2-absorbing $\delta$-semiprimary Ideals of Commutative Rings

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Abstract. Let $R$ be a commutative ring with nonzero identity, $\mathfrak{J}(R)$ the set of all ideals of $R$ and $\delta: \mathfrak{J}(R) \to \mathfrak{J}(R)$ an expansion of ideals of $R$. In this paper, we introduce the concept of 2-absorbing $\delta$-semiprimary ideals in commutative rings which is an extension of 2-absorbing ideals. A proper ideal $I$ of $R$ is called 2-absorbing $\delta$-semiprimary ideal if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$. Many properties and characterizations of 2-absorbing $\delta$-semiprimary ideals are obtained. Furthermore, 2-absorbing $\delta_1$-semiprimary avoidance theorem is proved.

1. Introduction

In this paper, all rings are commutative with nonzero identity. Let $I$ be a proper ideal of a ring $R$ and let $\mathfrak{J}(R)$ denote the set of all ideals of $R$. The radical of $I$ is defined by \( \{ a \in R : a^n \in I \text{ for some } n \in \mathbb{N} \} \), denoted by $\sqrt{I}$. Let $J$ be an ideal of $R$. Then the ideal $(I : J)$ consists of $r \in R$ with $rJ \subseteq I$, that is, $(I : J) = \{ r \in R : rJ \subseteq I \}$. For undefined notations and terminology refer to [10].

Various generalizations of prime and primary ideals are studied extensively in [1]-[3],[13],[14]. Recall from [4] and [5] that a proper ideal $I$ of $R$ is called a (weakly) 2-absorbing ideal if whenever $a, b, c \in R$ and $(0 \neq abc \in I)$ $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. A proper ideal $I$ of $R$ is called a (weakly) 2-absorbing primary ideal as in [6] and [7] if whenever $a, b, c \in R$ and $(0 \neq abc \in I)$ $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. As in the recent study [13], a proper ideal $I$ of $R$ is said to be a 2-absorbing quasi-primary if $\sqrt{I}$ is a 2-absorbing ideal; or equivalently, if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. D. Zhao [14] introduced the concept of expansions of ideals and extended many results of prime and primary ideals into the new concept. From [14], a function $\delta$ from $\mathfrak{J}(R)$ to $\mathfrak{J}(R)$ is an ideal expansion if it has the following properties: $I \subseteq \delta(I)$ and if
 Let \( \delta : \mathcal{I}(R) \to \mathcal{I}(R) \) be an expansion of ideals of a ring \( R \). We call a proper ideal \( I \) of \( R \) a (weakly) 2-absorbing \( \delta \)-semiprimary ideal if whenever \( a, b, c \in R \) and \( (0 \neq abc \in I) \) \( ab \in I \), then \( a \in \delta(I) \) or \( b \in \delta(I) \). It is shown (Theorem 2.24.) that if \( I \not\subseteq \sqrt{0} \), then the implication (2) is reversible.

It is shown (Theorem 2.24.) that if \( \delta(0) \) a 2-absorbing \( \delta \)-semiprimary ideal with \( \delta(\delta(0)) = \delta(0) \) and \( I \) is a weakly 2-absorbing \( \delta \)-semiprimary ideal, then either \( I \) is a 2-absorbing \( \delta \)-semiprimary ideal of \( R \) or \( I^2 \) is a 2-absorbing \( \delta \)-semiprimary ideal of \( R \). It is shown (Theorem 2.8.) that if \( I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n \) where \( I_1, I_2, \ldots, I_n \) \((n \geq 2)\) are ideals of \( R \) such that at most two of them are not 2-absorbing \( \delta \)-semiprimary, then \( \sqrt{I} \subseteq \sqrt{I_i} \) for some \( 1 \leq i \leq n \). From Theorem 2.30. to Theorem 2.34., we characterize 2-absorbing \( \delta \)-semiprimary ideals and weakly 2-absorbing \( \delta \)-semiprimary ideals of \( R = R_1 \times \cdots \times R_n \) where \( n \geq 2 \). Moreover, we state and prove 2-absorbing \( \delta_i \)-semiprimary avoidance theorem (Theorem 2.9.). In Section 3, we search an answer to the question that if \( I \) is a weakly 2-absorbing \( \delta \)-semiprimary ideal of \( R \) and if \( 0 \neq JKL \subseteq I \) for some ideals \( J, K, L \) of \( R \), then does it imply either \( JKL \subseteq \delta(I) \) or \( KL \subseteq \delta(I) \) or \( JL \subseteq \delta(I) \) ?

### 2. Properties of 2-absorbing \( \delta \)-semiprimary Ideals

**Definition 2.1.** Let \( \delta : \mathcal{I}(R) \to \mathcal{I}(R) \) be an expansion of ideals of \( R \) and \( I \) a proper ideal of \( R \).

1. We call \( I \) a 2-absorbing \( \delta \)-semiprimary ideal if whenever \( a, b, c \in R \) and \( abc \in I \), then either \( ab \in \delta(I) \) or \( bc \in \delta(I) \) or \( ac \in \delta(I) \).

2. We call \( I \) a weakly 2-absorbing \( \delta \)-semiprimary ideal if whenever \( a, b, c \in R \) and \( 0 \neq abc \in I \), then either \( ab \in \delta(I) \) or \( bc \in \delta(I) \) or \( ac \in \delta(I) \).

We start with trivial relations, hence we omit the proof.

**Theorem 2.2.** Let \( I \) be a proper ideal of \( R \). Then the following statements hold:

1. If \( I \) is a 2-absorbing \( \delta \)-semiprimary ideal, then \( I \) is a weakly 2-absorbing \( \delta \)-semiprimary ideal.
2. Let $I$ be a (weakly) $2$-absorbing $\delta_0$-semiprimary ideal if and only if $I$ is a (weakly) $2$-absorbing ideal.

3. Let $I$ be a (weakly) $2$-absorbing $\delta_1$-semiprimary ideal if and only if $I$ is a (weakly) $2$-absorbing quasi primary ideal.

4. If $I$ is a (weakly) $\delta$-semiprimary ideal, then $I$ is a (weakly) $2$-absorbing $\delta$-semiprimary ideal.

5. If $I$ is a (weakly) $2$-absorbing $\delta$-primary ideal, then $I$ is a (weakly) $2$-absorbing $\delta$-semiprimary ideal.

6. Let $\delta$ and $\gamma$ be two ideal expansions with $\delta(I) \subseteq \gamma(I)$. If $I$ is a (weakly) $2$-absorbing $\delta$-semiprimary ideal of $R$, then $I$ is a (weakly) $2$-absorbing $\gamma$-semiprimary ideal of $R$.

The following example shows that the concepts of weakly $2$-absorbing $\delta$-semiprimary and $2$-absorbing $\delta$-semiprimary are different.

**Example 2.3.** Let $R = \mathbb{Z}_{210}$ and $\delta : \mathcal{I}(\mathbb{R}) \rightarrow \mathcal{I}(\mathbb{R})$ an expansion of ideals of $R$ defined by $\delta(I) = (I : J)$ where $J = 7R$. Then $I = 0$ is a weakly $2$-absorbing $\delta$-semiprimary ideal of $R$ by definition. Observe that $\delta(I) = 30R$. However, $I$ is not $2$-absorbing $\delta$-semiprimary since $3 \cdot 5 \cdot 14 \in I$ but neither $3 \cdot 5 \in \delta(I)$ nor $3 \cdot 14 \in \delta(I)$ nor $5 \cdot 14 \in \delta(I)$.

For a $2$-absorbing $\delta$-semiprimary ideal which is not $2$-absorbing $\delta$-primary, see the next example:

**Example 2.4.** ([6, Example 2.9]) Let $R = \mathbb{Z}[X,Y,Z]$ and consider an ideal $I = (XYZ, X^3Y^4)R$ of $R$. Then $I$ is a $2$-absorbing $\delta_1$-semiprimary ideal of $R$ but $I$ is not a $2$-absorbing $\delta_1$-primary ideal of $R$ since $XYZ \in I$ but neither $XY \in I$ nor $YZ \in \delta_1(I)$ nor $XZ \in \delta_1(I)$ where $\delta_1(I) = (XY)R$.

