ZERO DISTRIBUTION OF SOME DELAY-DIFFERENTIAL POLYNOMIALS

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Abstract. Let $f$ be a meromorphic function of finite order $\rho$ with few poles in the sense $S_\lambda(r,f) := O(r^{\lambda+\varepsilon}) + S(r,f)$, where $\lambda < \rho$ and $\varepsilon \in (0, \rho - \lambda)$, and let $g(f) := \sum_{j=1}^{k} b_j(z) f(c_j + z)$ be a linear delay-differential polynomial of $f$ with small meromorphic coefficients $b_j$ in the sense $S_\lambda(r,f)$. The zero distribution of $f^n (g(f))^s - b_0$ is considered in this paper, where $b_0$ is a small function in the sense $S_\lambda(r,f)$.

1. Introduction

In this paper, we use key notions of the Nevanlinna theory and related results, as to those, we refer the reader to [6, 7, 9]. A meromorphic function $\alpha$ is said to be a $\lambda$-small function of a meromorphic function $f$ of finite order $\rho$, if there exists $\lambda < \rho$, such that for any $\varepsilon \in (0, \rho - \lambda)$,

$$T(r,\alpha) = O(r^{\lambda+\varepsilon}) + S(r,f),$$

outside a possible exceptional set $F$ of finite logarithmic measure. Here, $S(r,f)$ is any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside a set $F$. For the sake of simplicity, the right hand side in (1) will be denoted by $S_\lambda(r,f)$. In addition, we say that $f$ has few poles in the sense of (1), if $N(r,f) = S_\lambda(r,f)$.

The first author studied in [8] the zero distribution of $f^n (h(f))^s - b_0$, where $n, s$ are positive integers, $b_0$ is a $\lambda$-small function of $f$, and $h(f)$ is a shift polynomial given by

$$h(f)(z) := \sum_{j=1}^{k} b_j(z) f(z + c_j),$$

where $b_j$ are $\lambda$-small functions of $f$ and $c_j$ are complex numbers. A similar problem had been considered in [13] for $fg(f) - b_0$, where $b_0$ is a non-zero...
polynomial, and \( g(f) \) is a delay-differential polynomial given by

\[
g(f) := \sum_{j=1}^{k} b_j(z) f^{(k_j)}(z + c_j),
\]

where \( b_j \) are small functions of \( f \) in the sense \( T(r, b_j) = S(r, f) \), \( c_j \) are complex numbers and \( k_j \) are non-negative integers.

Our purpose is to improve and extend the results in [8, 13] for meromorphic function \( f \) with \( N(r, f) = S_\lambda(r, f) \) by considering the zero distribution of \( f^n(g(f))^r - b_0 \), where \( b_0 \) is a \( \lambda \)-small function of \( f \), and \( g(f) \) is a delay-differential polynomial given in (2) with coefficients \( b_j \) being \( \lambda \)-small functions of \( f \). In particular, we generalize some other results in [1, 4, 10, 11] and [3, Chapter 4].

The rest of the paper is organized as follows. Section 2 contains the results concerning the zero distribution of \( f^n(g(f))^r - b_0 \) in case \( b_0 \neq 0 \), while the results related to the case \( b_0 = 0 \) are given in Section 3. The lemmas needed for proving the main results are presented in Section 4, and proofs for the main results are given in Sections 5 and 6.

2. The case \( b_0 \neq 0 \)

Our starting point is the following two examples that show the incompleteness of [8, Theorem 4.4] and [13, Theorem 1.1]. The first example shows that some exceptional cases may occur in [8, Theorem 4.4].

**Example 2.1.** Let \( g_1(f) \equiv 1 \) and \( g_2(f) = f(z + \pi i) \), with \( f(z) = e^z + 1 \). Then

\[
f g_1(f) - 1 = e^z \quad \text{and} \quad f g_2(f) - 1 = -e^{2z}
\]

have no zeros.

Regarding [13, Theorem 1.1], we find that an exceptional case may occur as shown by the following example.

**Example 2.2.** Let \( f(z) = e^{z/2} + e^{-z/2} \) and let the delay-differential polynomials

\[
g_1(f) := \frac{1}{2} f(z + 4\pi i) + f'(z), \quad g_2(f) := \frac{1}{2} f(z + 4\pi i) - f'(z).
\]

Then,

\[
f g_1(f) - 1 = e^z \quad \text{and} \quad f g_2(f) - 1 = e^{-z}
\]

have no zeros.

Due to the above examples, we tried to complete [8, Theorem 4.4] and [13, Theorem 1.1]. In fact, we proved the following theorem which extends and completes these results.

**Theorem 2.3.** Let \( f \) be a transcendental meromorphic function of finite order \( \rho \) with \( N(r, f) = S_\lambda(r, f) \), \( b_0 \) be a non-vanishing \( \lambda \)-small function of \( f \) and \( g_1(f), g_2(f) \) be non-vanishing linear delay-differential polynomials as in (2)
with \( \lambda \)-small coefficients of \( f \) such that \( g_1(f) \not\equiv g_2(f) \). Then for the two functions \( F_1 := fg_1(f) - b_0 \) and \( F_2 := fg_2(f) - b_0 \), we have \( \max \{ \lambda(F_1), \lambda(F_2) \} = \rho \), except when one of the of the following cases holds:

(i) \( g_1(f) = L_1(z)f + M(z)f' \) and \( g_2(f) = L_2(z)f - M(z)f' \), where \( L_1, L_2 \) and \( M \) are non vanishing \( \lambda \)-small functions of \( f \), and \( L_1 + L_2 \not\equiv 0 \).

(ii) Only one of \( g_i(f), i = 1, 2 \), is a \( \lambda \)-small function of \( f \).

If \( g_1 \) and \( g_2 \) are shift polynomials, then \([13, \text{Theorem 1.1}]\) is correct. Moreover, if both of \( g_1 \) and \( g_2 \) are not small functions of \( f \), then \([8, \text{Theorem 4.4}]\) is correct.

The condition \( g_1(f) \not\equiv g_2(f) \) cannot be dropped out of Theorem 2.3. For example, if \( f(z) = e^z + z \) and \( g_k(f)(z) = 2f^{(k+1)}(z + \pi i) + f(z) \) for \( k = 1, 2 \), then \( f(z)g_k(f)(z) - z^2 = -e^{2z} \) has no zeros.

Three recent papers should be mentioned here related to Theorem 2.3: The paper \([12]\) is considering zeros of expressions of type \( f f^{(k)} - b \). The paper \([10]\) is involving the shifts \( f(z+c_1) \) and \( f(z+c_2) \) instead of \( g_1(f) \) and \( g_2(f) \). In the paper \([2]\), iterated differences replace \( g_1(f) \) and \( g_2(f) \). Moreover, \( b_0 \) is taken to be a non-zero polynomial in \([2]\) and \([10]\).

The next result extends \([8, \text{Theorem 2.1}]\). The proof is a simple modification of the corresponding proof of \([8, \text{Theorem 2.1}]\).

**Theorem 2.4.** Let \( f \) be a transcendental meromorphic function of finite order \( \rho \) with \( N(r, f) = S_\lambda(r, f) \), \( b_0 \) be a non-vanishing \( \lambda \)-small function of \( f \), \( g(f) \) be a non-vanishing delay-differential polynomial as in \((2)\) with \( \lambda \)-small coefficients of \( f \), \( n \geq 2 \) and \( s \geq 1 \). Then \( F := f^n g(f)^s - b_0 \) has sufficiently many zeros to satisfy \( \lambda(F) = \rho \).

The condition \( N(r, f) = S_\lambda(r, f) \) is necessary in Theorem 2.4. For example, the function \( f(z) = \tan z \) is of order 1 and \( N(r, f) = O(r) \). If \( g(f)(z) = f(z + \pi/2) = -\cot z \), then \( F(z) := f^2(z)g(f)^2(z) - 2 = -1 \) has no zeros.

During preparing this paper, Z. Huang offered us the following example, which shows that \([8, \text{Theorem 3.1}]\) does not hold always.

**Example 2.5.** Take \( f(z) = e^z + z \) and define

\[
g(f)(z) := 2f(z) - f(z + \log 2) = z - \log 2.
\]

Then, for every integer \( s \geq 1 \), the delay polynomial

\[
F(z) := f(z)g(f)^s(z) - z(z - \log 2)^s = (z - \log 2)^s e^z
\]

has finitely many zeros only.

We give the following extension and a complete version of \([8, \text{Theorem 3.1}]\).

**Theorem 2.6.** Let \( f \) be a transcendental meromorphic function of finite order \( \rho \) with \( N(r, f) = S_\lambda(r, f) \), \( b_0 \) be a non-vanishing \( \lambda \)-small function of \( f \) and \( g(f) \) be a non-vanishing delay-differential polynomial as in \((2)\) with \( \lambda \)-small coefficients of \( f \). If \( s \geq 2 \), then \( F := fg(f)^s - b_0 \) satisfies

\[
\max \{ \lambda(F), \lambda(f) \} = \rho.
\]
In particular, if $\rho \notin \mathbb{N}$, then $\lambda(F) = \lambda(f) = \rho$.

