respect to \( ≤ \), then \( T \) is well-ordered with respect to \( ≤ \). Hence for any \( f \in R[[S,ω]] \setminus \{0\} \) there exists a smallest element \( s_0 \) of \( \text{supp}(f) \) with respect to \( ≤ \), which will be denoted by \( π(f) \).

To study when the skew generalized power series ring \( R[[S,ω]] \) is a clean ring, we will need the following results on units which plays a key role in this paper.

**Proposition 2.1** ([23, Proposition 2.2]). Let \( R \) be a ring, \((S,≤)\) a strictly ordered monoid, \( ω : S → \text{End}(R) \) a monoid homomorphism and \( A = R[[S,ω]] \). Let \( f \in A \) and assume that there exists a smallest element \( s_0 \) in \( \text{supp}(f) \). If \( s_0 \in U(S) \) and \( f(s_0) \in U(R) \), then \( f \in U(A) \).

**Proposition 2.2** ([23, Lemma 2.5]). Let \( R \) be a ring, \((S,·,≤)\) an ordered abelian torsion-free group such that \( ≤ \) is subtotal, \( ω : S → \text{End}(R) \) a monoid homomorphism, \( A = R[[S,ω]] \) and \( ≤ \) the total order associated with \( ≤ \). If \( f \in A \setminus \{0\} \) and for the smallest element \( s_0 \) of \( \text{supp}(f) \) with respect to \( ≤ \) we have \( f(s_0) \in U(R) \), then \( f \in U(A) \).

According to Krempa [8], an endomorphism \( α \) of a ring \( R \) is said to be rigid if \( αa(α) = 0 \) implies \( a = 0 \), for \( a \in R \). A ring \( R \) is said to be \( α \)-rigid if there exists a rigid endomorphism \( α \) of \( R \).

In [6], the authors introduced \( α \)-compatible rings and studied their properties. An endomorphism \( α \) of a ring \( R \) is said to be compatible (and the ring \( R \) is called an \( α \)-compatible ring) if for each \( a,b \in R \), \( ab = 0 \) if and only if
$aa(b) = 0$. Basic properties of rigid and compatible endomorphisms, proved by Hashemi and Moussavi in [6, Lemmas 2.2 and 2.1] are summarized in the following lemma:

**Lemma 2.3.** Let $\alpha$ be an endomorphism of a ring $R$. Then:

(i) if $\alpha$ is compatible, then $\alpha$ is injective;

(ii) $\alpha$ is compatible if and only if for all $a, b \in R$, $\alpha(a)b = 0 \iff ab = 0$;

(iii) the following conditions are equivalent:

1. $\alpha$ is rigid;
2. $\alpha$ is compatible and $R$ is reduced;
3. for every $a \in R$, $\alpha(a)a = 0$ implies that $a = 0$.

Marks, Mazurek and Ziembowski in [18] extended these notions as follows:

**Definition 2.4 ([18]).** Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. The ring $R$ is said to be $S$-compatible (resp. $S$-rigid) if $\omega_s$ is compatible (resp. rigid) for every $s \in S$.

A ring $R$ is called 2-primal if $P(R) = \text{nil}(R)$. It is obvious that commutative rings and reduced rings are 2-primal. Shin in [44, Proposition 1.11] showed that a ring $R$ is 2-primal if and only if every minimal prime ideal $P$ of $R$ is completely prime (i.e., $R/P$ is a domain). Also he proved that the minimal-prime spectrum of a 2-primal ring is a Hausdorff space with a basis of closed-and-open sets [44, Proposition 4.7] (for further information on 2-primal rings, see [15], [16], [37] and the references therein).

In the proof of the next results we will need the following theorem. Some characterizations of the Jacobson radical of skew generalized power series rings prove in [21].

**Theorem 2.5 ([21]).** Let $R$ be a 2-primal ring, $(S, \leq)$ a nontrivial ordered group and $\omega : S \to \text{Aut}(R)$ a group homomorphism. Assume that $\leq$ is total or $(S, \leq)$ is an abelian torsion-free group such that $\leq$ is subtotal, and $A = R[[S, \omega]]$. Suppose that $R$ is $S$-compatible and $P(R)$ is a nilpotent ideal of $R$. Then $J(A)$ is a nilpotent ideal of $A$ and coincides with $P(R)[[S, \omega]].$

### 3. Clean rings of skew generalized power series

In this section we will characterize the cleanness of a skew generalized power series ring $R[[S, \omega]]$ under various assumptions on $R, S$ and $\omega$. We also study when $R[[S, \omega]]$ is (uniquely) clean. We start with the following lemma, which plays a key role in the sequel.

**Lemma 3.1.** Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to \text{End}(R)$ a monoid homomorphism. If $R$ is $S$-compatible and $e$ is an idempotent of $R$, then we have the following statements:

1. $\omega_s(e) = e$ for all $s \in S$;
2. $eR[[S, \omega]]e = (eRe)[[S, \omega]].$
Proof. (1) Since \( e(1 - e) = (1 - e)e = 0 \) and \( R \) is \( S \)-compatible, it concludes that \( e = e\omega_s(e) \) and \( \omega_s(e) = e\omega_s(e) \) for all \( s \in S \). Hence \( \omega_s(e) = e \) for all \( s \in S \).