**Theorem 2.5.** Let $\delta$ be an expansion function of $\mathcal{I}(\mathbb{R})$ and $I$ a proper ideal of $R$.

1. If $\delta(I)$ is a (weakly) $2$-absorbing ideal of $R$, then $I$ is a (weakly) $2$-absorbing $\delta$-semiprimary ideal of $R$.

2. Let $\delta(\delta(I)) = \delta(I)$. Then $\delta(I)$ is a (weakly) $2$-absorbing $\delta$-semiprimary ideal of $R$ if and only if $\delta(I)$ is a (weakly) $2$-absorbing ideal of $R$. Moreover, if $\delta(I)$ is $2$-absorbing $\delta$-semiprimary, then $|\text{Min}(\delta(I))| \leq 2$.

**Proof.** (1) Suppose that $0 \neq abc \in I$ and $ab \notin \delta(I)$. Since $I \subseteq \delta(I)$ and $\delta(I)$ is $2$-absorbing, we have $ac \in \delta(I)$ or $bc \in \delta(I)$. Thus $I$ is a (weakly) $2$-absorbing $\delta$-semiprimary ideal.

(2) Suppose that $\delta(I)$ is a (weakly) $2$-absorbing $\delta$-semiprimary ideal of $R$, since $\delta(\delta(I)) = \delta(I)$, $\delta(I)$ is (weakly) $2$-absorbing by the definition. The converse part is clear from (1). Suppose that $\delta(I)$ is a $2$-absorbing $\delta$-semiprimary ideal of $R$. Then $\delta(I)$ is $2$-absorbing; and so, $|\text{Min}(\delta(I))| \leq 2$ by [4, Theorem 2.3].

$\square$
The converse of Theorem 2.5. (1) is also true for \( \delta = \delta_1 \) by [13, Proposition 2.5].

**Theorem 2.6.** Let \( \delta \) be an expansion of \( J(\mathcal{R}) \) and \( I \) a 2-absorbing \( \delta \)-semiprimary ideal of \( R \). If \( \sqrt[\delta]{(I)} \subseteq \delta(\sqrt{T}) \), then \( \sqrt{T} \) is a 2-absorbing \( \delta \)-semiprimary ideal of \( R \). In particular, if \( \delta = \delta_1 \), then \( \sqrt{T} \) is a 2-absorbing ideal of \( R \).

**Proof.** Let \( a,b,c \in R \) with \( abc \in \sqrt{T} \) and \( ab \notin \delta(\sqrt{T}) \). Then \( a^nb^nc^n \in I \) for some positive integer \( n \geq 1 \). Since \( \sqrt[\delta]{(I)} \subseteq \delta(\sqrt{T}) \), we conclude \( a^nb^n \notin \delta(I) \). Since \( I \) is 2-absorbing \( \delta \)-semiprimary, we have \( b^nc^n \in \delta(I) \) or \( a^nc^n \in \delta(I) \). Since \( \sqrt[\delta]{(I)} \subseteq \delta(\sqrt{T}) \), we conclude that \( bc \in \delta(\sqrt{T}) \) or \( ac \in \delta(\sqrt{T}) \), we are done. The particular case is clear from [13, Proposition 2.5]. \( \square \)

The next example shows that the converse of Theorem 2.6. is not satisfied in general.

**Example 2.7.** Consider the ideal \( I = (X^3)/(X^4) \) of \( R = \mathbb{Z}_2[X]/(X^4) \). Then \( \sqrt{T} = (6,X)/(X^4) \) is a 2-absorbing ideal (2-absorbing \( \delta_0 \)-semiprimary ideal). However, \( I \) is not a 2-absorbing \( \delta_0 \)-semiprimary ideal since \( 0 \neq (X+X^3)(X+X^3)(X+X^3) \in I \) but \( X^2 + X^4 \notin \delta_0(I) = I \).

**Theorem 2.8.** Let \( I_1, I_2, \ldots, I_n \) \((n \geq 2)\) be ideals of \( R \) such that at most two of them are not 2-absorbing \( \delta_1 \)-semiprimary. If \( I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n \), then \( \sqrt{T} \subseteq \sqrt{T_I} \) for some \( 1 \leq i \leq n \).

**Proof.** From the covering \( I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n \), we conclude that \( \sqrt{T} \subseteq \sqrt{T_{I_1}} \cup \sqrt{T_2} \cup \cdots \cup \sqrt{T_{I_n}} \). By our hypothesis, we may assume that \( I_k \) is a 2-absorbing \( \delta_1 \)-semiprimary ideal for all \( k \geq 3 \). Hence \( \sqrt{T_k} \) is 2-absorbing for all \( k \geq 3 \) by Theorem 2.6. Thus \( \sqrt{T} = \sqrt[\delta]{\sqrt{T_I}} \subseteq \sqrt[\delta]{\sqrt{T_k}} = \sqrt{T_k} \) for some \( 1 \leq i \leq n \) by [12, Theorem 3.1]. \( \square \)

Let \( I, I_1, I_2, \ldots, I_n \) be ideals of \( R \). Recall that an efficient covering of \( R \) is a covering \( I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n \) in which no \( I_j \) where \( 1 \leq j \leq n \) satisfies \( I \subseteq I_j \) (i.e., no \( I_j \) is superfluous.) In the following result, we obtain 2-absorbing \( \delta_1 \)-semiprimary avoidance theorem.

**Theorem 2.9.** (2-absorbing \( \delta_1 \)-semiprimary avoidance theorem) Let \( I_1, I_2, \ldots, I_n \) \((n \geq 2)\) be ideals of \( R \) such that at most two of them not 2-absorbing \( \delta_1 \)-semiprimary. Suppose that \( \sqrt{T_i} \subseteq \sqrt{T_j} : x \) for all \( x \in R \setminus \sqrt{T_j} \) for all \( i \neq j \). If \( I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n \), then \( I \subseteq I_i \) for some \( 1 \leq i \leq n \).

**Proof.** Assume on the contrary that \( I \nsubseteq I_i \) for all \( 1 \leq i \leq n \). Hence \( I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n \) is an efficient covering of ideals of \( R \). So, it is clear that \( I = (I \cap I_1) \cup (I \cap I_2) \cup \cdots \cup (I \cap I_n) \) is an efficient union. From [11, Lemma 2.1], \( \bigcap_{i \neq k} (I \cap I_i) \subseteq I \cap I_k \). Since at most two of the \( I_i \) are not \( 2 \)-absorbing \( \delta_1 \)-semiprimary ideals, we may assume that \( I_k \) is a 2-absorbing \( \delta_1 \)-semiprimary ideal. From Theorem 2.6., we conclude that \( \sqrt{T_k} \) is 2-absorbing; and so \( \sqrt{T_k} : x \) is a prime ideal of \( R \) for all \( x \notin \sqrt{T_k} \) by [4, Theorem 2.5.]. On the other hand, our hypothesis implies that there exist \( a_i \in \sqrt{T_k} \setminus (\sqrt{T_k} : x) \)
for all $i \neq k$. Hence there are positive integers $m_i$ such that $a_i^{m_i} \in I_i$. Put $a = \prod_{i \neq k} a_i$ and $m = \max\{m_1, m_2, ..., m_n\}$. It is clear that $a^m x \in \bigcap_{i \neq k} (I_i \cap I_k)$. We show that $a^m x \not\in I \cap I_k$. Assume that $a^m x \in I \cap I_k$. Hence $a^m x \in \sqrt{I_k}$, and so $a^m \in (\sqrt{I_k} : x)$. Since $(\sqrt{I_k} : x)$ is prime, we conclude that $a_i \in (\sqrt{I_k} : x)$ for some $i \neq k$, a contradiction. Thus $\left(\bigcap_{i \neq k} (I_i \cap I_k)\right) \setminus (I \cap I_k)$ is nonempty which contradicts with $\bigcap_{i \neq k} (I_i \cap I_k) \subseteq I \cap I_k$. Thus $I \subseteq I_k$ for some $1 \leq i \leq n$. 