Example 2.5 illustrates Theorem 2.6 in the case when $\lambda(F) < \rho$ and $\lambda(f) = \rho \in \mathbb{N}$.

If $\rho(f) \notin \mathbb{N}$, we see that [8, Theorem 3.1] is correct. This leads to ask, in case $\rho(f) \in \mathbb{N}$, what are the conditions on $g(f)$ that ensure $\lambda(F) = \rho$ in Theorem 2.6? To give a partial answer, we consider a particular form of the delay-differential polynomial $g(f)$, which is given by

$$
\tilde{g}(f) := \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j}(z)f^{(j)}(z + c_i),
$$

where $b_{i,j}$ are $\lambda$-small functions of $f$, and $b_{im} \equiv 1$ for every $0 \leq i \leq n$. We prove the following result, which may be seen as another variant of [8, Theorem 3.1], besides Theorem 2.6.

**Theorem 2.7.** Let $f$ be a transcendental meromorphic function of finite order $\rho$ with $N(r,f) = S_\lambda(r,f)$, $b_0$ be a non-vanishing $\lambda$-small function of $f$ and $\tilde{g}(f)$ is given in (3) satisfying

$$
T(r,\tilde{g}(f)) \neq S_\lambda(r,f),
$$

and

$$
T(r,w) = S_\lambda(r,f)
$$

for every meromorphic solution $w$ of $\tilde{g}(w) = 0$. If $s \geq 2$, then $F := f(\tilde{g}(f))^s - b_0$ satisfies $\lambda(F) = \rho$.

Example 2.5 above shows that Theorem 2.7 could fail without the condition (4). Meanwhile, the next example shows that Theorem 2.7 could also fail without the condition (5).

**Example 2.8 ([1]).** Suppose that $f(z) = e^z - \frac{2}{3}e^{-z/2}$ and let

$$
\tilde{g}(f)(z) := f'(z + 4\pi i) - f(z) = e^{-z/2}.
$$

Clearly $T(r,\tilde{g}(f)) \neq S_\lambda(r,f)$ and the function $w = e^z$ is a solution of $\tilde{g}(w) = 0$ without satisfying the condition (5). Finally, we can see that

$$
F(z) := f(z)\tilde{g}(f)^2(z) - 1 = \left(e^z - \frac{2}{3}e^{-z/2}\right)e^{-z} - 1 = -\frac{2}{3}e^{-3z/2}
$$

has no zeros.

3. The case $b_0 = 0$

In this section, we generalize some results from [3, Chapter 4] and [11]. In [3], the difference operator $\Delta f$ has been used instead of $g(f)$, while in [11], $g(f)$ was considered to be a shift polynomial with constant coefficients.
Theorem 3.1. Let $f$ be a transcendental meromorphic function of finite order $\rho$ such that $N(r, f) = S_\lambda(r, f)$. Let $g(f)$ be a non-vanishing linear delay-differential polynomial as in (2) with at least two terms and $\lambda$-small coefficients of $f$. Suppose $n \geq 1$ and define $F := f^n g(f)$. Then

1. If $\lambda(f) = \rho$, then $\lambda(F) = \rho$ as well.
2. If $\lambda(f) < \rho$, then $\lambda(F) < \rho$. Furthermore
   (i) If $\lambda(f) \leq \rho - 1$, $\lambda < \rho - 1$ and $\rho \neq 1$, then $\lambda(F) = \rho - 1$.
   (ii) If $\rho - 1 < \lambda(f) = \lambda^* < \rho$ and $\lambda < \lambda^*$, then $\lambda(F) = \lambda^*$.
   (iii) If $\lambda(f) = \lambda = 0$ and $\rho = 1$, then $\lambda(F) = 0$.

3.2. The case (1) in Theorem 3.1 holds for $F := f^n g(f)^s$, $s \geq 1$.

The following example illustrates the case (2) in Theorem 3.1.

Example 3.3. (1) The function $e^z$ is of order 2 and has no zeros. Define

$$g(f)(z) := f(z) + f(z + 1) = e^z (e^{2z+1} + 1).$$

Then, for any integer $n$, $\lambda(F) = 1 = \rho(f) - 1$. This illustrates the case (2)-(i) in Theorem 3.1.

(2) The function $f(z) = e^z \cosh \sqrt{z}$ is an entire function of order 1 and $\lambda(f) = 1/2$. Let

$$g(f)(z) := f''(z) + \left(\frac{1}{2z} - 2\right) f'(z) + \left(2 - \frac{3}{4z}\right) f(z) = e^z \cosh \sqrt{z}.$$ 

Then, for every integer $n$, $\lambda(F) = \lambda(f) = 1/2$. This illustrates the case (2)-(ii) in Theorem 3.1.

The condition $\lambda < \rho - 1$ for $S_\lambda(r, f)$ is necessary for the case (2)-(i) in Theorem 3.1. For example, the function $f(z) = (e^z + 1)e^{2z}$ is of order $\rho(f) = 2$ and $\lambda(f) = 1$. Let

$$g(f)(z) := \frac{1}{e^z + 1} f'(z) - \frac{2z e^{4z^2 - 4iz}}{e^z + 1} f(z + 2\pi i) = \frac{e^z}{e^z + 1} e^{2z}.$$ 

Clearly, the coefficients of $g(f)$ are of growth at most $S_\lambda(r, f)$, where $\lambda = 1 = \rho(f) - 1$. Then $F(z) := f(z) g(f)(z) = e^{2z^2 + z}$ such that $\lambda(F) = 0 \neq \rho(f) - 1$.

Theorem 3.4. Let $f$ be a transcendental meromorphic function of finite order $\rho$ with a finite Borel exceptional value $d$ and $N(r, f) = S_\lambda(r, f)$. Let $g(f)$ be a non-constant delay-differential polynomial as in (2) with $\lambda$-small coefficients of $f$. Defining $F := fg(f)$, the following statements hold:

(i) If $d \neq 0$ and

$$\sum_{j=1, k_j=0}^k b_j(z) \neq 0,$$ 

(ii) If $d = 0$ and $\lambda(f) = 0$, then

$$\lambda(F) = 0,$$ 

(iii) If $d = 0$ and $\lambda(f) \neq 0$, then

$$\lambda(F) = \lambda(f).$$
then $F(z)$ has at most one finite Borel exceptional value $d^* \neq 0$, which satisfies

$$\frac{d^* - F(z)}{(d - f(z))^2} = \frac{d^*}{d^2} = \sum_{j=1,k_j=0}^{k} b_j(z).$$

(ii) If $d \neq 0$ and

$$\sum_{j=1,k_j=0}^{k} b_j(z) \equiv 0,$

then $F(z)$ has no finite Borel exceptional values.

(iii) If $d = 0$, then 0 is a Borel exceptional value of $F(z)$ as well.

The case (i) of Theorem 3.4 may occur. For example, the function $f(z) = e^z + 1$ has a Borel exceptional value 1. If $g(f) = f(z + \pi i)$, then $F(z) = 1 - e^{2z}$ has 1 as a Borel exceptional value as well, and $\frac{1-F(z)}{(1-f(z))^{2}} = 1$.

Remark 3.5. The case (iii) of Theorem 3.4 is in fact a special case of the case (2) of Theorem 3.1.

The following consequence of Theorem 3.4 generalizes [4, Theorem 1.2].

**Corollary 3.6.** Under the hypotheses of Theorem 3.4, $F(z)$ has no Borel exceptional value $b$ such that

$$b - d^2 \sum_{j=1,k_j=0}^{k} b_j(z) \neq 0.$$

The following example illustrates Corollary 3.6.

**Example 3.7.** (1) The function $f(z) = e^z + 1$ has the Borel exceptional value 1. If

$$g(f)(z) := f(z + \pi i) - 2f(z) + 4f'(z) = e^z - 1,$$

then $F(z) := f(z)g(f)(z) = e^{2z} - 1$, and for every $b \neq 1(1 - 2) = -1$, we have $\lambda(F - b) = 1$.

(2) The function $f(z) = e^z$ has 0 as Borel exceptional value. If

$$g(f)(z) := 2f(z + \pi i) + 2f'(z) + f(z) = e^z,$$

then $F(z) := f(z)g(f)(z) = e^{2z}$, and for every $b \neq 0(2 + 1) = 0$, we have $\lambda(F - b) = 1$.