(2) By an easy computation and using part (1) the result proves. \( \Box \)

Recall from [17] that an ordered monoid \((S, \leq)\) is called \textit{quasitotally ordered} (and that \( \leq \) is a quasitotal order on \( S \)) if \( \leq \) can be refined to an order \( \preceq \) with respect to which \( S \) is a strictly totally ordered monoid. The class of quasitotally ordered monoids is quite large and important. For example, this class includes the totally ordered monoids, submonoids of a free group, and torsion-free nilpotent groups (see [30, Lemma 13.1.6 and Corollary 13.2.8]). Also, every commutative, torsion-free, and cancellative ordered monoid is quasitotally ordered monoid (e.g. see [39, 3.3]).

Proposition 3.2. Let \( R \) be a ring, \((S, \leq)\) a quasitotally ordered monoid, and \( \omega : S^+ \to \text{End}(R) \) a monoid homomorphism. Assume that \( R \) is \( S \)-rigid. If \( f \) is an idempotent of \( R[[S, \omega]] \), then \( f(1) \) is an idempotent of \( R \) and \( f = e_{f(1)} \).

Proof. Suppose that \( f \) is a nonzero idempotent of \( R[[S, \omega]] \). By hypothesis, the order \((S, \leq)\) can be refined to a strict total order \( \preceq \) on \( S \). It implies that there exists \( u_0 \in \text{supp}(f) \) such that \( u_0 \) is a minimal element of \( \text{supp}(f) \) under the total order \( \preceq \). For any \((u, v) \in X_{u_0^2}(f, f)\), \( u_0 \preceq u \), \( u_0 \preceq v \). If \( u_0 \prec u \), since \( \preceq \) is a strict order, \( u_0^2 \prec uu_0 \preceq uv = u_0^2\), a contradiction. Thus \( u = u_0 \). Similarly, \( v = u_0 \). Hence:

\[
(3.1) \quad f^2(u_0^2) = \sum_{(u, v) \in X_{u_0^2}(f, f)} f(u)\omega_u(f(v)) = f(u_0)\omega_{u_0}(f(u_0)).
\]

Assume that \( u_0 < 1 \). Since \( \preceq \) is a strict order, it follows that \( u_0^2 \prec u_0 \). Hence the minimality of \( \text{supp}(f) \) implies that \( f(u_0^2) = 0 \). From \( f^2 = f \) and Equation 3.1, we infer that \( f(u_0)\omega_{u_0}(f(u_0)) = 0 \). Since \( R \) is \( S \)-rigid, we obtain \( f(u_0) = 0 \) which contradicts to the fact that \( u_0 \) is a minimal element of \( \text{supp}(f) \). Hence \( 1 \preceq u_0 \).

Suppose that there exists \( 1 \prec s_0 \) such that \( f(s_0) \neq 0 \). We can assume that \( s_0 \) is the smallest with the condition under the total order \( \preceq \). Therefore \( f(s) = 0 \) for all \( 1 \prec s \prec s_0 \). From \( f^2 = f \), it implies that

\[
(3.2) \quad f(1)^2 = f(1) \quad \text{and} \quad f(s_0) = f(1)f(s_0) + f(s_0)\omega_{s_0}(f(1)).
\]

Since \( f(1) \) is an idempotent element of the ring \( R \), from Lemma 3.1(1) we infer \( f(s_0) = f(1)f(s_0) + f(s_0)f(1) \). Multiplying the last equation on the left by \( f(1) \) we have \( f(1)f(s_0) = f(1)f(s_0) + f(1)f(s_0)f(1) \) and thus \( f(1)f(s_0)f(1) = 0 \). Since \( R \) is \( S \)-rigid, \( f(1)f(s_0) = f(s_0)\omega_{s_0}(f(1)) = 0 \) and \( f(s_0) = 0 \) follows, which is a contradiction. Consequently we have \( f(s) = 0 \) for all \( s \in S \setminus \{1\} \). Thus \( f = e_{f(1)} \), as desired. \( \Box \)

If \((S, \leq)\) is a totally ordered monoid, then the \textit{positive cone} of \( S \) is denoted by \( S^+ \), i.e., \( S^+ = \{s \in S : s \geq 1\} \). Since \( S^+ \) is a submonoid of \( S \), for any ring
$R$ and $\omega : S \to \text{End}(R)$ a monoid homomorphism the skew generalized power series ring $R[[S^+,\omega]]$ is a subring of the ring $R[[S,\omega]]$.

**Lemma 3.3.** Let $R$ be a ring, $(S,\leq)$ a quasitotally ordered monoid such that for any $s \in S$ with $s \leq 1$ we have $s \in U(S)$, and let $\omega : S \to \text{End}(R)$ be a monoid homomorphism. Then for any $f \in R[[S,\omega]]$ there exist $u \in U(S)$ and $g \in R[[S^+,\omega]]$ such that $f = ge_u$.