Recall from [14] that an ideal expansion $\delta$ of $\mathcal{I}(R)$ is said to be intersection preserving if it satisfies $\delta(I_1 \cap I_2 \cap \cdots \cap I_n) = \delta(I_1) \cap \delta(I_2) \cap \cdots \cap \delta(I_n)$ for any ideals $I_1, I_2, ..., I_n$ of $R$.

**Theorem 2.10.** Let $\delta$ be an intersection preserving expansion function of $\mathcal{I}(R)$. If $I_1, I_2, ..., I_n$ are 2-absorbing $\delta$-semiprimary ideals of $R$ with $\delta(I_i) = K$ for all $i \in \{1, 2, ..., n\}$, then $I = \bigcap_{i=1}^{n} I_i$ is a 2-absorbing $\delta$-semiprimary ideal of $R$.

**Proof.** Suppose that $abc \in I$, $ab \notin \delta(I)$ and $bc \notin \delta(I)$ for some $a, b, c \in R$. Since $\delta(I) = \delta(\bigcap_{i=1}^{n} I_i) = \bigcap_{i=1}^{n} \delta(I_i) = K$, we have $ab \notin K$ and $bc \notin K$. Since $abc \in I_i$ and $I_i$ is 2-absorbing $\delta$-semiprimary, $ac \in \delta(I_i) = K = \delta(I)$; so we are done.

**Theorem 2.11.** Let $\delta$ be an expansion function of $\mathcal{I}(R)$ and $I$ a (weakly) 2-absorbing $\delta$-semiprimary ideal of $R$. Then the following hold:

1. If $J \subseteq I$ and $\delta(J) = \delta(I)$, then $J$ is a (weakly) 2-absorbing $\delta$-semiprimary ideal of $R$.

2. If $J \subseteq I \subseteq \sqrt{\delta}$, then $J$ is a (weakly) 2-absorbing quasi-primary ideal of $R$.

3. Let $I \subseteq \sqrt{\delta}$. If $K$ is an ideal of $R$, then $IK, I \cap K$ and $I^n$ are (weakly) 2-absorbing quasi-primary ideals of $R$ for all positive integers $n \geq 1$.

**Proof.** (1) Suppose that $(0 \neq abc \in J)$ $abc \in J$ for some $a, b, c \in R$. Since $J \subseteq I$, we have $(0 \neq abc \in I)$ $abc \in I$. Since $I$ is (weakly) 2-absorbing $\delta$-semiprimary, we have either $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$. Since $\delta(I) = \delta(J)$, $J$ is a (weakly) 2-absorbing $\delta$-semiprimary ideal of $R$.

(2) Since $J \subseteq I \subseteq \sqrt{\delta}$, we have $\delta(J) = \sqrt{J} = \sqrt{\delta} = \delta(I) = \sqrt{\delta}$. Thus the result is clear from (1).

(3) is a particular result of (2).

**Theorem 2.12.** Let $\delta$ be an expansion of $\mathcal{I}(R)$. Every proper principal ideal is a 2-absorbing $\delta$-semiprimary ideal of $R$ if and only if every proper ideal is a 2-absorbing $\delta$-semiprimary ideal of $R$.

**Proof.** Suppose that every proper principal ideal is a 2-absorbing $\delta$-semiprimary ideal of $R$. Let $I$ be a proper ideal of $R$ and $a, b, c \in R$ with $abc \in I$. Then $abc \in (abc)$
and since \((abc)\) is 2-absorbing \(\delta\)-semiprimary ideal of \(R\) by our assumption, we have either \(ab \in \delta(abc) \subseteq \delta(I)\) or \(bc \in \delta(abc) \subseteq \delta(I)\) or \(ac \in \delta(abc) \subseteq \delta(I)\). Thus \(I\) is a 2-absorbing \(\delta\)-semiprimary ideal of \(R\). The converse part is obvious. 

**Theorem 2.13.** Let \(\delta\) be an ideal expansion such that \(\delta(I) \subseteq \sqrt{\mathcal{T}}\) and \(\delta(I)\) a semiprime ideal of \(R\) for every ideal \(I\). If \(I\) is a 2-absorbing \(\delta\)-semiprimary ideal, then \(\delta(I) = \sqrt{\mathcal{T}}\).

**Proof.** Suppose that \(a \in \sqrt{\mathcal{T}}\). Then there exists \(n\) which is the least positive integer \(n\) with \(a^n \in I\). If \(n = 1\), then \(a \in I \subseteq \delta(I)\). For \(n \geq 2\), \(a^n = a^{n-2}aa \in I\). Since \(a^{n-1} \notin I\), we have \(a^2 \in \delta(I)\). Since \(\delta(I)\) is semiprime, \(a \in \delta(I)\). Thus \(\delta(I) = \sqrt{\mathcal{T}}\). □

Recall that a ring \(R\) is said to be a Boolean ring if \(x^2 = x\) for every \(x \in R\). Since \(\sqrt{\mathcal{T}} = I\) for every proper ideal \(I\) of \(R\), we have the following result.

**Theorem 2.14.** Let \(R\) be a Boolean ring and \(I\) a proper ideal of \(R\). Then the following are equivalent:

1. \(I\) is a (weakly) 2-absorbing quasi-primary ideal of \(R\).
2. \(I\) is a (weakly) 2-absorbing primary ideal of \(R\).
3. \(I\) is a (weakly) 2-absorbing ideal of \(R\).

**Proof.** Since \(\sqrt{\mathcal{T}} = I\), the claim is clear. □

**Definition 2.15.** Let \(\delta\) be an expansion function of \(\mathcal{J}(R)\) and \(I\) a weakly 2-absorbing \(\delta\)-semiprimary ideal of \(R\). We call a triple \((a, b, c)\) a \(\delta\)-triple zero of \(I\) if \(abc = 0\) for some elements \(a, b, c\) of \(R\) (not necessarily distinct) and neither \(ab \in \delta(I)\) nor \(bc \in \delta(I)\) nor \(ac \in \delta(I)\).

**Remark 2.16.** Let \(\delta\) be an expansion function of \(\mathcal{J}(R)\) and \(I\) a weakly 2-absorbing \(\delta\)-semiprimary ideal of \(R\). Then \(I\) is a not 2-absorbing \(\delta\)-semiprimary ideal of \(R\) if and only if there exists at least one \(\delta\)-triple zero of \(I\).

**Theorem 2.17.** Let \(\delta\) be an expansion function of \(\mathcal{J}(R)\) and \(I\) a weakly 2-absorbing \(\delta\)-semiprimary ideal of \(R\). If \(I\) is not a 2-absorbing \(\delta\)-semiprimary ideal of \(R\), then \(I^3 = 0\); that is \(I \subseteq \sqrt{0}\).