Before we state the final result, we recall the definition of the multi-order exponent of convergence of zeros of $f$ by

$$\lambda_{(2)}(f) := \lim_{r \to \infty} \sup r \log + \frac{N_{(2)} \left( r, \frac{1}{f} \right)}{\log r},$$

where $N_{(2)} \left( r, \frac{1}{f} \right)$ denotes the counting function of zeros of $f$ whose multiplicities are not less than 2.
Theorem 3.8. Let \( f \) be a transcendental meromorphic function of finite order \( \rho \) such that \( \lambda_2(f) = \rho \) and that \( N(r, f) = S_\lambda(r, f) \), \( g(f) \) be a non-constant linear delay-differential polynomial as in (2) with \( \lambda \)-small coefficients of \( f \). If \( n \geq 1 \), then \( F := f^n g(f) \) takes every value \( a \in \mathbb{C} \) infinitely often and such that \( \lambda(F - a) = \rho \).

4. Auxiliary results

In this section, we collect the results that are needed for proving the main results.

Using the same reasoning as in the proof of [7, Lemma 2.4.2], we easily get the following lemma.

Lemma 4.1. Let \( f \) be a transcendental meromorphic solution of finite order \( \rho \) of a differential-difference equation

\[
f^n P(z, f) = Q(z, f),
\]

where \( P(z, f) \) and \( Q(z, f) \) are delay-differential polynomials in \( f \) with \( \lambda \)-small coefficients of \( f \). If the total degree of \( Q(z, f) \) is \( \leq n \), then for each \( \varepsilon > 0 \)

\[
m(r, P(z, f)) = O(r^{\rho - 1 + \varepsilon}) + S_\lambda(r, f).
\]

Lemmas 4.2 and 4.3 below are, respectively, slight modifications of [5, Theorem 3] and [8, Lemma 2.5].

Lemma 4.2. Let \( f \) be a transcendental meromorphic function of finite order \( \rho \), and let \( g \) be a \( \lambda \)-small function of \( f \). Then for all \( z \) such that \( |z| \notin E \cup [0, 1] \), where \( E \) is of finite logarithmic measure, and for all \( k > j \)

\[
\left| \frac{g^{(k)}(z)}{g^{(j)}(z)} \right| \leq |z|^{(k-j)(\lambda - 1 + \varepsilon)}.
\]

Lemma 4.3. Let \( f \) be a transcendental meromorphic function of finite order \( \rho \) with \( N(r, f) = S_\lambda(r, f) \), and let \( g(f) \) be a non-vanishing linear delay-differential polynomial with \( \lambda \)-small coefficients of \( f \). If \( n \geq 1 \) and \( s \geq 1 \), then we have \( \rho(f^n g(f)^s) = \rho \).

The following lemma is the complete version of [8, Proposition 4.1].

Lemma 4.4. Let \( f \) be a transcendental meromorphic function of finite order \( \rho \) with \( N(r, f) = S_\lambda(r, f) \), \( b_0 \) be a non-vanishing \( \lambda \)-small function of \( f \) and \( g_1(f) \), \( g_2(f) \) be non-vanishing linear delay-differential polynomials as in (2) with \( \lambda \)-small coefficients of \( f \) such that \( g_1(f) \neq g_2(f) \). Suppose that

\[
f g_j(f) = b_0 + \beta_j e^{h_j}, \quad j = 1, 2,
\]

where \( \beta_1, \beta_2 \) are \( \lambda \)-small functions of \( f \), and \( h_1, h_2 \) are polynomials. Then,

\[
\deg h_1 = \deg h_2 = \rho.
\]
Furthermore, if \( \text{deg}(h_1 + h_2) < \rho \), then the delay-differential polynomials \( g_1(f) \) and \( g_2(f) \) reduce to
\[
g_1(f) = L_1(z)f + M(z)f' \quad \text{and} \quad g_2(f) = L_2(z)f - M(z)f',
\]
where \( L_1, L_2, M \) are non-vanishing \( \lambda \)-small functions of \( f \), and \( L_1 + L_2 \neq 0 \).

**Proof.** As to the claims \( \text{deg} h_1 = \text{deg} h_2 = \rho \), the proof in [8] may be repeated, verbatim.

Now, suppose that \( \text{deg}(h_1 + h_2) < \rho \). Differentiating (6) and eliminating exponentials, we obtain
\[
f' + \frac{g_j(f)'}{g_j(f)} - \frac{\beta_j'}{\beta_j} - h'_j = A_j \frac{1}{fg_j(f)},
\]
where
\[
A_j = b_0 \left( \frac{b_0'}{b_0} - \frac{\beta_j'}{\beta_j} - h'_j \right).
\]
Moreover, \( A_1 \) and \( A_2 \) are not vanishing identically by the reasoning used in the proof of [8, Theorem 2.1]. From (7), we obtain
\[
m\left( r, \frac{1}{fg_j(f)} \right) = S_\lambda(r, f), \quad j = 1, 2
\]
and
\[
N_2\left( r, \frac{1}{fg_j(f)} \right) = S_\lambda(r, f), \quad j = 1, 2,
\]
where \( N_2(r, \cdot) \) stands for the non-simple zeros. Therefore
\[
T(r, fg_j(f)) = N_1\left( r, \frac{1}{fg_j(f)} \right) + S_\lambda(r, f), \quad j = 1, 2,
\]
where \( N_1(r, \cdot) \) counts the simple zeros only.

Clearly, we also have \( N_2(r, 1/f) = S_\lambda(r, f) \), hence
\[
N(r, 1/f) = N_1(r, 1/f) + S_\lambda(r, f).
\]
Making use of the identity \( \frac{1}{f''} = \frac{g_j(f)'}{fg_j(f)} \), we obtain \( m(r, 1/f) = S_\lambda(r, f). \)
Assuming, as we may, that \( \rho - 1 \leq \lambda < \rho \), we obtain
\[
T(r, f) = N_1(r, 1/f) + S_\lambda(r, f).
\]
Writing now (7) in the form
\[
f'g_j(f) + f(g_j(f))' - \left( \frac{\beta_j'}{\beta_j} + h'_j \right) fg_j(f) = A_j, \quad j = 1, 2,
\]
we observe that \( f'(z_0)g_j(f)(z_0) - A_j(z_0) = 0 \) as soon \( z_0 \) is a simple zero of \( f \), outside of all possible zeros and poles of \( A_j, \beta_j \). Since (8) holds for both of \( j = 1, 2 \), it is easy to see that all possible poles of
\[
H = \frac{A_1 g_2(f) - A_2 g_1(f)}{f}
\]
are multiple except perhaps at the shift values \( f(z_0 + c_j) \) and the poles of \( A_j \), hence \( N(r, H) = S_\lambda(r, f) \). Moreover, \( m(r, H) = S_\lambda(r, f) \), hence, \( T(r, H) = S_\lambda(r, f) \).

Since \( \deg(h_1 + h_2) \leq \rho - 1 \leq \lambda \), we have \( T(r, e^{h_1 + h_2}) = O(r^{\lambda + \epsilon}) \). Moreover, for \( \varphi := \beta_1 \beta_2 e^{h_1 + h_2} \), we have \( T(r, \varphi) = S_\lambda(r, f) \).

Consider now a simple zero, say \( z_0 \), of \( f \). At the same time, we may assume that \( b_0, \beta_1, \beta_2, \varphi \) as well as all coefficients of \( g_1(f), g_2(f) \) are non-zero and finite at \( z_0 \). Write now (6) in the form

\[
fg_1(f) = b_0 + \beta_1 e^{h_1}, \quad fg_2(f) = b_0 + \frac{\varphi}{\beta_1} e^{-h_1}.
\]

Thus, we obtain

\[
e^{h_1} = -\frac{b_0}{\beta_1} = -\frac{\varphi}{b_0 \beta_1}
\]

and so \( b_0^2 = \varphi \) at \( z_0 \). If \( b_0^2 - \varphi \) is not vanishing identically, we conclude that

\[
N_{11}(r, 1/f) \leq N \left( r, \frac{1}{b_0^2 - \varphi} \right) + S_\lambda(r, f) = S_\lambda(r, f).
\]

Hence, \( T(r, f) = S_\lambda(r, f) \), resulting in a contradiction. It remains to consider the case that \( b_0^2 \equiv \varphi \). We have

\[
b_0^2 = \beta_1 \beta_2 e^{h_1 + h_2} = (fg_1(f) - b_0)(fg_2(f) - b_0),
\]

resulting in

\[
fg_1(f)g_2(f) = b_0(g_1(f) + g_2(f)).
\]

But now, from

\[
g_1(f)g_2(f) = b_0 \frac{g_1(f) + g_2(f)}{f},
\]

we see that \( m(r, g_1(f)g_2(f)) = S_\lambda(r, f) \), hence, \( T(r, g_1(f)g_2(f)) = S_\lambda(r, f) \), as well. Denote now \( \psi := g_1(f)g_2(f) \). Making use of (10), we get

\[
b_0 + \beta_1 e^{h_1} = fg_1(f) = \frac{b_0}{\psi} g_1(f)(g_1(f) + g_2(f)) = \frac{b_0}{\psi} g_1(f)^2 + b_0.
\]

Thus we obtain

\[
g_1(f)^2 = \frac{\beta_1 \psi}{b_0} e^{h_1}.
\]

Similarly,

\[
g_2(f)^2 = \frac{\beta_2 \psi}{b_0} e^{h_2},
\]

and, further

\[
\left( \frac{g_1(f)}{g_2(f)} \right)^2 = \frac{\beta_1}{\beta_2} e^{h_1 - h_2}.
\]

Recalling the identity (9), we now proceed to considering

\[
A_1 g_2(f) - A_2 g_1(f) = H f.
\]
If $H$ vanishes identically, we see that
\[
\left( \frac{A_1}{A_2} \right)^2 = \left( \frac{g_1(f)}{g_2(f)} \right) = \frac{\beta_1}{\beta_2} e^{h_1 - h_2},
\]
hence $T(r, e^{h_1 - h_2}) = S_\lambda(r, f)$. Therefore, $\deg(h_1 - h_2) \leq \rho - 1$. Combining with $\deg(h_1 + h_2) \leq \rho - 1$, we obtain $\deg h_1 < \rho$, contradicting $\deg h_1 = \rho$.