**Proof.** Our proof follows the method employed in [26, Lemma 2.3]. By hypothesis, the order $(S,\leq)$ can be refined to a strict total order $\preceq$ on $S$. If $f = 0$, then we can put $u = 1$ and $g = 0$. Thus we assume that $f \neq 0$. Since $\text{supp}(f)$ is a non-empty artinian and narrow subset of $S$, the set of minimal elements of $\text{supp}(f)$ is finite and non-empty. Thus there exists a minimal element of $\text{supp}(f)$ under the total order $\preceq$, which will be denoted by $s$. If $s \geq 1$, then $\text{supp}(f) \subseteq S^+$, so $f \in R[[S^+,\omega]]$ and we can set $u = 1$ and $g = f$. We are left with the case where $s < 1$. Then $s \in U(S)$. Furthermore, if $t$ is any element of $\text{supp}(f)$, then $s \preceq t$, so $1 \preceq ts^{-1}$ and thus $ts^{-1} \in S^+$. Hence, for the function $g : S \to R$ defined by

$$g(x) = f(xs) \quad \text{for any} \quad x \in S$$

we have $\text{supp}(g) \subseteq \text{supp}(f) \cdot s^{-1} \subseteq S^+$, which implies that $g \in R[[S^+,\omega]]$. Moreover, for any $x \in S$ we have

$$f(x) = g(xs^{-1}) = g(xs^{-1})\omega_{xs^{-1}}(e_s(s)) = ge_s(x),$$

which shows that $f = ge_s$. Hence we can put $u = s$, and the result follows. \(\square\)

A ring $R$ is (von Neumann) regular (resp. unit-regular) if $a \in aRa$ (resp. $a \in aU(R)a$) for all $a \in R$. A ring $R$ is semiregular if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. Recall that a monoid $S$ is cyclic if for some $s \in S$ we have $S = \{s^n : n \in \mathbb{N} \cup \{0\}\}$.

**Proposition 3.4.** Let $R$ be a ring, $(S,\cdot,\leq)$ a quasitotally ordered cyclic group generated by $s$, and let $\omega : S \to \text{End}(R)$ be a group homomorphism. Assume that $R$ is $S$-rigid.

1. If $e_s + e_se_s$ is a unit of $R[[S,\omega]]$, then there exists $c \in R$ such that $a = ac$.  
2. If $R[[S,\omega]]$ is a clean ring, then $R$ is (von Neumann) regular.

**Proof.** (1) By hypothesis, the order $(S,\leq)$ can be refined to a strict total order $\preceq$ on $S$. Since $S$ is a cyclic group, Lemma 3.3 implies that there exist $n \in \mathbb{N} \cup \{0\}$ and $f \in R[[S^+,\omega]]$ such that

$$c_a + c_a e_s f = e_{sn},$$

where $S^+$ is the positive cone of $S$. Without loss of generality, we can assume that $1 < s$. Then computing both sides of Equation (3.2) at 1 we obtain:

$$af(1) + b\omega_s(f(s^{-1})) = e_{sn}(1).$$
Since $s^{-1} \prec 1$ and $f \in R[[S^+, \omega]]$, $f(s^{-1}) = 0$. Then Equation (3.3) becomes:

$$af(1) = e_{s^n}(1).$$

(3.4)

If $n = 0$, then Equation (3.4) implies that $af(1) = 1$ and so $a = af(1)a$. Thus, we will assume that $1 \leq n$. Then Equation (3.4) becomes:

$$af(1) = 0.$$ 

Indeed, after computing both sides of Equation (3.2) at $s$ we obtain:

$$af(s) + b\omega_s(f(1)) = 0.$$ 

(3.6)

Multiplying Equation (3.6) by $a$ from the left yields $a^2f(s) + ab\omega_s(f(1)) = 0$. Since $R$ is $S$-rigid, from Equation (3.5), we concludes $ab\omega_s(f(1)) = 0$. This implies $a^2f(s) = 0$. Since $R$ is reduced, we have

$$af(s) = 0.$$ 

(3.7)

We see easily by induction and an argument similar above that $af(1) = af(s) = \cdots = af(s^{n-1}) = 0$. Since $R$ is $S$-rigid, we have $\omega_s(f(s^{n-1}))a = 0$. On the other hand, computing both sides of Equation (3.2) at $s^n$ we obtain:

$$af(s^n) + b\omega_s(f(s^{n-1})) = 1.$$ 

(3.8)

Now multiplying Equation (3.8) by $a$ from the right gives $a = af(s^n)a$.

(2) Let $a \in R$. Then $-c_b e_{s-1} = e + u$ where $e^2 = e \in R[[S, \omega]]$ and $u$ is a unit of $R[[S, \omega]]$. By Proposition 3.2, there exists $b \in R$ such that $e = c_b$. So $c_b e_{s-1} + c_b$ is a unit of $R[[S, \omega]]$, and hence $c_a + c_b e_s$ is a unit of $R[[S, \omega]]$. From part (1) it follows that there exists $c \in R$ such that $a = aca$. This proves that $R$ is (von Neumann) regular, and the proof is complete. \qed

In the proof of the next result we will need the following criterion for cleanness of a ring which is due to Zhou and Ziembowski [46].

**Lemma 3.5** ([46, Lemma 2.3]). Let $a \in R$ and $e^2 = e \in R$ such that both $ea(1-e)$ and $(1-e)ae$ are contained in $J(R)$. If $ea$ is clean in $Re$ and $(1-e)(1-e)$ is clean in $(1-e)R(1-e)$, then $a$ is clean in $R$.

The following proposition provides us with a method of constructing clean rings.

**Theorem 3.6.** Let $R$ be a 2-primal ring, $(S, \leq)$ a nontrivial ordered group and $\omega : S \to \text{Aut}(R)$ a group homomorphism. Assume that $\leq$ is total or $(S, \leq)$ is an abelian torsion-free group such that $\leq$ is subtotal. Suppose that $R$ is $S$-compatible and $J(R)$ is a nilpotent ideal of $R$. If $R$ is semiregular, then the skew generalized power series ring $R[[S, \omega]]$ is clean.