**Proof.** Suppose that \(I\) is a weakly 2-absorbing \(\delta\)-semiprimary ideal of \(R\) which is not 2-absorbing \(\delta\)-semiprimary. Hence there exists a \(\delta\)-triple zero \((a, b, c)\) of \(I\) by Remark 2.16. Then \(I\) and neither \(ab \in \delta(I)\) nor \(bc \in \delta(I)\) nor \(ac \in \delta(I)\). For some \(b, c \in R\). First, we show that \(abI = 0\). Assume that \(abi \neq 0\) for some \(i \in I\). Since \(0 \neq ab(c + i) \in I\) and \(I\) is a weakly 2-absorbing \(\delta\)-semiprimary, we conclude \(ab \in I\) or \(a(c + i) \in I\) or \(b(c + i) \in I\), a contradiction. Similarly, it is easy to show that \(bcI = acI = 0\). Now, we show that \(aI^2 = 0\). Assume that \(ai_1i_2 \neq 0\) for some \(i_1, i_2 \in I\). Since \(abi = bcI = acI = 0\), we have \(0 \neq a(b + i_1)(c + i_2) = ai_1i_2 \in I\). Since \(I\) is a weakly 2-absorbing \(\delta\)-semiprimary, this contradicts our assumption that \(ab \in \delta(I)\) nor \(bc \in \delta(I)\) nor \(ac \in \delta(I)\). Thus \(aI^2 = 0\). One can easily show
symmetrically that $bI^2 = cI^2 = 0.$ Lastly, we show that $I^3 = 0.$ Assume that $i_1i_2i_3 \neq 0$ for some $i_1, i_2, i_3 \in I$. Since $abI = bcI = acI = aI^2 = bI^2 = cI^2 = 0,$ observe that $0 \neq (a + i_1)(b + i_2)(c + i_2) = i_1i_2i_3 \in I.$ Since $I$ is a weakly $2$-absorbing $δ$-semiprimary, again we conclude a contradiction. Thus $I^3 = 0.$

The following example shows that for an ideal $I$ of $R$ with $I^3 = 0$, $I$ needs not to be a weakly $2$-absorbing $δ$-semiprimary ideal.

**Example 2.18.** Let $R = \mathbb{Z}_{60}$ and $I = 30R$. Then $I^3 = 0$. However, $I$ is not a weakly $2$-absorbing $δ$-semiprimary ideal (for $δ = δ_0$ or $δ = δ_1$) since $2 \cdot 3 \cdot 5 \in I$ but neither $2 \cdot 3 \in δ(I)$ nor $3 \cdot 5 \in δ(I)$ nor $2 \cdot 5 \in δ(I)$.

As a conclusion of Theorem 2.17, we have the following two results.

**Corollary 2.19.** Let $R$ be a reduced ring. Then every nonzero weakly $2$-absorbing $δ$-semiprimary ideal of $R$ is a $2$-absorbing $δ$-semiprimary ideal of $R$.

**Corollary 2.20.** Let $M$ be a finitely generated $R$-module. Let $I$ be a weakly $2$-absorbing $δ$-semiprimary ideal of $R$ that is not $2$-absorbing $δ$-semiprimary. If $IM = M$, then $M = 0$.

**Theorem 2.21** Let $δ$ be an expansion function of $3(\mathcal{R})$ and $I$ a weakly $2$-absorbing $δ$-semiprimary ideal of $R$ with $δ(I) = δ(0)$. Then $I$ is not $2$-absorbing $δ$-semiprimary ideal of $R$ if and only if there exists a $δ$-triangular zero of $0$.

**Proof.** Suppose that $I$ is not a $2$-absorbing $δ$-semiprimary ideal. Hence $abc = 0$ but $ab \notin δ(I)$, $ac \notin δ(0)$ and $bc \notin δ(I)$ for some $a, b, c \in R$. Since $δ(I) = δ(0)$, $(a, b, c)$ is a $δ$-triangular zero of $0$. The converse part is obvious.

In particular, suppose that $I \subseteq √0$ is a weakly $2$-absorbing quasi primary ideal of $R$. A consequence of Theorem 2.21, there exists a $δ$-triangular zero of $0$ if and only if $I$ is not a $2$-absorbing quasi primary ideal of $R$. The following example shows that if $δ(I) \neq δ(0)$ and there exists a $δ$-triangular zero of $0$, then $I$ may be a $2$-absorbing $δ$-semiprimary ideal of $R$.

**Example 2.22.** Consider $R = \mathbb{Z}_{24}$, $δ : 3(\mathcal{R}) \rightarrow 3(\mathcal{R})$ is defined by $δ(I) = δ_1(I)$ for every nonzero proper ideal $I$ of $R$, and $δ(0) = δ_0(0) = 0$. Let $I = 12R$. Then $δ(I) = 6R$ is $2$-absorbing by [4, Theorem 3.15], $I$ is a $2$-absorbing $δ$-semiprimary ideal by Theorem 2.5. Since $2 \cdot 3 \cdot 4 = 0$ but neither $2 \cdot 3 \in δ(0)$ nor $2 \cdot 4 \in δ(0)$ nor $3 \cdot 4 \in δ(0)$, we conclude that $(2, 3, 4)$ is a $δ$-triangular zero of $0$.

**Theorem 2.23.** $δ$ be an intersection preserving expansion function of $3(\mathcal{R})$. If $I_1, I_2, ..., I_n$ are weakly $2$-absorbing quasi-primary (weakly $2$-absorbing $δ_1$-semiprimary) ideals of $R$ that are not $2$-absorbing quasi-primary, then $I = \bigcap_{i=1}^{n} I_i$ is a weakly $2$-absorbing quasi-primary ideal of $R$.

**Proof.** Since each $I_i$ is a $2$-absorbing quasi-primary ideal of $R$ that is not $2$-absorbing quasi-primary, we have $δ(I_i) = √{I_i} = √0$ by Theorem 2.17. Thus, remain of the proof is easily concluded similar to the proof of Theorem 2.10.
Theorem 2.24. Let $\delta$ be an expansion function of $\mathcal{J}(R)$ and $\delta(0)$ a 2-absorbing $\delta$-semiprimary ideal such that $\delta(\delta(0)) = \delta(0)$. Suppose that $I$ is a weakly 2-absorbing $\delta$-semiprimary ideal. Then $I$ is a 2-absorbing $\delta$-semiprimary ideal of $R$ or $I^2$ is a 2-absorbing $\delta$-semiprimary ideal of $R$.

Proof. Suppose that $I$ is a weakly 2-absorbing $\delta$-semiprimary ideal that is not 2-absorbing $\delta$-semiprimary. Hence $I^3 \subseteq 0 \subseteq \delta(0)$ by Theorem 2.17. Since $\delta(0)$ is a 2-absorbing ideal of $R$ by Theorem , we conclude that $I^2 \subseteq \delta(0)$ by [4, Theorem 2.13]. Since $0 \subseteq I^2 \subseteq \delta(0)$ and $\delta(\delta(0)) = \delta(0)$, we have $\delta(I^2) = \delta(0)$. Since $\delta(I^2)$ is a 2-absorbing ideal of $R$, $I^2$ is a 2-absorbing $\delta$-semiprimary ideal of $R$ by Theorem 2.5. □

Let $R$ and $S$ be commutative rings with $1 \neq 0$, and let $\delta, \gamma$ be two expansion functions of $\mathcal{J}(R)$ and $\mathcal{J}(S)$, respectively. Then a ring homomorphism $f : R \to S$ is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for all ideals $I$ of $S$. For example, if $\gamma_1$ is a radical operation on ideals of $S$ and $\delta_1$ is a radical operation on ideals of $R$. Then every homomorphism from $R$ to $S$ is a $\delta\gamma_1$-homomorphism. Additionally, if $f$ is a $\delta\gamma$-epimorphism and $I$ is an ideal of $R$ containing $\ker(f)$, then $\gamma(f(I)) = f(\delta(I))$ [3]

Theorem 2.25. Let $f : R \to S$ be a $\delta\gamma$-homomorphism, where $\delta$ and $\gamma$ are expansion functions of $\mathcal{J}(R)$ and $\mathcal{J}(S)$, respectively. Then the following statements hold:

1. If $J$ is a 2-absorbing $\gamma$-semiprimary ideal of $S$, then $f^{-1}(J)$ is a 2-absorbing $\delta$-semiprimary ideal of $R$.

2. If $J$ is a weakly 2-absorbing $\gamma$-semiprimary ideal of $S$, and $\ker(f)$ is a weakly 2-absorbing $\delta$-semiprimary ideal of $R$, then $f^{-1}(J)$ is a weakly 2-absorbing $\delta$-semiprimary ideal of $R$.

3. Let $f$ be an epimorphism and $I$ a proper ideal of $R$ with $\ker(f) \subseteq I$. Then $I$ is (weakly) 2-absorbing $\delta$-semiprimary ideal of $R$ if and only if $f(I)$ is a (weakly) 2-absorbing $\gamma$-semiprimary ideal of $S$.