Now, we need to consider (13), assuming that $H$ does not vanish identically. Squaring (13), and making use of (11) and (12), we get
\[
f^2 = A_1^2 \beta_2 \psi e^{h_2} + A_2^2 \frac{\beta_1 \psi}{b_0} e^{h_1} - \frac{2A_1 A_2}{H^2} f.
\]
Multiplying by $g_1(f)^2$, making use of (11) and (12) again, and recalling that $f^2 g_1(f)^2 = (b_0 + \beta_1 e^{h_1})^2$, an elementary computation results in
\[
\left( \beta_1^2 - \frac{A_2^2}{H^2} \left( \frac{\beta_1 \psi}{b_0} \right)^2 \right) e^{2h_1} + \left( 2b_0 \beta_1 + \frac{2A_1 A_2 \beta_1}{H^2 b_0} \psi \right) e^{h_1} + \left( b_0^2 - \frac{A_2^2}{H^2} \psi^2 \right) = 0.
\]
(14)

Then, we have $T(r, e^{h_1}) = S_\lambda(r, f)$, resulting in a contradiction $\deg h_1 < \rho$, provided not all coefficients in (14) are vanishing identically. Suppose finally that $\kappa$ where
\[
\kappa = \frac{H}{\beta_1^2}. \quad \text{Differentiating (15), we get}
\]
(16)

\[
(g_1(f))' = \kappa'(z) f + \kappa(z) f' - (g_2(f))'.
\]

Substituting (15) and (16) into (8) for $j = 1$, we obtain
\[
\left( \kappa' - \kappa \left( \frac{\beta_1'}{\beta_1} + h_1' \right) \right) f^2 + 2\kappa f' f + \left( \frac{\beta_1'}{\beta_1} + h_1' \right) f g_2(f) - (g_2(f))' f = A_1.
\]
(17)

By adding (17) to the equation (8) for $j = 2$ and keeping in mind $A_1 + A_2 \equiv 0$, we get
\[
\left( \frac{\beta_1'}{\beta_2} - \frac{\beta_1'}{\beta_1} + (h_2 - h_1)' \right) g_2(f) = \left( \kappa' - \kappa \left( \frac{\beta_1'}{\beta_1} + h_1' \right) \right) f + 2\kappa f'.
\]
(18)

It’s easy to see that
\[
\frac{\beta_1'}{\beta_2} - \frac{\beta_1'}{\beta_1} + (h_2 - h_1)' \neq 0 \quad \text{and} \quad \kappa' - \kappa \left( \frac{\beta_1'}{\beta_1} + h_1' \right) \neq 0.
\]
Otherwise, we get $\deg(h_1 - h_2) < \rho$ and $\deg h_1 < \rho$, respectively, which is a contradiction. Therefore, (18) can be rewritten as
\[
g_2(f) = L_2(z) f - M(z) f',
\]
(19)
where

\[ L_2 = -\kappa' - \kappa \left( \frac{\beta_1'}{\beta_1} + h'_1 \right) \quad \text{and} \quad M = \frac{2\kappa}{\beta_1} - \frac{\beta_2'}{\beta_2} + (h_1 - h_2)'. \]

Similarly, we have

\[ g_1(f) = L_1(z)f + M(z)f', \]

where

\[ L_1 = \frac{\kappa' - \kappa \left( \frac{\beta_1'}{\beta_1} + h'_1 \right)}{\frac{\beta_1}{\beta_1} - \frac{\beta_2'}{\beta_2} + (h_1 - h_2)'} . \]

This completes the proof of Lemma 4.4. □

5. Proofs of theorems of Section 2

The proof of Theorem 2.3 follows, to large extent, the corresponding proof of [8, Theorem 4.4].

Proof of Theorem 2.3. It suffices to show that the only cases which may occur, when \( \max\{\lambda(F_1), \lambda(F_2)\} < \rho \), are (i) and (ii).

Suppose that \( \max\{\lambda(F_1), \lambda(F_2)\} < \lambda \) for some \( \lambda < \rho \). Then

\[ f g_j(f) - b_0 = \beta_j e^{h_j}, \quad j = 1, 2, \]

where \( \beta_j \) are non-vanishing \( \lambda \)-small functions of \( f \) and \( h_1, h_2 \) are polynomials.

Therefore, from Lemma 4.4, we have \( \deg h_1 = \deg h_2 = \rho \).

Now, if \( \deg(h_1 + h_2) < \rho \), then, from Lemma 4.4, we obtain the exceptional case (i) in Theorem 2.3.

Next, we consider the case \( \deg(h_1 + h_2) = \rho \). In this case, we show that the exceptional case (ii) in Theorem 2.3 is the only possible one. To this end, we proceed as follows.

(a) Suppose that \( g_1(f) \) and \( g_2(f) \) both are \( \lambda \)-small functions of \( f \). Then from the second main theorem of Nevanlinna, we know that

\[ T(r, f) \leq N(r, f) + N \left( r, \frac{1}{f - \frac{b_0}{g_1(f)}} \right) + N \left( r, \frac{1}{f - \frac{b_0}{g_2(f)}} \right) + S(r, f) \]

\[ = N \left( \frac{1}{\beta_1} \right) + N \left( \frac{1}{\beta_2} \right) + S_{\lambda}(r, f) = S_{\lambda}(r, f), \]

which is impossible.

(b) Suppose that both of \( g_1(f) \) and \( g_2(f) \) are not \( \lambda \)-small functions of \( f \). First, we claim that \( \deg(h_1 - h_2) = \rho \). If this is not the case, that is, \( \deg(h_1 - h_2) < \rho \), then from (21), we have

\[ f \left( g_1(f) - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} g_2(f) \right) = b_0 \left( 1 - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} \right) . \]
By Lemma 4.1, we obtain
\[ m \left( r, g_1(f) - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} g_2(f) \right) = O(r^{\rho - 1 + \varepsilon}) + S_\lambda(r, f). \]

Without loss of generality, we may assume that \( \rho - 1 < \lambda < \rho \). Then
\[ T \left( r, g_1(f) - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} g_2(f) \right) = S_\lambda(r, f). \]

If \( g_1(f) - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} g_2(f) \equiv 0 \), then from (22), we get
\[ f \left( g_2(f) - g_1(f) \right) = \beta_2 e^{h_2} - \beta_1 e^{h_1} = 0, \]
which contradicts the assumption \( g_1(f) \not\equiv g_2(f) \). Thus \( g_1(f) - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} g_2(f) \not\equiv 0 \), and therefore, (22) and (23) yield
\[ T(r, f) = T \left( r, \frac{b_0 \left( 1 - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} \right)}{g_1(f) - \frac{\beta_1}{\beta_2} e^{h_1 - h_2} g_2(f)} \right) = S_\lambda(r, f), \]
which is a contradiction. Thus \( \deg(h_1 - h_2) = \rho \).

Second, recall that the function \( H \) defined in (9), i.e.,
\[ H := \frac{A_1(g_2(f)) - A_2(g_1(f))}{f} \]
is a \( \lambda \)-small function of \( f \), where \( A_1 \) and \( A_2 \) are non-vanishing \( \lambda \)-small functions of \( f \).

Suppose that \( H \) is vanishing identically. Then (21) implies that
\[ -\beta_1 A_2 e^{h_1} + \beta_2 A_1 e^{h_2} + (A_1 - A_2)b_0 = 0. \]
Since \( \deg(h_1 - h_2) = \rho \), we see from [14, Theorem 1.51] that all coefficients in this equation are vanishing identically, which gives a contradiction.