**Proof.** We set $A := R[[S, \omega]]$ and $\tilde{R} := R/J(R)$. We only consider the case $(S, \leq)$ is an abelian torsion-free group such that $\leq$ is subtotal because the proof of the other case is similar. Suppose that $\leq$ is the total order associated with $\leq$. Suppose that $f$ is a nonzero element of $A$. We will show that $f$ is clean in $A$. Since $\text{supp}(f)$ is a non-empty artinian and narrow subset of $S$, the
set of minimal elements of \( \text{supp}(f) \) is finite and non-empty. Thus there exists a minimal element of \( \text{supp}(f) \) under the total order \( \preceq \). Then by Lemma 3.3, there exist \( s \in S \) and \( g \in R[[S^+, \omega]] \) such that \( f = ge_s \), where \( S^+ \) is the positive cone of \( S \). Without loss of generality, we can assume that \( 1 \preceq s \). Since \( \text{supp}(g) \) is a non-empty artinian and narrow subset of \( S \), the set \( \{ t \in \text{supp}(g) : 1 \preceq t \preceq s \} \) is finite, say equal to \( \{ t_1 = 1, t_2, t_3, \ldots, t_k = s \} \). Since \( J(R) \) is nilpotent and \( R \) is 2-primal, from Theorem 2.5 it implies that \( J(A) = J(R)[[S, \omega]] \). As \( R \) is 2-primal, the assumption implies that \( \hat{R} \) is strongly regular. It follows that \( g(t_1) = \hat{c}_0 \hat{u}_0 \), where \( \hat{c}_0 \) is a central idempotent of \( \hat{R} \) and \( \hat{u}_0 \) is a unit of \( \hat{R} \). Since \( J(R) \) is nil, [10, Theorem 21.28] concludes that idempotents of \( \hat{R} \) can be lifted to idempotents of \( R \), so we can assume that \( c_0^2 = e_0 \). Therefore, \( g(t_1) = c_0 u_0 c_0 + j_0 \) and so \( c_0 g(t_1) e_0 = c_0 u_0 c_0 + c_0 j_0 e_0 \), where \( c_0 u_0 e_0 \) is a unit of \( c_0 R e_0 \) and \( j_0 \in J(R) \). Applying Lemma 3.1, we obtain

\[
 c_{e_0} f c_{e_0} = c_{e_0} g c_{e_0} e_0,
\]

\[
 (c_0 R e_0)[[S, \omega]] = c_{e_0} A c_{e_0}.
\]

As \( c_{e_0} g c_{e_0} (t_1) = c_0 u_0 c_0 + c_0 j_0 e_0 \) is a unit of the ring \( c_0 R e_0 \), from Proposition 2.2 it follows that \( c_{e_0} f c_{e_0} \) is a unit (and hence a clean element of the ring \( (c_0 R e_0)[[S, \omega]] = c_{e_0} A c_{e_0} \)). Also, since \( \hat{c}_0 \) is a central element in \( \hat{R} \) and \( R \) is an \( S \)-compatible ring, Lemma 3.1(2) implies that \( c_{e_0} f - f c_{e_0} \in J(R)[[S, \omega]] = J(A) \). Therefore, we have

\[
 c_{e_0} f c_{(1-e_0)}, c_{(1-e_0)} f c_{e_0} \in J(A).
\]

By Lemma 3.5, to prove that \( f \) is clean in \( A \) it suffices to show that

\[
 c_{(1-e_0)} f c_{(1-e_0)}
\]

is clean in \( ((1-e_0) R (1-e_0))[[S, \omega]] = c_{(1-e_0)} A c_{(1-e_0)} \). Note that

\[
 f_1 := c_{(1-e_0)} f c_{(1-e_0)} = c_{(1-e_0)} g c_{(1-e_0)} e_0,
\]

and also we have \( c_{(1-e_0)} g c_{(1-e_0)} (t_1) = (1-e_0) j_0 (1-e_0) \).

Since \( R_1 := (1-e_0) R (1-e_0) \) is again 2-primal and \( S \)-compatible, strongly regular modulo \( J(R_1) \) with \( J(R_1) \) nil and \( P(R_1) \) is a nilpotent ideal of \( R_1 \) as argued above we have \( (1-e_0) g (t_2) (1-e_0) = e_1 u_1 \) otherwise, \( e_1 e_1 \) is an idempotent of \( R_1 \) which is central modulo \( J(R_1) \), \( e_1 u_1 e_1 \) is a unit of \( e_1 R_1 e_1 \) and \( j_1 \in J(R_1) \). Applying Lemma 3.1, we obtain \( c_{e_1} f_1 c_{e_1} = c_{e_1} g c_{e_1} e_0 \). We set:

\[
 g_1(t) := \begin{cases} 
 g(t) & t_1 \preceq t < t_2 \\
 0 & \text{otherwise}
\end{cases}, \quad g_2(t) := \begin{cases} 
 g(t) & t_2 \preceq t \\
 0 & \text{otherwise}
\end{cases}
\]

It is easy to see that \( g_1, g_2 \in A \) and also we have

\[
 c_{e_1} f_1 c_{e_1} = c_{e_1} g_1 c_{e_1} e_0 + c_{e_1} g_2 c_{e_1} e_0.
\]