Proof. (1) Let $abc \in f^{-1}(J)$ for some $a, b, c \in R$. Then $f(abc) = f(a)f(b)f(c) \in J$, which implies $f(a)f(b) = f(ab) \in \gamma(J)$ or $f(b)f(c) = f(bc) \in \gamma(J)$ or $f(a)f(c) = f(bc) \in \gamma(J)$. Thus we have $ab \in f^{-1}(\gamma(J))$ or $bc \in f^{-1}(\gamma(J))$ or $ac \in f^{-1}(\gamma(J))$. Since $f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$, $f^{-1}(J)$ is a 2-absorbing $\delta$-semiprimary ideal of $R$.

(2) Let $0 \neq abc \in f^{-1}(J)$ for some $a, b, c \in R$. Then $f(abc) = f(a)f(b)f(c) \in J$. If $f(abc) \neq 0$, it can be easily proved similar to (1) that $f^{-1}(J)$ is a weakly 2-absorbing $\delta$-semiprimary ideal of $R$. Assume that $f(abc) = 0$. Hence $abc \in \ker(f)$. Since $\ker(f)$ is weakly 2-absorbing $\delta$-semiprimary, we have $ab \in \delta(\ker(f))$ or $bc \in \delta(\ker(f))$ or $ac \in \delta(\ker(f))$. Since $\delta(\ker(f)) = \delta(f^{-1}(0)) \subseteq \delta(f^{-1}(J))$, we are done.

(3) Let $0 \neq xyz \in f(I)$ for some $x, y, z \in S$. Then there are some elements $a, b, c \in I$ such that $x = f(a), y = f(b)$ and $z = f(c)$. Then $f(a)f(b)f(c) = f(abc) \in f(I)$ and since $\ker(f) \subseteq I$, we conclude $0 \neq abc \in I$. Since $I$ is (weakly) 2-absorbing $\delta$-semiprimary, we have either $ab \in \delta(I)$ or $bc \in \delta(I)$.
or $ac \in \delta(I)$. Thus $xy \in f(\delta(I))$ or $yz \in f(\delta(I))$ or $yz \in f(\delta(I))$. Since $f(\delta(I)) = \delta(f(I))$, we are done. \hfill \Box

**Remark 2.26.** Let $\delta$ be an expansion function of $\mathfrak{J}(R)$ and $I$ a proper ideal of $R$. Then the function $\delta_q : R/I \to R/I$ defined by $\delta_q(J/I) = \delta(J)/I$ for all ideals $I \subseteq J$, becomes an expansion function of $R/I$ [3]. Consider the natural homomorphism $\pi : R \to R/J$. Then for ideals $I$ of $R$ with $\ker(\pi) \subseteq I$, we have $\delta_q(\pi(I)) = \pi(\delta(I))$.

From Theorem 2.25. and Remark 2.26., we have the following result.

**Corollary 2.27.** Let $\delta$ be an expansion function of $\mathfrak{J}(R)$.

1. Let $I$ and $J$ be ideals of $R$ with $I \subseteq J$. Then $J$ is a 2-absorbing $\delta_q$-primary ideal of $R$ if and only if $J/I$ is a 2-absorbing $\delta_q$-primary ideal of $R/I$.

2. If $I$ is a 2-absorbing $\delta_q$-primary ideal of $R$ and $R'$ is a subring with $R' \not\subseteq I$, then $I \cap R'$ is a 2-absorbing $\delta_q$-primary ideal of $R'$.

Let $\delta$ be an expansion function of ideals of a polynomial ring $R[X]$ where $X$ is an indeterminate. Observe that the function as in Remark 2.26., $\delta_q : R[X]/(X) \to R[X]/(X)$ defined by $\delta_q(J/(X)) = \delta(J)/(X)$ for all ideals $J$ of $R[X]$ with $(X) \subseteq J$, is an expansion function of ideals of $R$ as $R[X]/(X) \cong R$. According to these expansions, we have the following equivalent situations:

**Theorem 2.28.** Let $\delta$ be an expansion function of $\mathfrak{J}(R)$ and $I$ a proper ideal of $R$. Then the following are equivalent:

1. $I$ is a 2-absorbing $\delta_q$-primary ideal of $R$.

2. $(I, X)$ is a 2-absorbing $\delta_q$-primary ideal of $R[X]$.

**Proof.** From Corollary 2.27., we conclude that $(I, X)$ is a 2-absorbing $\delta_q$-primary ideal of $R[X]$ if and only if $(I, X)/(X)$ is a 2-absorbing $\delta_q$-primary ideal of $R[X]/(X)$. Since $(I, X)/(X) \cong I$ and $R[X]/(X) \cong R$, the result is obtained. \hfill \Box

Let $S$ be a multiplicatively closed subset of a ring $R$ and let $\delta$ be an expansion function of $\mathfrak{J}(R)$. Note that $\delta_S$ is an expansion function of $\mathfrak{J}(R_S)$ such that $\delta_S(I_S) = (\delta(I))_S$. In the next theorem, we investigate 2-absorbing $\delta_S$-primary ideals of the localization $R_S$.

**Theorem 2.29.** Let $\delta$ be an expansion function of $\mathfrak{J}(R)$ and $S$ a multiplicatively closed subset of $R$. If $I$ is a (weakly) 2-absorbing $\delta_q$-primary ideal of $R$ with $I \cap S = \emptyset$, then $I_S$ is a (weakly) 2-absorbing $\delta_S$-primary ideal of $R_S$.

**Proof.** Let $(0 \neq \frac{xy}{s_1s_2s_3} \in I_S) \frac{xy}{s_1s_2s_3} \in I_S$ for some $x, y, z \in R; s_1, s_2, s_3 \in S$. Then we have $(0 \neq sxz \in I) sxz \in I$ for some $s \in S$. Then $sxz \in \delta(I)$ or $yz \in \delta(I)$ or $xsz \in \delta(I)$. Hence $\frac{sxz}{s_1s_2s_3} \in \delta(I)_S$ or $\frac{xsz}{s_1s_3s_3} \in \delta(I)_S$ or $\frac{xsz}{s_1s_1s_3} \in \delta(I)_S$. Since $(\delta(I))_S = \delta_S(I_S)$, $I_S$ is a (weakly) 2-absorbing $\delta_S$-primary ideal of $R_S$. \hfill \Box

Let $R = R_1 \times \cdots \times R_n$ ($n \geq 2$) where $R_1, R_2, ..., R_n$ are commutative rings with nonzero identity, let $\delta_i$ be an expansion function of $\mathfrak{J}(R_i)$ for each $i \in \{1, 2, ..., n\}$.
For a proper ideal $I_1 \times \cdots \times I_n$, the function $\delta_x$ defined by $\delta_x(I_1 \times I_2 \times \cdots \times I_n) = \delta_\alpha(I_1) \times \delta_\beta(I_2) \times \cdots \times \delta_\alpha(I_n)$ is an expansion function of $\mathcal{J}(R)$. In the next four theorems, we characterize 2-absorbing $\delta_x$-semiprimary ideals and weakly 2-absorbing $\delta_x$-semiprimary ideals of $R_1 \times \cdots \times R_n$.

**Theorem 2.30.** Let $R_1$ and $R_2$ be commutative rings with $1 \neq 0$ and $R = R_1 \times R_2$, and let $\delta_\alpha$, $\delta_\beta$ be expansion functions of $\mathcal{J}(R_1)$ and $\mathcal{J}(R_2)$, respectively. Suppose that $\delta_x(I)$ is a proper ideal of $R$ for any proper ideal of $R$. Then the following statements are equivalent:

1. $I = I_1 \times I_2$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R$.

2. Either $I_1$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R_1$ and $\delta_x(I_2) = R_2$ or $I_2$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R_2$ and $\delta_x(I_1) = R_1$ or $I_1, I_2$ are $\delta_1, \delta_2$-semiprimary ideals of $R_1, R_2$, respectively.