Suppose now that \( H \) is not vanishing identically. From here on, we follow the same reasoning of the proof of [8, Theorem 4.4] omitting most of the details. From (24), we have
\[ g_1(f) = \frac{A_1}{A_2} g_2(f) - \frac{H}{A_2} f. \]
Differentiating (25), substituting \( g_1(f) \) and \( (g_1(f))' \) into (8), and then adding the result to (8), in the case \( j = 2 \), multiplied by \(-A_1/A_2\), we obtain
\[ \left( \frac{B_1 H}{A_2} - \left( \frac{H}{A_2} \right)' \right) f + \left( \frac{2H}{A_2} \right)' f' - D g_2(f) = 0, \]
where
\[ D := \frac{B_1 A_1}{A_2} - \left( \frac{A_1}{A_2} \right)' - \frac{A_1 B_2}{A_2}. \]
and $B_j := \beta_j/\beta + h_j$, $j = 1, 2$. The coefficients here, denoted as $T_1D$ for $f$ and $T_2D$ for $f'$ (and $D$ for $g_2(f)$) are not vanishing identically, see [8, p. 818]. Hence, we may write (25) in the form

$$g_2(f) = T_1f + T_2f'$$

with $\lambda$-small coefficients of $f$. Differentiating now (26), substituting this expression and (26) into (8) with $j = 2$ results in

$$g_2(f) = T_1f + T_2f'$$

Differentiate (27). By a careful analysis of simple zeros of $f$ at this expression and at (27), we obtain

$$f'' = \frac{\tilde{H}}{3A_2T_2} f - \frac{2A_2T_1 + 2A_2T_2 - A_2' - A_2B_2T_2}{3A_2T_2} f',$nolabel$$

where

$$\tilde{H} := \frac{(2A_2T_1 + 2A_2T_2 - A_2' - A_2B_2T_2)f' + 3A_2T_2f''}{f}$$

is a $\lambda$-small function of $f$. To continue, substitute (28) into (27), implying

$$Q_1f^2 + Q_2f f' + T_2(f')^2 = A_2,$nolabel$$

where

$$Q_1 := T_1' - B_2T_1 + 3 \frac{\tilde{H}}{3A_2} \quad \text{and} \quad Q_2 := \frac{1}{3} \left( 4T_1' + T_2' - 2T_2B_2 + \frac{A_2'}{A_2} \right).$$

Here, in particular, $Q_2$ is not vanishing identically, as one may easily see. Differentiation of (30) now results in

$$Q_1'f^2 + (2Q_1 + Q_2')f f' + (Q_2 + T_2')(f')^2 + Q_2f f'' + 2T_2f' f''' = A_2'.$$

Analyzing simple zeros of $f$ at (27) and (31) we obtain

$$f''' = \frac{\tilde{\tilde{R}}}{2A_2T_2} f + \frac{A_2' - A_2(Q_2 + T_2')}{2A_2T_2} f',$nolabel$$

where

$$\tilde{\tilde{R}} := \frac{(-A_2' + A_2(Q_2 + T_2'))f' + 2A_2T_2f''}{f}$$

is a $\lambda$-small function of $f$. Substitute now (32) into (31) to obtain

$$\left( Q_1' + \frac{Q_2\tilde{\tilde{R}}}{2A_2T_2} \right) f^2 + \left( 2Q_1 + Q_2' - \frac{1}{2} \frac{Q_2}{T_2}(Q_2 + T_2') + \frac{1}{2} \frac{A_2'}{A_2} Q_2 + \frac{\tilde{\tilde{R}}}{A_2} \right) f f'$$

$$+ \frac{A_2'}{A_2} T_2(f')^2 = A_2.'$$

(33)
We are now approaching to the final reasoning for a contradiction. If (4)

\[ Q_1 + \frac{Q_2 R}{2A_2 T_2} - \frac{A'_2}{A_2} Q_1 \] 

(37) does not vanish identically, it is not difficult to conclude that $e^{h_1+h_2}$ is of order less than $\rho$, a contradiction with the assumption $\deg(h_1 + h_2) = \rho$.

Therefore, we must have $4Q_1 T_2 = Q_2^2$. Denoting $h_1(z) = \alpha z^\rho + \cdots$ and $h_2(z) = \beta z^\rho + \cdots$, we may repeat the reasoning in [8, pp. 821–822], to see that

\[
\lim_{|z| \to \infty} \frac{b_1 h_2'}{b_1 + h_2'} = \frac{\alpha \beta}{(\alpha + \beta)z^\rho} = \frac{2}{\rho}.
\]

Solving the equation $\frac{\alpha \beta}{(\alpha + \beta)z^\rho} = \frac{2}{\rho}$ results in either $\alpha = 2\beta$ or $\alpha = \frac{1}{2} \beta$. We proceed to considering the case $\alpha = 2\beta$. We may now write

\[ e^{h_1(z)} = e^{2\beta z^\rho} e^{P_1(z)}, \quad e^{h_2(z)} = e^{\beta z^\rho} e^{P_2(z)}, \]

where $P_1(z)$ and $P_2(z)$ are two polynomials of degree $\rho - 1$ at most. Recall that we have $g_2(f) = T_1 f + T_2 f'$ and

\[ g_1(f) = \left( \frac{A_1}{A_2} T_1 - \frac{H}{A_2} \right) f + \frac{A_1}{A_2} T_2 f'. \]

Therefore,

\[ b_0 + \beta e^{2\beta z^\rho} e^{P_2(z)} = f g_2(f) = T_1 f^2 + T_2 f' f \]

and

\[ b_0 + \beta_1 e^{2\beta z^\rho} e^{P_1(z)} = f g_1(f) = \left( \frac{A_1}{A_2} T_1 - \frac{H}{A_2} \right) f^2 + \frac{A_1}{A_2} T_2 f' f. \]

By a simple computation,

\[ g_2(f) = \frac{b_0}{f} + \beta e^{P_2 - \frac{1}{2} P_2} \left( \frac{1}{\beta_1} \left( \frac{A_1}{A_2} T_1 - \frac{H}{A_2} + \frac{A_1}{A_2} T_2 f' f - \frac{b_0}{f^2} \right) \right)^{1/2}. \]
We next show that \( g_2(f) \) is a small function by computing \( T(r, T_1f + T_2f') \). Indeed,

\[
T(r, T_1f + T_2f') = m(r, T_1f + T_2f') + S_\lambda(r, f) \\
= \frac{1}{2\pi} \int_{E_1} \log^+ |T_1f + T_2f'|d\theta + \frac{1}{2\pi} \int_{E_2} \log^+ |T_1f + T_2f'|d\theta \\
+ S_\lambda(r, f),
\]

where \( E_1, \text{ resp. } E_2 \), means the part on the circle of radius \( r \) such that \( |f| \leq 1 \), resp. \( |f| > 1 \). Again, we may repeat the reasoning in [8, pp. 823–824], to get

\[
\frac{1}{2\pi} \int_{E_1} \log^+ |T_1f + T_2f'|d\theta = S_\lambda(r, f) \text{ and } \frac{1}{2\pi} \int_{E_2} \log^+ |T_1f + T_2f'|d\theta = S_\lambda(r, f).
\]

Thus, we obtain

\[
T(r, g_2(f)) = T(r, T_1f + T_2f') = S_\lambda(r, f),
\]

which is a contradiction. Similarly, for the case \( \alpha/\beta = 1/2 \), we obtain \( T(r, g_1(f)) = S_\lambda(r, f), \) which is a contradiction too. This shows that the case, when \( g_1(f) \) and \( g_2(f) \) are not \( \lambda \)-small functions of \( f \), is not possible. Thus the case (ii) in Theorem 2.3 is the only possible case.

This completes the proof of Theorem 2.3. \( \square \)

**Proof of Theorem 2.4.** Suppose, contrary to the assertion, that \( \lambda(F) = \lambda < \rho \). Since \( N(r, f) = S_\lambda(r, f) \), we have \( N(r, F) = S_\lambda(r, f) \) as well. By the standard Hadamard representation, we may write

\[
f^n g(f)^s = b_0 + \beta e^h,
\]

where \( \beta \) is a non-vanishing \( \lambda \)-small function of \( f \) and \( h \) is a polynomial of degree \( \leq \rho \). Actually, \( \deg h = \rho \). Indeed, if \( \deg h \leq \mu < \rho \), then

\[
T(r, f^n g(f)^s) = O(r^{\mu+\varepsilon}) + S_\lambda(r, f),
\]

leading to \( \rho(f) \leq \max \{ \mu, \lambda \} < \rho \), a contradiction with Lemma 4.3. Differentiating now (38) and eliminating \( e^h \), we obtain

\[
f^n\frac{f'}{f} + s g(f)\frac{g'}{g} = \frac{\beta'}{\beta} - h' = \frac{A}{f^n(g(f))^s},
\]

where \( A := b'_0 - b_0 \frac{\beta'}{\beta} - b_0 h' \) cannot vanish identically as shown in [8, p. 811].