Since \( c_{e_1} g_1 c_{e_1} (t_1) = e_1 j_0 e_1 \in e_1 J(R_1) e_1 \), we have

\[
 c_{e_1} g_1 c_{e_1} e_0 \in J(c_{e_1} R_1[[S, \omega]] c_{e_1}).
\]
Furthermore, since \( e_1g_2e_1(t_2) = e_1u_1e_1 + e_1j_1e_1 \) is a unit of the ring \( e_1R_1e_1 \), Proposition 2.2 implies that \( c_1g_2e_1e_s \) is a unit of the ring \( (e_1R_1e_1)[[S, \omega]] \). It follows from Equation (3.9) that \( c_1f_1e_1i \) is a unit (and hence a clean element of \( c_1R_1[[S, \omega]]e_1 \)). Since \( e_1a - ae_1 \in J(R_1) \) for all \( a \in R_1 \), we have

\[ c_1f_1c_1(1-e_0-e_1), c_1(1-e_0-e_1)f_1c_1, J(R_1)[[S, \omega]] \].

Therefore to show that \( f_1 \) is clean in \( R_1[[S, \omega]] \), by Lemma 3.5 it suffices to prove that \( f_2 := c_1(1-e_0-e_1)f_1c_1(1-e_0-e_1) \) is clean in \( c_1(1-e_0-e_1)R[[S, \omega]]c_1(1-e_0-e_1) \).

Consider that \( (1-e_0-\cdots-e_{i-1})g(t_i)(1-e_0-\cdots-e_{i-1}) = e_iu_ie_i + j_i \) such that \( e_i \) is an idempotent of \( R_0 := (1-e_0-\cdots-e_{i-1})R(1-e_0-\cdots-e_{i-1}) \) which is central modulo \( J(R_i) \), \( e_iu_ie_i \) is a unit of \( e_iR_i e_i \) and \( j_i \in J(R_i) \) for \( i = 1, 2, \ldots, k-1 \). We set \( f_i := c_1(1-e_0-e_1-\cdots-e_{i-1})f_1c_1(1-e_0-e_1-\cdots-e_{i-1}) \) for \( i = 1, 2, \ldots, k-1 \).

A similar argument as above yields to show that \( f_i \) is clean in \( R_i[[S, \omega]] \), by Lemma 3.5 it suffices to prove that \( f_{i+1} \) is a clean element in \( R_{i+1}[[S, \omega]] \) for \( i = 1, 2, \ldots, k-1 \). Now we set:

\[ g_1(t) := \begin{cases} g(t) & t_1 \leq t \leq t_{k-1} \\ 0 & \text{otherwise} \end{cases}, \quad g_2(t) := \begin{cases} g(t) & s \leq t \\ 0 & \text{otherwise} \end{cases}. \]

It is easy to see that \( g_1, g_2 \in A \) and also we have \( f_k = h_1 + h_2 \), where

\begin{align*}
(3.10) \quad h_1 &= c_1(1-e_0-e_1-\cdots-e_{k-1})g_1c_1(1-e_0-e_1-\cdots-e_{k-1})e_s, \\
(3.11) \quad h_2 &= c_1(1-e_0-e_1-\cdots-e_{k-1})g_2c_1(1-e_0-e_1-\cdots-e_{k-1})e_s.
\end{align*}

Since for \( i = 1, 2, \ldots, k-1 \), \( c_1(1-e_0-e_1-\cdots-e_{k-1})g_1c_1(1-e_0-e_1-\cdots-e_{k-1})(t_i) \in J(R_k) \) it follows that \( h_1 \in J(R_k)[[S, \omega]] = J(R_k)[[S, \omega]] \). Furthermore, since \( \text{supp}(h_2) \subseteq S^+ \), we have \( h_2 \in [[S^+, \omega]] \). As \( R_k \) is clean, \([35, \text{Theorem 3.2.8}]\) implies that \( R_k[[S^+, \omega]] \) is clean. But \( h_2 \) is an element of the ring \( R_k[[S^+, \omega]] \). Hence \( h_2 \) is a clean element of \( R_k[[S^+, \omega]] \). Therefore \( h_2 \) is a clean element of \( R_k[[S, \omega]] \). Thus \( h_2 = w + e \), where \( w \) is a unit of \( R_k[[S, \omega]] \) and \( e \) is an idempotent of \( R_k[[S, \omega]] \). On the other hand, \( h_1 \in J(R_k[[S, \omega]]) \). Therefore, \( h_1 + w \) is a unit of \( R_k[[S, \omega]] \), and so \( f_k = (h_1 + w) + e \) is a clean element of \( R_k[[S, \omega]] \). Hence by Lemma 3.5, we infer that \( f \) is clean in \( A \), and the proof is complete. 