**Proof.** (1) $\implies$ (2): Suppose that $I = I_1 \times I_2$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R$. Since $I$ is proper, $\delta_x(I) = \delta_x(I_1) \times \delta_x(I_2)$ is a proper ideal of $R$ from the hypothesis. Hence we have three cases:

**Case 1:** Let $\delta_x(I_1) \neq R_1$ and $\delta_x(I_2) = R_2$. We show that $I_1$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R_1$. Suppose that $ab \in I_1$ and $ab \notin \delta_x(I_1)$. Then $(a,0)(b,0)(c,0) \in I$ and $(a,0)(b,0) \notin \delta_x(I)$ implies that $(b,0)(c,0) \in \delta_x(I)$ or $(a,0)(c,0) \in \delta_x(I)$. Thus $bc \in \delta(I_1)$ or $ac \in \delta(I_1)$, we are done.

**Case 2:** Let $\delta_x(I_2) \neq R_2$ and $\delta_x(I_1) = R_1$. One can easily obtain similar to Case 1 that $I_2$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R_2$.

**Case 3:** Let $\delta_x(I_1) \neq R_1$ and $\delta_x(I_2) \neq R_2$. Suppose that $ab \in I_1$ and $a \notin \delta_x(I_1)$ for some $a, b \in R_1$. Observe that $(a,1)(b,1)(1,0) \in I$, $(a,1)(b,1) \notin \delta_x(I)$, and $(a,1)(1,0) \notin \delta_x(I)$. Since $I$ is 2-absorbing $\delta_x$-semiprimary, we conclude $(b,1)(1,0) \in \delta_x(I)$. Thus $b \in \delta(I_1)$; and so $I_1$ is a $\delta_1$-semiprimary ideal of $R_1$. It can be shown by a symmetric way that $I_2$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R_2$.

(2) $\implies$ (1): If $I_1$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R_1$ and $\delta_x(I_2) = R_2$ or $I_2$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R_2$ and $\delta_x(I_1) = R_1$, then clearly $I$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R$. Now, suppose that $I_1$ and $I_2$ are $\delta_1, \delta_2$-semiprimary ideals of $R_1, R_2$, respectively. Suppose that $(a_1, a_2)(b_1, b_2)(c_1, c_2) \in I = I_1 \times I_2$, $(a_1, a_2)(b_1, b_2) \notin \delta_x(I)$ and $(a_1, a_2)(c_1, c_2) \notin \delta_x(I)$. Here we have four cases.

**Case 1:** Let $a_1b_1 \notin \delta(I_1)$ and $a_1c_1 \notin \delta(I_1)$. Since $a_1b_1c_1 \in I_1$, it contradicts with the assumption that $I_1$ is a $\delta_1$-semiprimary ideal.

**Case 2:** Let $a_2b_2 \notin \delta(I_2)$ and $a_2c_2 \notin \delta(I_2)$. Since $a_2b_2c_2 \in I_2$, it contradicts with the assumption that $I_2$ is a $\delta_2$-semiprimary ideal.

**Case 3:** Let $a_1b_1 \notin \delta(I_1)$ and $a_2c_2 \notin \delta(I_2)$. Since $a_1b_1c_1 \in I_1$ and $I_1$ is a $\delta_1$-semiprimary, we have $c_1 \in \delta(I_1)$. Since $a_2b_2c_2 \in I_2$ and $I_2$ is $\delta_2$-semiprimary, we have $b_2 \in \delta(I_2)$. Thus $(b_1, b_2)(c_1, c_2) \in \delta_x(I)$. **Case 4:** Let $a_1c_1 \notin \delta(I_1)$ and $a_2b_2 \notin \delta(I_2)$. Since $a_1b_1c_1 \in I_1$ and $I_1$ is $\delta_1$-semiprimary, we have $b_1 \in \delta(I_1)$. Since $a_2b_2c_2 \in I_2$ and $I_2$ is $\delta_2$-semiprimary, we have $c_2 \in \delta(I_2)$. Thus $(b_1, b_2)(c_1, c_2) \in \delta_x(I)$. Therefore, $I$ is a 2-absorbing $\delta_x$-semiprimary ideal of $R$. 

\[ \square \]
Theorem 2.31. Let \( R_1, R_2, \ldots, R_n \) be commutative rings with nonzero identity and \( R = R_1 \times \cdots \times R_n \) where \( n \geq 2 \). Let \( \delta_i \) be an expansion function of \( \mathcal{J}(R_i) \) for each \( i = 1, \ldots, n \). Then the following statements are equivalent:

1. \( I \) is a 2-absorbing \( \delta_x \)-semiprimary ideal of \( R \).
2. \( I = I_1 \times \cdots \times I_n \) and either for some \( k \in \{1, \ldots, n\} \) such that \( I_k \) is a 2-absorbing \( \delta_x \)-semiprimary ideal of \( R_k \) and \( \delta_j(I_j) = R_j \) for all \( j \in \{1, \ldots, n\}\setminus\{k\} \) or \( I_k \) and \( I_t \) are \( \delta_{k,t} \)-semiprimary ideals of \( R_k \) and \( R_t \), respectively for some \( k, t \in \{1, 2, \ldots, n\} \) and \( \delta_j(I_j) = R_j \) for all \( j \in \{1, \ldots, n\}\setminus\{k, t\} \).

Proof. It can be obtained from Theorem 2.30 by using mathematical induction on \( n \).

Theorem 2.32. Let \( R_1 \) and \( R_2 \) be commutative rings with identity, \( R = R_1 \times R_2 \), and let \( \delta_1, \delta_2 \) be expansion functions of \( \mathcal{J}(R_1) \) and \( \mathcal{J}(R_2) \), respectively. Then the following statements are equivalent:

1. \( I = I_1 \times I_2 \) is a weakly 2-absorbing \( \delta_x \)-semiprimary ideal of \( R \).
2. \( I = I_1 \times I_2 \) is a 2-absorbing \( \delta_x \)-semiprimary ideal of \( R \).
3. \( I_1 \) is a 2-absorbing \( \delta_1 \)-semiprimary ideal of \( R_1 \).

Proof. (1)\( \Rightarrow \) (2): Suppose that \( I = I_1 \times I_2 \) is a weakly 2-absorbing \( \delta_x \)-semiprimary ideal of \( R \). Since \( I^2 \neq 0 \), \( I = I_1 \times I_2 \) is a 2-absorbing \( \delta_x \)-semiprimary ideal of \( R \) by Theorem 2.17.

(2)\( \Rightarrow \) (3)\( \Rightarrow \) (1) is clear from Theorem 2.30.

Definition 2.33. Let \( R \) be a ring and \( \delta \) an expansion function of \( \mathcal{J}(R) \). We say \( \delta \) has \((\star)\) property if the following condition is satisfied for all ideals \( J \) of \( R \):

\((\star)\) \( \delta(J) = R \) if and only if \( J = R \).

Theorem 2.34. Let \( R_1, R_2, \ldots, R_n \) be commutative rings with identity and \( R = R_1 \times \cdots \times R_n \) where \( n \geq 3 \). Let \( \delta_i \) be an expansion function of \( \mathcal{J}(R_i) \) which has \((\star)\) property for each \( i = 1, \ldots, n \). For a nonzero ideal \( I \) of \( R \), the following statements are equivalent:

1. \( I \) is a weakly 2-absorbing \( \delta_x \)-semiprimary ideal of \( R \).
2. \( I \) is a 2-absorbing \( \delta_x \)-semiprimary ideal of \( R \).
3. \( I = I_1 \times \cdots \times I_n \) and either for some \( k \in \{1, \ldots, n\} \) such that \( I_k \) is a 2-absorbing \( \delta_x \)-semiprimary ideal of \( R_k \) and \( I_j = R_j \) for all \( j \in \{1, \ldots, n\}\setminus\{k\} \) or \( I_k \) and \( I_t \) are \( \delta_{k,t} \)-semiprimary ideals of \( R_k \) and \( R_t \), respectively for some \( k, t \in \{1, 2, \ldots, n\} \) and \( I_j = R_j \) for all \( j \in \{1, \ldots, n\}\setminus\{k, t\} \).
Proof.