Since \( n \geq 2 \) and \( N(r, f) = S_\lambda(r, f) \), the equation (39) gives \( N(r, 1/f) = S_\lambda(r, f) \). If \( s \geq 2 \), we similarly observe that \( N(r, 1/g(f)) = S_\lambda(r, f) \). By the second main theorem,

\[
T(r, f^n g(f)^s) \leq N(r, f^n g(f)^s) + N(r, 1/f^n g(f)^s) + N(r, 1/F) + S(r, f) = S_\lambda(r, f),
\]

contradicting Lemma 4.3.
It remains to consider the case \( s = 1 \). Since \( f \) is a meromorphic function of finite order \( \rho \) such that \( \max \{ N(r, f), N(r, 1/f) \} = S_\lambda(r, f) \), \( f \) may be represented as \( f(z) = \gamma(z)e^{Q(z)} \), where \( T(r, \gamma) = S_\lambda(r, f) \) and \( Q \) is a polynomial of degree \( \deg Q = \rho \). Write now \( g(f)(z) = G(f)(z)e^{Q(z)} \), where

\[
T(r, G(f)) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f).
\]

Hence

\[
N \left( r, \frac{1}{g(f)} \right) = N \left( r, \frac{1}{G(f)} \right) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f).
\]

Recalling \( N(r, f) = S_\lambda(r, f) \), the second main theorem may be applied again to obtain

\[
T(r, f^n g(f)) \leq nN \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g(f)} \right) + N \left( r, \frac{1}{F} \right) + N(r, f^n g(f)) + S(r, f')
\]

\[
= O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f).
\]

Thus, \( \rho(f^n g(f)) \leq \max \{ \rho - 1, \lambda \} \), contradicting Lemma 4.3. \( \square \)

**Proof of Theorem 2.6.** Suppose, contrary to the assertion that \( \max \{ \lambda(F), \lambda(f) \} < \rho \).

We may write

\[
f(g(f))^s = b_0 + \beta e^h,
\]

where \( \beta \) is a non-vanishing \( \lambda \)-small function of \( f \) and \( h \) is a polynomial of degree \( \rho \). Differentiating (40) and eliminating \( e^h \), we obtain

\[
\frac{f'}{f} + s \frac{g(f)'}{g(f)} - \frac{\beta'}{\beta} - h' = \frac{b_0 \left( \frac{h'}{h} - \frac{a'}{a} - h' \right)}{f(g(f))^s}.
\]

Since \( s \geq 2 \), we conclude from (41) that

\[
N \left( r, \frac{1}{f(g(f))^s} \right) = S_\lambda(r, f).
\]

By this and (41), we conclude again that

\[
N \left( r, \frac{1}{f(g(f))^s} \right) \leq \mathcal{N} \left( r, \frac{1}{f} \right) + S_\lambda(r, f).
\]

By using the second main theorem of Nevanlinna, we get

\[
T(r, f(g(f))^s) \leq N(r, f(g(f))^s) + N \left( r, \frac{1}{f(g(f))^s} \right) + N \left( r, \frac{1}{F} \right) + S(r, f)
\]

\[
\leq \mathcal{N} \left( r, \frac{1}{f} \right) + S_\lambda(r, f).
\]

From (44) and Lemma 4.3, we obtain \( \rho = \rho(f(g(f))^s) \leq \lambda(f) < \rho \), a contradiction.
Suppose now that $\rho \notin \mathbb{N}$. Then, clearly, $\lambda(f) = \rho$. On the other hand, by Lemma 4.3 again, we have $\rho(F) = \rho \notin \mathbb{N}$. Hence $\lambda(F) = \rho$. \hfill \Box

Proof of Theorem 2.7. As one may clearly see, our reasoning here is to some part similar to the reasoning applied by Alotaibi in [1].

Suppose, contrary to the assertion that $\lambda(F) < \rho$. On the other hand, since $b_0$ is a non-vanishing $\lambda$-small function of $f$, we have $F = b_0F^*$, where

$$F^* := \frac{1}{b_0} f\tilde{g}(f)^s - 1.$$  

Clearly, $\lambda(F) = \lambda(F^*)$ and $\rho(F) = \rho(F^*) = \rho$. So, in the following, we consider only $F^*$. 

Put $b_{i,1} = b_{i,m+1} = 0$ and since $b_{i,m}(z) \equiv 1$ for every $0 \leq i \leq n$, we have

$$\tilde{g}(f) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j} f^{(j)}(z + c_i) = \sum_{i=0}^{n} \sum_{j=-1}^{m+1} b_{i,j} f^{(j)}(z + c_i).$$

Using the fact that $b_{i,m+1} = 0$, we have

$$\tilde{g}(f)' = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j}' f^{(j)}(z + c_i) + \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j} f^{(j+1)}(z + c_i)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m+1} (b_{i,j}' + b_{i,j-1}) f^{(j)}(z + c_i).\tag{45}$$

Since $\tilde{g}(f) = \sum_{i=0}^{n} \sum_{j=0}^{m+1} b_{i,j} f^{(j)}(z + c_i)$, we find that $w = f$ solves the delay-differential equation

$$\sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} w^{(j)}(z + c_i) = 0,\tag{46}$$

where

$$d_{i,j} = b_{i,j}' + b_{i,j-1} - \frac{\tilde{g}(f)'}{\tilde{g}(f)} b_{i,j}, \quad \text{and} \quad d_{0,m+1} = 1$$

for every $0 \leq i \leq n$, $0 \leq j \leq m + 1$.

Let $w = uv$, where $v = b_0/\tilde{g}(f)^s$. By using Leibniz' rule in (46) and the convention $C_j^k = 0$ for $k > j$, we get

$$\sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} (uv)^{(j)}(z + c_i) = \sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} \sum_{k=0}^{j} C_j^k u^{(k)}(z + c_i) v^{(j-k)}(z + c_i)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} \sum_{k=0}^{m+1} C_j^k u^{(k)}(z + c_i) v^{(j-k)}(z + c_i) = 0.$$
Dividing the right hand side by \( v(z) \) we get

\[
0 = \sum_{i=0}^{n} \sum_{k=0}^{m+1} u^{(k)}(z + c_i) \sum_{j=0}^{m+1} C_j^k d_{i,j} \frac{v^{(j-k)}(z + c_i)}{v(z)} = \sum_{i=0}^{n} \sum_{k=0}^{m+1} A_{i,k}(z) u^{(k)}(z + c_i),
\]

where, again since \( C_j^k = 0 \) for \( k > j \),

\[
A_{i,k}(z) = \sum_{j=k}^{m+1} C_j^k d_{i,j} \frac{v^{(j-k)}(z + c_i)}{v(z)}.
\]

In particular, this gives

\[
\sum_{i=0}^{n} A_{i,0} = \sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} \frac{v^{(j)}(z + c_i)}{v(z)} = \sum_{i=0}^{n} \sum_{j=0}^{m+1} \left( b'_{i,j} + b_{i,j-1} - \frac{\tilde{g}(f)'}{\tilde{g}(f)} b_{i,j} \right) \frac{v^{(j)}(z + c_i)}{v(z)} = \frac{\tilde{g}(v)'}{\tilde{g}(v)} \tilde{g}(v).
\]

We claim now that

\[
\sum_{i=0}^{n} A_{i,0} \neq 0.
\]

To prove this, we suppose the contrary. By using (48), we get

\[
\tilde{g}(v)' = \frac{\tilde{g}(f)'}{\tilde{g}(f)} \tilde{g}(v).
\]

We consider two cases:

**Case 1:** If \( \tilde{g}(v) \neq 0 \), then by simple integration of the above equation, we get

\[
\tilde{g}(v) = c \tilde{g}(f),
\]

where \( c \) is a non-zero constant. Defining \( H := v - cf \), linearity of \( \tilde{g} \) implies that \( \tilde{g}(H) = 0 \). By assumption, \( T(r, H) = S_{\lambda}(r, f) \). Further defining \( G := f + \frac{1}{c} H \), we see that

\[
v = cG \text{ and } \tilde{g}(f) = \tilde{g} \left( f + \frac{1}{c} H \right) = \tilde{g}(G).
\]

Therefore, \( T(r, G) = T(r, f) + S_{\lambda}(r, f) \). On the other hand, since \( v = b_0/\tilde{g}(f)^s \), we get

\[
1 = \frac{c}{b_0} G \tilde{g}(f)^s = \frac{c}{b_0} G \tilde{g}(G)^s.
\]

This leads to \( T(r, G \tilde{g}(G)^s) = S_{\lambda}(r, f) \), which is a contradiction with Lemma 4.3.