Recall that a module \( RM \) has the (full) exchange property if for every module \( RA \) and any two decompositions \( A = M' \oplus N = \bigoplus_{i \in I} A_i \) with \( M' \cong M \), there exist submodules \( A_i' \subseteq A_i \) such that \( A = M' \oplus \bigoplus_{i \in I} A_i' \). A module \( RM \) has the finite exchange property if the above condition is satisfied whenever the index set \( I \) is finite. Exchange rings were introduced by Warfield [45] via the exchange property of modules. By [27], every clean ring is an exchange ring and it is shown in [27, Proposition 1.8] that a ring with central idempotents is clean if and only if it is an exchange ring [45]. A ring \( R \) is said to be strongly regular if for any \( a \in R \) there exists \( b \in R \) such that \( a = a^2b \). Strongly regular rings are exactly von Neumann regular rings in which all idempotents are central.
The following theorem provides a characterization of the cleanness of a skew generalized power series ring $R[[S, \omega]]$ in the case where $(S, \leq)$ an ordered cyclic group and $\leq$ is total or $(S, \leq)$ is an abelian torsion-free group such that $\leq$ is subtotal.

**Theorem 3.7.** Let $R$ be a 2-primal ring, $(S, \leq)$ an ordered cyclic group and $\omega : S \to \text{Aut}(R)$ a group homomorphism. Assume that $\leq$ is total or $(S, \leq)$ is an abelian torsion-free group such that $\leq$ is subtotal. Suppose that $R$ is $S$-compatible and $P(R)$ is a nilpotent ideal of $R$. Then the following statements are equivalent:

1. $R[[S, \omega]]$ is a clean ring;
2. $R[[S, \omega]]/J(R[[S, \omega]])$ is a clean ring;
3. $R[[S, \omega]]$ is an exchange ring;
4. $R[[S, \omega]]/J(R[[S, \omega]])$ is an exchange ring;
5. $R$ is semiregular with $J(R)\ nil$;
6. $R/J(R)$ is strongly regular with $J(R)\ nil$.

**Proof.** We set $A := R[[S, \omega]]$, $\overline{R} := R/P(R)$ and $\overline{R} := R/J(R)$.

(1) $\Rightarrow$ (2). The result follows from [1, Proposition 7], since $R$ is a clean ring if and only if $\overline{R}$ is a clean ring and all idempotents of the ring $\overline{R}$ can be lifted to idempotents of the ring $R$.

(2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). The result follows from [27, Corollary 2.4], since $R$ is an exchange ring if and only if $\overline{R}$ is an exchange ring and all idempotents of the ring $\overline{R}$ can be lifted to idempotents of $R$.

(4) $\Rightarrow$ (5). By Theorem 2.5, $J(A) = P(R)[[S, \omega]]$. So there exists a natural ring isomorphism

$$\overline{R}[[S, \overline{\omega}]] \cong A/(P(R)[[S, \omega]]) = A/J(A)$$

is an exchange ring, where $\overline{\omega} : S \to \text{Aut}(R/P(R))$ is the induced group homomorphism (i.e., $\overline{\omega}(r + P(R)) = \omega_s(r) + P(R)$ for any $s \in S$ and $r \in R$). The hypothesis implies that $P(R) = \text{nil}(R)$, and so $\overline{R}$ is a reduced ring. We prove that $\overline{R}$ is $(S, \overline{\omega})$-rigid. To prove this, consider any $\overline{a} \in \overline{R}$ and all $s \in S$ with $\overline{a} \cdot \overline{\omega}_s(\overline{a}) = \overline{0}$. So $s \omega(a) \in P(R)$. Therefore $a^2 \in \text{nil}(R)$, by Lemma 2.3(ii) and hence $\overline{a} = \overline{0}$. Thus $\overline{R}$ is an $(S, \overline{\omega})$-rigid ring. Hence all idempotents of $\overline{R}[[S, \overline{\omega}]]$ are central by Proposition 3.2. Therefore $\overline{R}[[S, \overline{\omega}]]$ is a clean ring by [27, Proposition 1.8]. Hence $\overline{R}$ is regular by Proposition 3.4. It follows that $J(R) \subseteq P(R)$. So $J(R) = \text{nil}(R)$, which is nil.

(5) $\Rightarrow$ (6). The hypothesis implies that $J(R) = \text{nil}(R)$, and so $\overline{R}$ is a strongly regular ring.

(6) $\Rightarrow$ (1). This follows from Theorem 3.6. $\square$

The following corollaries are immediate consequences of Theorem 3.7.

**Corollary 3.8.** Let $R$ be a domain, $(S, \leq)$ an ordered cyclic group and $\omega : S \to \text{Aut}(R)$ a group homomorphism. Assume that $\leq$ is total or $(S, \leq)$ is an
abelian torsion-free group such that $\leq$ is subtotal. Then $R[[S,\omega]]$ is a clean ring if and only if $R$ is a division ring.

Proof. It is easy to show that $R$ is $S$-compatible. Now, the result follows from Theorem 3.7. □

Let $R$ be a ring, $\alpha$ an automorphism of $R$, and $(S,\leq)$ an infinite cyclic group generated by $x$ with the ordering $x^m \leq x^n$ if and only if $m \leq n$. By setting $\omega x^n = \alpha^n$ for any $n \in \mathbb{Z}$ we obtain a monoid homomorphism $\omega : S \to \text{End}(R)$. The ring $R[[S,\omega]]$ is called the \textit{skew Laurent series ring} and denoted by $R[[x,x^{-1};\alpha]]$. The following result generalizes [46, Theorem 2.5].