(1)$\Leftrightarrow$(2): Suppose that $I = I_1 \times \cdots \times I_n$ is a weakly 2-absorbing $\delta_\times$-semiprimary ideal of $R$. Since $I$ is nonzero, there exists an element $0 \neq (x_1, x_2, \ldots, x_n) \in I$. Hence $0 \neq (x_1, 1, \ldots, 1)(1, x_2, \ldots, 1) \cdots (1, 1, \ldots, x_n) \in I$ implies that $1 \in \delta(I_k)$ for some $k \in \{1, \ldots, n\}$. Thus $I_k = R_k$ for some $k \in \{1, \ldots, n\}$; so $I^3$ can not be 0. Therefore, $I$ is a 2-absorbing $\delta_\times$-semiprimary ideal of $R$ by Theorem 2.17. The converse is obvious.

(2)$\Leftrightarrow$(3): From Theorem 2.31., the claim is clear. □

Let $R$ be a commutative ring and $M$ an $R$-module. The idealization $R(+)M = \{(r, m) : r \in R, m \in M\}$ is a commutative ring with addition and multiplication, respectively: $(r, m)(s, m') = (r + s, m + m')$ and $(r, m)(s, m') = (rs, rm' + sm)$ for each $r, s \in R$; $m, m' \in M$. Additionally, $I(+)M$ is an ideal of $R(+)M$ where $I$ is an ideal of $R$ and $N$ is a submodule of $M$ if and only if $IM \subseteq N$ ((2) and [9]). In this circumstances, $I(+)N$ is called a homogeneous ideal of $R(+)M$. Let $\delta$ be an expansion function of $R$. Clear that $\delta_\times$ is defined as $\delta_\times(I(+)N) = \delta(I)(+)M$ for all ideal $I(+)N$ of $R(+)M$ is an expansion function of $R(+)M$.

Theorem 2.35. Let $\delta$ be an expansion function of $R$ and $I(+)N$ be a homogeneous ideal of $R(+)M$. Then, $I$ is a 2-absorbing $\delta$-semiprimary ideal of $R$ if and only if $I(+)N$ is a 2-absorbing $\delta_\times$-semiprimary ideal of $R(+)M$.

Proof. Let $(r_1, m_1)(r_2, m_2)(r_3, m_3) = (r_1r_2r_3, r_2r_3m_1 + r_1r_3m_2 + r_1r_2m_3) \in I(+)N$. Then $r_1r_2r_3 \in I$. Since $I$ is 2-absorbing $\delta$-semiprimary, we have $r_1r_2 \in \delta(I)$ or $r_2r_3 \in \delta(I)$ or $r_1r_3 \in \delta(I)$. Since $\delta_\times(I(+)N) = \delta(I)(+)M$, we conclude that $(r_1, m_1)(r_2, m_2) \in \delta_\times(I(+)N)$ or $(r_2, m_2)(r_3, m_3) \in \delta_\times(I(+)N)$ or $(r_1, m_1)(r_3, m_3) \in \delta_\times(I(+)N)$. Conversely, suppose that $r_1r_2r_3 \in I$ for some $r_1, r_2, r_3 \in R$. Then $(r_1, 0)(r_2, 0)(r_3, 0) = (r_1r_2r_3, 0) \in I(+)N$. The remain of the proof is clear. □

3. Strongly (weakly) 2-absorbing $\delta$-semiprimary ideal

First, we state the following theorem which gives a characterization for 2-absorbing $\delta$-semiprimary ideals in terms of ideals of $R$.

Theorem 3.1. Let $\delta$ be an expansion function of $\mathcal{J}(R)$ and $I$ a proper ideal of $R$. Then the following are equivalent:

1. $I$ is a 2-absorbing $\delta$-semiprimary ideal of $R$.

2. For every elements $a, b \in R$ with $ab \notin \delta(I)$, $(I : ab) \subseteq (\delta(I) : a) \cup (\delta(I) : b)$.

3. For every elements $a, b \in R$ with $ab \notin \delta(I)$, $(I : ab) \subseteq (\delta(I) : a)$ or $(I : ab) \subseteq (\delta(I) : b)$.

4. For every elements $a, b \in R$ with $abJ \subseteq I$ and $ab \notin \delta(I)$ implies either $aJ \subseteq \delta(I)$ or $bJ \subseteq \delta(I)$.

5. For any ideals $J, K$ and $L$ of $R$ with $JKL \subseteq I$ implies $JL \subseteq \delta(I)$ or $KL \subseteq \delta(I)$. 

Definition 3.5. Let \( JKL \) respect to \( JKL \) and need to be a strongly weakly 2-absorbing.

Theorem 3.4. Assume that neither \( JKL \) and \( JKL \) with (0
\( I \subseteq (I : ab) \subseteq (\delta(I) : a) \) or \( JKL \subseteq (I : ab) \subseteq (\delta(I) : b) \) by our assumption. Thus, 
\( aJ \subseteq \delta(I) \) or \( bJ \subseteq \delta(I) \).

(4)\( \Rightarrow \)(5): Suppose that \( JKL \subseteq I \) and \( KKL \nsubseteq \delta(I) \). Then \( ab \nsubseteq \delta(I) \) for some
\( a \in K \) and \( b \in L \). Hence \( aJ \subseteq \delta(I) \) or \( bJ \subseteq \delta(I) \). Assume that \( aJ \subseteq \delta(I) \) and \( bJ \nsubseteq \delta(I) \). We show that \( JKL \subseteq \delta(I) \). If \( kJ \nsubseteq \delta(I) \) for some \( k \in K \), then
\( (a + k)bJ \subseteq I \). Since \( bJ \nsubseteq \delta(I) \), we have \( (a + k)J \subseteq \delta(I) \); and so, we get \( kJ \subseteq \delta(I) \), a contradiction. Assume that \( aJ \nsubseteq \delta(I) \) and \( bJ \subseteq \delta(I) \). Similar to the previous argument, we conclude that \( JKL \subseteq \delta(I) \). Now, suppose that \( JKL \subseteq \delta(I) \) and \( JKL \subseteq \delta(I) \).

Assume that neither \( JKL \subseteq \delta(I) \) nor \( JKL \subseteq \delta(I) \). Then there exist \( k \in K \) and \( l \in L \) such that \( kJ \nsubseteq \delta(I) \) and \( lJ \nsubseteq \delta(I) \). Since \( kJ \subseteq I \), we conclude \( kl \subseteq \delta(I) \). Since \( (a + k)lJ \subseteq I \), \( lJ \nsubseteq \delta(I) \) and \( (a + k)J = aJ + kJ \nsubseteq \delta(I) \), we have \( (a + k)J \subseteq \delta(I) \).

Now, suppose that \( aJ \subseteq \delta(I) \), we have \( aJ \subseteq \delta(I) \). Since \( (b + l)kJ \subseteq I \), \( kJ \nsubseteq \delta(I) \) and \( (b + l)J = bJ + lJ \nsubseteq \delta(I) \), we have \( (b + l)J \subseteq I \). Since \( kJ \subseteq \delta(I) \), we have \( bk \subseteq \delta(I) \).

(5)\( \Rightarrow \)(1): Suppose that \( abc \subseteq I \) for some \( a, b, c \in R \). Put \( J = (a) \), \( K = (b) \) and
\( L = (c) \) in (5). Hence, the result is clear.

Now, we define strongly (weakly) 2-absorbing \( \delta \)-semiprimary ideals as follows:

Definition 3.2. Let \( \delta \) be an expansion function of \( J(R) \). We call a proper ideal \( I \) of
\( R \) a strongly (weakly) 2-absorbing \( \delta \)-semiprimary ideal if whenever \( JKL \subseteq \delta(I) \) with \((0 \nsubseteq KKL \leq I \) \( JKL \subseteq \delta(I) \) implies \( JKL \subseteq \delta(I) \) or \( KKL \subseteq \delta(I) \) or \( KKL \subseteq \delta(I) \).