**Case 2:** If \( \tilde{g}(v) \equiv 0 \), then

\[
S_{\lambda}(r, f) = T(r, v) = sT(r, 1/\tilde{g}(f)) + S_{\lambda}(r, f)
\]
a contradiction with the condition \( T(r, \tilde{g}(f)) \neq S_\lambda(r, f) \).

Returning to our proof now, we have
\[
u = \frac{w}{v} = \frac{1}{b_0} f \hat{g}(f)^* = F^* + 1.
\]
So \( F^* + 1 \) solves the linear delay-differential equation
\[
\sum_{i=0}^{n} \sum_{k=0}^{m+1} A_{i,k}(z) u^{(k)}(z + c_i) = 0.
\]
Hence
\[
\sum_{i=0}^{n} \sum_{k=0}^{m+1} A_{i,k}(z) F^*(k)(z + c_i) = -\sum_{i=0}^{n} A_{i,0}(z).
\]
From (47) and (42) we deduce that
\[
T(r, A_{i,k}) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f).
\]
Since \( \lambda(F^*) < \rho \), we may write
\[
F^*(z) = \beta^*(z) e^{h(z)},
\]
where \( T(r, \beta^*) = S_\lambda(r, f) \), and \( h \) is a polynomial of degree equal to \( \rho \). Obviously
\[
F^*(k)(z + c_i) = \psi_k(z + c_i) e^{h(z+c_i)},
\]
where \( \psi_k \) \((k = 0, \ldots, m + 1)\) are differential polynomials in \( \beta \) and \( h \). By substituting (51) and (52) into (50) and since \( \sum_{i=0}^{n} A_{i,0} \neq 0 \), we get
\[
\sum_{i=0}^{n} \sum_{k=0}^{m+1} A_{i,k}(z) \psi_k(z + c_i) e^{h(z+c_i) - h(z)} = -e^{-h(z)}.
\]
Hence
\[
T(r, e^h) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f),
\]
resulting in a contradiction \( \deg h = \rho(f) \leq \max \{ \rho - 1, \lambda \} < \rho \). This completes the proof of Theorem 2.7. \( \Box \)

6. Proofs of results from Section 3

Proof of Theorem 3.1. (1) Suppose that \( \lambda(F) < \rho \). Writing \( F = f^n g(f) = \beta e^h \), where \( T(r, \beta) = S_\lambda(r, f) \) and \( h \) is a polynomial, we get
\[
n f' f + (g(f))' - \frac{g'}{\beta} = h'.
\]
If \( f \) vanishes at \( z_0 \), then \( F(z_0) = 0 \), unless \( g(f) \) has a pole at \( z_0 \). This may happen at the coefficients of \( g(f) \) and the poles of \( f(z_0 + c_j) \) only, contributing at most by \( S_\lambda(r, f) \). Therefore, \( \lambda(f) < \rho \), and the claim follows.

(2) In this case, we may write \( f(z) = \tau(z) e^{\alpha z^r} \), where \( \alpha \) \((\alpha \neq 0)\) is a constant and \( \tau \) is a \( \lambda \)-small function of \( f \). Then, of course, \( f(z + c_j) = \tau(z + c_j) \tau_j(z) e^{\alpha z^r} \),
where $\tau_j$ is a meromorphic function of order $\rho - 1$. Therefore, it is not difficult to see that $g(f) = T(z)e^{\alpha z^\rho}$, where $T(z)$ is a differential polynomial of $\tau$, of its shifts and of its derivatives. Therefore $\rho(T) \leq \lambda$, implying that $\lambda(F) < \rho$.

We divide the proof of Part (2)-(i) into three parts:

(a) If $\lambda(f) < \rho - 1$, then $\rho(F) = \rho$ by Lemma 4.3. Write now $f(z) = \tau(z)e^{h(z)}$, where $h$ is a polynomial of degree $\geq 2$, and $\tau$ is a $\lambda$-small function of $f$ where $\lambda < \rho - 1$. Recalling that

$$g(f) := \sum_{j=1}^{k} b_j(z)(f^{(k_j)}(z) + c_j)$$

with $k \geq 2$, we may write

$$g(f) = \sum_{j=1}^{k} \tau_j(z)f(z + c_j) := \sum_{j=1}^{k} b_j(z) \frac{f^{(k_j)}(z) + c_j}{f(z + c_j)} f(z + c_j).$$

Suppose first, contrary to the claim, that $\lambda(F) < \rho - 1$. Writing, as we may, $F(z) = \sigma(z)Q(z)$, where $\sigma$ is a $\lambda$-small function of $f$ where $\lambda < \rho - 1$ and $Q$ is a polynomial of degree $\rho$. Therefore, we now have

$$\sigma(z) = \sum_{j=1}^{k} \alpha_j(z)e^{\beta_j(z)},$$

where $\alpha_j(z) = \tau^n(z)\tau_j(z)\tau(z + c_j)$ are $\lambda$-small functions of $f$ with $\lambda < \rho - 1$ and $\beta_j(z) = nh(z) + h(z + c_j) - Q(z)$ for every $1 \leq j \leq k$. Set also,

$$h(z) = b_mz^m + b_{m-1}z^{m-1} + \cdots + b_0, b_m \neq 0,$$

where $b_m, b_{m-1}, \ldots, b_0$ are constants and $m = \rho \geq 2$. Hence, for every $i \neq j$

$$\beta_i(z) - \beta_j(z) = h(z + c_i) - h(z + c_j) = mb_m(c_i - c_j)z^{m-1} + \cdots.$$

If for all $j$, $\deg \beta_j(z) \geq \rho - 1$, then, by [14, Theorem 1.51], $\sigma(z) \equiv 0$ and $\alpha_j(z) \equiv 0$ for $j = 1, \ldots, k$. Hence $F$ vanishes, a contradiction. If this is not the case, then $\deg \beta_j(z) < \rho - 1$ for some $1 \leq j_0 \leq k$, which is a contradiction. Therefore,

$$\sigma(z) - \alpha_{j_0}(z)e^{\beta_{j_0}(z)} = \sum_{j=1, j \neq j_0}^{k} \alpha_j(z)e^{\beta_j(z)}.$$

Since $\deg(\beta_i - \beta_j) = \rho - 1$ for every $1 \leq i \neq j \leq k$, by [14, Theorem 1.51] again, $\alpha_j(z) \equiv 0$ for all $j \neq j_0$, and $\sigma(z)e^{-\beta_{j_0}(z)}(z) \equiv \alpha_{j_0}(z)$. This implies that $\tau_j(z)(z + c_j)$ vanishes for all $j \neq j_0$, hence $g(f)$ includes just one term, a contradiction.

We point out to the reader that it is impossible to have $\deg \beta_j < \rho - 1$ for all $1 < j < k$, since $\deg(\beta_i - \beta_j) = \rho - 1$ for every $1 \leq i \neq j \leq k$. 

(b) Suppose next that \( \lambda(F) > \rho - 1 \). We may write \( g(f) \) as

\[
g(f) = \left( \sum_{j=1}^{k} \tau_j(z) \tau(z + c_j) e^{h(z+\epsilon_j) - h(z)} \right) e^{h(z)}.
\]

Since \( \deg(h(z + c_j) - h(z)) = \rho - 1 \) and \( \tau_j(z) (1 \leq j \leq k) \), \( \tau(z) \) are \( \lambda \)-small functions of \( f \) with \( \lambda < \rho - 1 \), then \( \lambda(g(f)) \leq \rho - 1 \). This inequality and \( \lambda(f) < \rho - 1 \) implies a contradiction

\[
\rho - 1 < \lambda(F) = \lambda(f^n g(f)) \leq \rho - 1.
\]

(c) If finally \( \lambda(f) = \rho - 1 \), then, from (53), \( \lambda(F) \leq \rho - 1 \). On the other hand, we have

\[
nN \left( r, \frac{1}{f} \right) = N \left( r, \frac{g(f)}{F} \right) \leq N \left( r, \frac{1}{F} \right) + S_\lambda(r, f).
\]

By (54) and since \( \lambda < \rho - 1 \) we obtain \( \rho - 1 = \lambda(f) \leq \lambda(F) \). This completes the proof of Part (2)-(i).

To prove Part (2)-(ii), we may write \( f(z) = \tau(z) e^{h(z)} \), where \( h \) is a polynomial of degree \( \rho \geq 1 \) and \( \tau := \frac{\tau_1}{\tau_2} \) is a meromorphic function where \( \tau_1, \tau_2 \) are the canonical products of zeros and poles, respectively. Since \( \lambda(f) = \lambda^* \) \((\rho - 1 < \lambda^* < \rho) \) and \( \lambda < \lambda^* \), we have \( \rho(\tau_1) = \lambda^* \) and \( \rho(\tau_2) < \lambda^* \). This leads to \( \rho(\tau) = \lambda^* \). By this and (53) we deduce that \( \lambda(g(f)) \leq \lambda^* \), hence \( \lambda(F) \leq \lambda^* \).