Corollary 3.9. Let $R$ be a 2-primal ring and $\alpha$ an automorphism of $R$. Suppose that $R$ is either a ring with ACC on both right and left annihilators, or is left or right Goldie, or has the ACC on ideals, or has right Krull dimension and $R$ is $S$-compatible. Then the following statements are equivalent:

(1) $R[[x,x^{-1};\alpha]]$ is a clean ring;
(2) $R[[x,x^{-1};\alpha]]/J(R[[x,x^{-1};\alpha]])$ is a clean ring;
(3) $R[[x,x^{-1};\alpha]]$ is an exchange ring;
(4) $R[[x,x^{-1};\alpha]]/J(R[[x,x^{-1};\alpha]])$ is an exchange ring;
(5) $R$ is semiregular with $J(R)$ nil;
(6) $R/J(R)$ is strongly regular with $J(R)$ nil.

Proof. If $R$ has the ACC on ideals or $R$ is right Goldie or satisfies the ascending chain condition on both right and left annihilators, then by [43, Lemma 2.6.22], [12, Theorem 1] and [7, Theorem 1], $\mathcal{P}(R)$ is nilpotent, respectively. If $R$ has right Krull dimension, then by [13], $\mathcal{P}(R)$ is nilpotent. Also, if $R$ is a ring with ACC on both right and left annihilators, then by [2, Theorem 1.34], $\mathcal{P}(R)$ is nilpotent. Now, the result follows by Theorem 3.7. □

A ring $R$ is called \textit{strongly $\pi$-regular} if for each $a \in R$ there exists $n \geq 1$ such that $a^n \in a^{n+1}R$. A commutative ring $R$ is strongly $\pi$-regular if and only if $R/J(R)$ is (strongly) regular with $J(R)$ nil (see [11, Exercise 4.15]).

Corollary 3.10. Let $R$ be a commutative ring, $(S,\leq)$ a totally ordered cyclic group and $\omega : S \to \text{Aut}(R)$ a group homomorphism. Assume that $R$ is $S$-compatible and $R$ is either a ring with ACC on both right and left annihilators, or is left or right Goldie, or has the ACC on ideals, or has right Krull dimension. Then the skew Malcev-Neumann series ring $R((S,\omega))$ is a clean ring if and only if $R$ is strongly $\pi$-regular.

Corollary 3.11. Let $R$ be a reduced ring and $(S,\leq)$ a totally ordered cyclic group. Then the Malcev-Neumann series ring $R((S))$ is a clean ring if and only if $R$ is strongly regular.

We close this paper by investigating the uniquely clean property of a skew generalized power series ring $R[[S,\omega]]$. 
Theorem 3.12. Let $R$ be a ring, $(S, \leq)$ an ordered group and $\omega : S \to \text{Aut}(R)$ a group homomorphism. Assume that $\leq$ is total or $(S, \leq)$ is an abelian torsion-free group such that $\leq$ is subtotal. Then $R[[S, \omega]]$ is not uniquely clean for any nontrivial ring $R$.

Proof. Our proof follows the method employed in [46, Proposition 2.11]. We set $A := R[[S, \omega]]$. Suppose on the contrary that $A$ is uniquely clean. Then $\overline{A} := A/J(A)$ is Boolean by [29, Theorem 20]. So $\overline{1}$ is the only unit of $\overline{A}$. Let $s$ be any element of $S$ such that $1 < s$. Therefore, $\overline{e_s} = \overline{1}$, that is, $1 - e_s \in J(A)$. From Propositions 2.1 and 2.2 it implies that $1 - e_s$ is a unit of $A$. This is a contradiction.

Corollary 3.13. The following rings is not uniquely clean for any nontrivial ring $R$:

(a) The skew Laurent series ring $R[[x, x^{-1}; \sigma]]$, where $\sigma$ is an automorphism of $R$.

(b) The skew Mal’cev-Neumann series ring $R((S, \omega))$, where $(S, \leq)$ is a totally ordered group and $\omega : S \to \text{Aut}(R)$ a group homomorphism.

To characterize skew generalized power series rings that are uniquely clean, we will need the following lemma.

Lemma 3.14. Let $R$ be a ring, $(S, \leq)$ a positively strictly ordered monoid, and $\omega : S \to \text{End}(R)$ a monoid homomorphism. If $e$ is an idempotent of $R[[S, \omega]]$ and $f(1)$ is a central element of $R$ such that $\omega_s(f(1)) = f(1)$ for all $s \in S$, then $e = f(1)$.

Proof. The proof is similar to Proposition 3.2. □

Below we provide a characterization of uniquely clean skew generalized power series rings $R[[S, \omega]]$ in the case where $(S, \leq)$ is a positively strictly ordered monoid. Recall from [24] that an endomorphism $\sigma$ of a ring $R$ is idempotent-stabilizing if $\sigma(e) = e$ for every idempotent $e$ of $R$. It is easy to show that every $\sigma$-compatible ring is idempotent-stabilizing. The following theorem generalizes [29, Example 9].

Theorem 3.15. Let $R$ be a ring, $(S, \leq)$ a positively strictly ordered monoid, and $\omega : S \to \text{End}(R)$ a monoid homomorphism. Then $R[[S, \omega]]$ is uniquely clean if and only if $R$ is uniquely clean and $\omega_s$ is idempotent-stabilizing for all $s \in S$.