As a result of Theorem 3.1., \( I \) is a 2-absorbing \( \delta \)-semiprimary ideal of \( R \) if and only if \( I \) is a strongly 2-absorbing \( \delta \)-semiprimary ideal of \( R \). Motivated from this result, we search the answer for the following question:

Question 3.3. If \( I \) is a weakly 2-absorbing \( \delta \)-semiprimary ideal of \( R \), then does \( I \) need to be a strongly weakly 2-absorbing \( \delta \)-semiprimary ideal of \( R \)?

Theorem 3.4. Let \( I \) be a weakly 2-absorbing \( \delta \)-semiprimary ideal of \( R \) and suppose
that \( 0 \neq JKL \subseteq I \) for some ideals \( J, K, L \) of \( R \). If \( I \) is a free \( \delta \)-triple-zero with
respect to \( JKL \), then \( JKL \subseteq \delta(I) \) or \( KKL \subseteq \delta(I) \) or \( KKL \subseteq \delta(I) \).

To prove the theorem above, we need the following definition and lemmas.

Definition 3.5. Let \( I \) be a weakly 2-absorbing \( \delta \)-semiprimary ideal of \( R \) and suppose
that \( JKL \subseteq I \) for some ideals \( J, K, L \) of \( R \). We call \( I \) a free \( \delta \)-triple-zero
with respect to \( JKL \) if \((a, b, c) \) is not a \( \delta \)-triple-zero of \( I \) for every \( a \in J, b \in K \)
and \( c \in L \). (Equivalently, if \( a \in J, b \in K, c \in L \), then \( ab \in \delta(I) \) or \( bc \in \delta(I) \) or
\( ac \in \delta(I) \).)
Lemma 3.6. Let $I$ be a weakly 2-absorbing $\delta$-semiprimary ideal of $R$ and let $abL \subseteq I$ for some $a, b \in R$ and an ideal $L$ of $R$. If $(a, b, l)$ is not a $\delta$-triple-zero of $I$ for all $l \in L$ and $ab \notin \mathfrak{d}(I)$, then $aL \subseteq \mathfrak{d}(I)$ or $bL \subseteq \mathfrak{d}(I)$.

Proof. Assume on the contrary that $abL \subseteq I$ but neither $ab \in \mathfrak{d}(I)$ nor $aL \subseteq \mathfrak{d}(I)$ nor $bL \subseteq \mathfrak{d}(I)$. Hence there exist $l_1, l_2 \in L$ such that $a_l \notin \mathfrak{d}(I)$ and $b_l \notin \mathfrak{d}(I)$. Since $ab l_1 \in I$ but neither $ab \in \mathfrak{d}(I)$ nor $a_l \in \mathfrak{d}(I)$, we have $b l_1 \in \mathfrak{d}(I)$ by our hypothesis that $(a, b, l_1)$ is not a $\delta$-triple-zero of $I$. Similarly, since $ab l_2 \in I$ but neither $ab \in \mathfrak{d}(I)$ nor $b_l \in \mathfrak{d}(I)$, we conclude that $a_l \in \mathfrak{d}(I)$. Now $(a, l_1 + l_2) \in I$ and since $ab \notin \mathfrak{d}(I)$, we have either $a(l_1 + l_2) \in \mathfrak{d}(I)$ or $b(l_1 + l_2) \in \mathfrak{d}(I)$. Thus, we conclude $aL \subseteq \mathfrak{d}(I)$ or $bL \subseteq \mathfrak{d}(I)$, a contradiction. Thus, $aL \subseteq \mathfrak{d}(I)$ or $bL \subseteq \mathfrak{d}(I)$. □

Lemma 3.7. Let $I$ be a weakly 2-absorbing $\delta$-semiprimary ideal of $R$ and let $aKL \subseteq I$ for some $a \in R$ and for an ideal $J$ of $R$. If $(a, k, l)$ is not a $\delta$-triple-zero of $I$ for all $k \in K$, $l \in L$, then $ak \subseteq \mathfrak{d}(I)$ or $al \subseteq \mathfrak{d}(I)$ or $KL \subseteq \mathfrak{d}(I)$.

Proof. Assume that neither $aK \subseteq \mathfrak{d}(I)$ nor $al \subseteq \mathfrak{d}(I)$ nor $KL \subseteq \mathfrak{d}(I)$. Thus there exist $k, k_1 \in K$ such that $ak \notin \mathfrak{d}(I)$ and $k_1L \notin \mathfrak{d}(I)$. Since $akL \subseteq I$, $ak \notin \mathfrak{d}(I)$ and $al \notin \mathfrak{d}(I)$, we have $kL \subseteq \mathfrak{d}(I)$ by Lemma 3.6. Since $ak_1L \subseteq I$, $kl \notin \mathfrak{d}(I)$ and $k_1L \notin \mathfrak{d}(I)$, we have by $ak_1 \notin \mathfrak{d}(I)$ Lemma 3.6. Now, since $a(k + k_1)L \subseteq I$ and $al \notin \mathfrak{d}(I)$, from Lemma 3.6. we conclude that either $a(k + k_1) \notin \mathfrak{d}(I)$ or $(k + k_1)L \subseteq I$. Hence $ak \in \mathfrak{d}(I)$ or $k_1L \subseteq I$, a contradiction. Thus, $aK \subseteq \mathfrak{d}(I)$ or $aL \subseteq \mathfrak{d}(I)$ or $KL \subseteq \mathfrak{d}(I)$. □

Proof of Theorem 3.4. Assume on the contrary that neither $JK \subseteq \mathfrak{d}(I)$ nor $KL \subseteq \mathfrak{d}(I)$ nor $JL \subseteq \mathfrak{d}(I)$. Hence there exists $a, a_1 \in J$ such that $aK \notin \mathfrak{d}(I)$ and $a_1L \notin \mathfrak{d}(I)$. Since $aKL \subseteq I$, $KL \notin \mathfrak{d}(I)$ and $aK \notin \mathfrak{d}(I)$, we have $aL \subseteq \mathfrak{d}(I)$ by Lemma 3.7. Since $aKL \subseteq I$, $KL \notin \mathfrak{d}(I)$ and $a_1L \notin \mathfrak{d}(I)$, we have $a_1K \subseteq \mathfrak{d}(I)$ by Lemma 3.7. Now $(a + a_1)KL \subseteq I$ and since $KL \notin \mathfrak{d}(I)$, we conclude either $(a + a_1)K \subseteq \mathfrak{d}(I)$ or $(a + a_1)L \subseteq \mathfrak{d}(I)$. Hence, we have $aK \subseteq \mathfrak{d}(I)$ or $a_1L \subseteq \mathfrak{d}(I)$, a contradiction. Therefore, $JK \subseteq \mathfrak{d}(I)$ or $KL \subseteq \mathfrak{d}(I)$ or $JL \subseteq \mathfrak{d}(I)$. □

More general than 2-absorbing $\delta$-semiprimary ideal of a commutative ring, the concept of $n$-absorbing $\delta$-semiprimary ideal where $n$ is a positive integer can be defined. We shall give just the definition of this concept which may be inspired the other work:

Definition 3.8. Let $R$ be a commutative ring with nonzero identity, $\delta : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R)$ an expansion of ideals of $R$ and $n$ a positive integer. We call a proper ideal $I$ of $R$ a (weakly) $n$-absorbing $\delta$-semiprimary ideal if whenever $(0 \neq x_1 \cdots x_{n+1} \in I)$ $x_1 \cdots x_{k+1} \in I$ for some $x_1, \ldots, x_{n+1} \in R$, then there exists $1 \leq k \leq n$ such that $x_1 \cdots x_k x_{k+1} \cdots x_{n+1} \in \mathfrak{d}(I)$. In particular, for $n = 1, 2$, it is $\delta$-semiprimary and 2-absorbing $\delta$-semiprimary ideal, respectively.

References

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