On the other hand, from (54) and since \( \lambda < \lambda^* \), we deduce that \( \lambda(F) \leq \lambda^* \).

As to Part (2)-(iii), we may use the same reasoning as in (2) above to obtain that \( g(f) = T(z) e^{\alpha z} \) where \( \alpha \) is a constant and \( T \) is a meromorphic function of order 0. Therefore, \( \lambda(F) = 0 \).

\[ \square \]

Proof of Theorem 3.4. (i) Suppose that \( d \neq 0 \) is the Borel exceptional value of \( f(z) \) and

\[
\sum_{j=1, k_j = 0}^{k} b_j(z) \neq 0.
\]

Clearly, the order \( \rho \) is an integer and \( f(z) \) can be written in the form

\[
f(z) = d + \pi(z) e^{\alpha z^\rho},
\]

where \( \alpha \neq 0 \) is a constant and \( \pi(z) \) is a non-vanishing meromorphic function satisfying \( \rho(\pi) < \rho \). Thus

\[
f(z + c_j) = d + \pi(z + c_j) \pi_j(z) e^{\alpha z^\rho},
\]

where \( \pi_j \) is a meromorphic function of order \( \rho - 1 \). On the other hand, we may write \( g(f) \) as

\[
g(f) = \sum_{j \in I_1} b_j(z) f(z + c_j) + \sum_{j \in I_2} b_j(z) f^{(k_j)}(z + c_j),
\]
where $I_1 = \{1 \leq j \leq k : k_j = 0\}$ and $I_2 = \{1 \leq j \leq k : k_j > 0\}$. Hence, for every $j \in I_2$
\begin{equation}
(58)
 f^{(k_j)}(z + c_j) = Q_{k_j}(z)e^{az^p},
\end{equation}
where $Q_{k_j}$ is a meromorphic function of order less than $\rho$. By substituting (56) and (58) into (57), we get
\begin{equation}
F(z) = d^2 A(z) + d(B(z) + C(z) + \pi(z)A(z))e^{az^p}
+ \pi(z)(B(z) + C(z))e^{2az^p},
\end{equation}
where
\[A(z) := \sum_{j \in I_1} b_j(z) (\neq 0), \quad B(z) := \sum_{j \in I_1} \pi(z + c_j)\pi_j(z)b_j(z)\]
and
\[C(z) := \sum_{j \in I_2} b_j(z)Q_{k_j}(z).\]
By Lemma 4.3, we know that $\rho(F) = \rho$. If $F(z)$ has a Borel exceptional value $d^*$, then
\begin{equation}
(60)
F(z) = d^* + \pi^*(z)e^{\beta z^p},
\end{equation}
where $\beta (\neq 0)$ is a constant, and $\pi^*(z) (\neq 0)$ is a meromorphic function of order less than $\rho$. By (59) and (60), we have
\begin{equation}
(61)
d(B(z)+C(z)+\pi(z)A(z))e^{az^p} + \pi(z)(B(z)+C(z))e^{2az^p} - \pi^*(z)e^{\beta z^p} = d^* - d^2 A(z).
\end{equation}

**Case 1.** If $\beta \neq \alpha$ and $\beta \neq 2\alpha$, then by (61) and [14, Theorem 1.51], we get $\pi^*(z) \equiv 0$, which is a contradiction.

**Case 2.** If $\beta = \alpha$ and $\beta \neq 2\alpha$, then the equation (61) may written as
\[d(B(z)+C(z)+\pi(z)A(z))e^{az^p} + \pi(z)(B(z)+C(z))e^{2az^p} = d^* - d^2 A(z).\]
By this and [14, Theorem 1.51], we get $\pi^*(z) = -\frac{d^*}{d^2}f(z)$. Substituting this into (60) and combining the result with (55), we obtain
\[f(z)g(f(z)) = F(z) = d^* + \frac{d^*}{d\pi(z)}e^{az^p} = \frac{d^*}{d^2}f(z).\]
Thus, $g(f(z)) = -\frac{d^*}{d^2}$, contradicting the assumption that $g(f)$ is non-constant.

**Case 3.** If $\beta \neq \alpha$ and $\beta = 2\alpha$, then the equation (61) may written as
\[d(B(z)+C(z)+\pi(z)A(z))e^{az^p} + (\pi(z)(B(z)+C(z)) - \pi^*(z))e^{2az^p} = d^* - d^2 A(z).\]
By this and [14, Theorem 1.51], we get $\pi^*(z) = -\frac{d^*}{d^2}\pi^2(z)$. Substituting this into (60) and combining the result with (55) , we get
\[F(z) = d^* - \frac{d^*}{d^2}(f(z) - d)^2.\]
Clearly \( d^* \neq 0 \) as we would have \( g(f) \equiv 0 \) otherwise, which is a contradiction. Hence

\[
(62) \quad \frac{d^* - F(z)}{(d - f(z))^2} = \frac{d^*}{d^2}.
\]

We prove now the second identity, from (62) we get

\[
\sum_{j \in I_1} b_j(z) \frac{f(z + c_j) - d}{f(z) - d} + \sum_{j \in I_2} b_j(z) \frac{f(k_j z + c_j)}{f(z) - d} + \frac{d^*}{d^2} = \frac{d^*}{d^2} - d \sum_{j \in I_1} b_j(z).
\]

Suppose contrary to the claim that \( d^* - d^2 \sum_{j \in I_1} b_j(z) \neq 0 \), then from the above equation, we deduce

\[
m \left( r, \frac{1}{f - d} \right) = O(r^{\rho - 1 + \epsilon}) + S_\lambda(r, f).
\]

By this and since \( d \) is a Borel exceptional value of \( f \), we obtain

\[
T \left( r, \frac{1}{f - d} \right) = O(r^{\rho - 1 + \epsilon}) + S_\lambda(r, f),
\]

which is a contradiction.

(ii) Suppose that \( d \neq 0 \) and

\[
\sum_{j=1, k_j=0}^k b_j(z) \equiv 0.
\]

Suppose that \( F \) has a finite Borel exceptional value \( d^* \). By the same proof as in (i), we obtain (61) as

\[
d(B(z) + C(z))e^{\alpha z^p} + \pi(z)(B(z) + C(z))e^{2\alpha z^p} - \pi^*(z)e^{\beta z^p} = d^*.
\]

If \( \beta \neq \alpha \) and \( \beta \neq 2\alpha \), \( \beta = \alpha \) and \( \beta \neq 2\alpha \) or \( \beta \neq \alpha \) and \( \beta = 2\alpha \), then by using [14, Theorem 1.51], we get \( \pi^*(z) \equiv 0 \) in all three cases, which is a contradiction.

(iii) Suppose that \( d = 0 \) is the Borel exceptional value of \( f \). Using the same method as above, we obtain (59) with \( d = 0 \):

\[
F(z) = \pi(z)(B(z) + C(z))e^{2\alpha z^p}.
\]

Since \( \rho(F) = \rho \), then \( \pi(z)(B(z) + C(z)) \neq 0 \) and since \( \rho(\pi(B + C)) < \rho \), we deduce that \( d = 0 \) is a Borel exceptional value of \( f \). \( \square \)

**Proof of Theorem 3.8.** We shall prove this theorem by contradiction. Suppose contrary to our assertion that \( \lambda(F - a) < \rho \), then \( \rho \) is an integer \( \geq 1 \).

If first \( a = 0 \), applying the principle of contraposition on the part (1) of Theorem 3.1, we get \( \lambda_2(f) \leq \lambda(f) < \rho \), which is a contradiction.

Suppose next that \( a \neq 0 \), then \( F(z) \) can be written as the form

\[
F(z) = f^n(z)g(f)(z) - a = \tau(z)e^{Q(z)},
\]

where
where \( \tau(z) \) is a \( \lambda \)-small function of \( f \) and \( Q(z) \) is a polynomial of degree \( \rho \geq 1 \). Differentiating (63) and eliminating \( e^{Q(z)} \) yields

\[
\frac{F'(z)}{F(z)} = \frac{\tau'(z)}{\tau(z)} + Q'(z) \left(1 - \frac{a}{F(z)}\right).
\]

Clearly \( \frac{\tau'(z)}{\tau(z)} + Q'(z) \neq 0 \). Indeed, if not, then \( F'(z) \equiv 0 \) which contradicts the fact \( \rho(F) = \rho \). Since \( \lambda_2(f) = \rho \geq 1 \), then there exists a multiple zero \( z_0 \) of \( f \) that is not a pole of the coefficients of \( g(f) \) and such that \( \frac{\tau'(z_0)}{\tau(z_0)} + Q'(z_0) \neq 0 \).

From the above equation, we observe that \( z_0 \) is a simple pole of \( \frac{F'}{F} \) and a pole of multiplicity at least 2 of \( aF \), which is a contradiction. \( \square \)

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