Proof. Suppose that $R$ is a uniquely clean ring and $\omega_s$ is idempotent-stabilizing for all $s \in S$. Let $f \in R[[S, \omega]]$. By [35, Theorem 3.2(8)], $R[[S, \omega]]$ is a clean ring. Let $f = g_1 + h_1 = g_2 + h_2$ where $g_1, g_2$ are units in $R[[S, \omega]]$ and $h_1, h_2$ are idempotents in $R[[S, \omega]]$. Since $h_1(0)$ and $h_2(0)$ are idempotent elements in $R$, by [29, Lemma 4] it follows that $h_1(0)$ and $h_2(0)$ are central. From Lemma 3.15 we deduce that $h_1 = c_{h_1(0)}$ and $h_2 = c_{h_2(0)}$. Now from $g_1 - g_2 = c_{h_2(0)} - c_{h_1(0)}$, it follows that $g_1(s) = g_2(s)$ for all $s \in S \setminus \{1\}$. 


Moreover, we have \( g_1(0) + h_1(0) = g_1(0) + h_2(0) \). Since \( g_1(0) \) and \( g_2(0) \) are unit elements in \( R \) and also \( R \) is a uniquely clean ring, it implies that \( g_1(0) = g_2(0) \). Hence \( g_1 = g_2 \) and so \( h_1 = h_2 \). This proves that \( R[[S, \omega]] \) is a uniquely clean ring. Conversely, suppose that \( R[[S, \omega]] \) is a uniquely clean ring and \( a \) is an idempotent in \( R \). From [29, Lemma 4] we infer that the idempotent element \( c_s \) is central in \( R[[S, \omega]] \). Thus \( c_s e_s = e_s c_s \) for all \( s \in S \) and so \( \omega_s(a) = a \) for all \( s \in S \). Therefore \( \omega_s \) is idempotent-stabilizing for all \( s \in S \). Furthermore, it is easy to show that the ring \( R \) is isomorphic to a factor ring of \( R[[S, \omega]] \). By [29, Theorem 22] every factor ring of a uniquely clean ring is again uniquely clean, therefore \( R \) is uniquely clean, which completes the proof. \( \Box \)

The following corollary provides a rich class of examples of uniquely clean rings.

**Corollary 3.16.** Let \( S \) be a submonoid of \((\mathbb{N} \cup \{0\})^n \) \((n \geq 2)\), endowed with the order \( \leq \) induced by the product order, or lexicographic order or reverse lexicographic order. Let \( R \) be a ring and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. Then \( R[[S, \omega]] \) is uniquely clean if and only if \( R \) is uniquely clean and \( \omega_s \) is idempotent-stabilizing for all \( s \in S \).

**Corollary 3.17.** Let \((S_1, \leq_1), \ldots, (S_n, \leq_n)\) be positively strictly ordered monoids. Denote by \((\text{lex} \leq)\) and \((\text{relex} \leq)\) the lexicographic order, the reverse lexicographic order, respectively, on the ordered monoid \( S_1 \times \cdots \times S_n \). Then \( R[[S_1 \times \cdots \times S_n, \text{lex} \leq]] \) is uniquely clean if and only if \( R[[S_1 \times \cdots \times S_n, \text{relex} \leq]] \) is uniquely clean.

Let \( \alpha \) and \( \beta \) be endomorphisms of \( R \) such that \( \alpha \circ \beta = \beta \circ \alpha \). Assume that \( S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \) is endowed with the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of \( \mathbb{N} \cup \{0\} \), and define \( \omega : S \to \text{End}(R) \) a monoid homomorphism via \( \omega(m, n) = \alpha^m \circ \beta^n \) for any \( m, n \in \mathbb{N} \cup \{0\} \). Then \( R[[S, \omega]] \cong R[[x, y; \alpha, \beta]] \), in which \( (a x^m y^n)(b x^p y^q) = a a^m \circ \beta^n(b)x^{m+p}y^{n+q} \) for any \( m, n, p, q \in \mathbb{N} \cup \{0\} \).

**Corollary 3.18.** Let \( \alpha \) and \( \beta \) be endomorphisms of a ring \( R \) such that \( \alpha \circ \beta = \beta \circ \alpha \). Then the ring \( R[[x, y; \alpha, \beta]] \) is uniquely clean if and only if \( R \) is uniquely clean and \( \alpha \) and \( \beta \) are idempotent-stabilizing.

**Corollary 3.19.** Let \( R \) be a ring and let \( S \) be any of the additive monoids \( \mathbb{Q}^+ = \{a \in \mathbb{Q} \mid a \geq 0\} \) or \( \mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\} \), where \( \leq \) is the usual order. Then the ring \( R[[S]] \) is uniquely clean if and only if \( R \) is uniquely clean.

Let \( R \) be a ring, and consider the multiplicative monoid \( \mathbb{N}^{\geq 1} \), endowed with the usual order \( \leq \). Then \( A = R[[\mathbb{N}^{\geq 1}]] \) is the ring of arithmetical functions with values in \( R \), endowed with the Dirichlet convolution:

\[
f g(n) = \sum_{d|n} f(d) g(n/d) \quad \text{for each } n \geq 1.
\]
Corollary 3.20. Let $R$ be a ring. Then the ring of arithmetical functions $R[[N^{≥1}]]$ is uniquely clean if and only if $R$ is uniquely clean.

Remark 3.21. The author does not know the answer to the following question: is it true that the assumption that $(S, \leq)$ is an ordered cyclic group is essential in Proposition 3.4 and Theorem 3.7?

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References

CLEANNESS OF SKEW GENERALIZED POWER SERIES RINGS


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