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GLOBAL LARGE SOLUTIONS FOR THE COMPRESSIBLE MAGNETOHYDRODYNAMIC SYSTEM

Jinlu Li, Yanghai Yu, and Weipeng Zhu

Abstract. In this paper we consider the global well-posedness of compressible magnetohydrodynamic system in $\mathbb{R}^d$ with $d \geq 2$, in the framework of the critical Besov spaces. We can show that if the initial data, the shear viscosity and the magnetic diffusion coefficient are small comparing with the volume viscosity, then the compressible magnetohydrodynamic system has a unique global solution. Our result improves the previous one by Danchin and Mucha [10] who considered the compressible Navier-Stokes equations.

1. Introduction

The present paper is devoted to the equations of magnetohydrodynamics (MHD) which describe the motion of electrically conducting fluids in the presence of a magnetic field. The barotropic compressible magnetohydrodynamic system can be written as

$$(1)\quad \begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = b \cdot \nabla b - \frac{1}{2}\nabla(|b|^2) & \\ \qquad + \mu\Delta u + \nabla\big((\mu+\lambda)\mathrm{div} u\big), & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d \\ \partial_t b + (\mathrm{div} u) b + u \cdot \nabla b - b \cdot \nabla u - \nu\Delta b = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d \\ \mathrm{div} b = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d, \end{cases}$$

where $\rho = \rho(t,x) \in \mathbb{R}^+$ denotes the density, $u = u(t,x) \in \mathbb{R}^d$ and $b = b(t,x) \in \mathbb{R}^d$ stand for the velocity field and the magnetic field, respectively. The barotropic assumption means that the pressure $P = P(\rho)$ is given and assumed to be strictly increasing. The constant $\nu > 0$ is the resistivity acting as the magnetic diffusion coefficient of the magnetic field. The shear and volume viscosity coefficients μ and λ are constant and fulfill the standard strong parabolicity assumption:

$$\mu > 0 \quad \text{and} \quad \kappa = \lambda + 2\mu > 0.$$

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To complete the system (1), the initial data are supplemented by

$$(u, b, \rho)(t, x)|_{t=0} = (u_0(x), b_0(x), \rho_0(x)) \tag{2}$$

and also, as the space variable tends to infinity, we assume

$$\lim_{|x|\to\infty} \big(u_0(x), b_0(x), \rho_0(x)\big) = (0, 0, 1). \tag{3}$$

The system of MHD involves various topics such as the evolution and dynamics of astrophysical objects, thermonuclear fusion, metallurgy and semiconductor crystal growth, see for example [2,18]. Roughly speaking, the system (1) is a coupling between the compressible Navier-Stokes equations with the magnetic equations (heat equations). When $b \equiv 0$, the system (1) reduces to the usual compressible Navier-Stokes system for baratropic fluids. We should mentioned that the system (1) contains much richer structures than the Navier-Stokes system and offer us new opportunities. Due to its physical importance, complexity, rich phenomena and mathematical challenges, there have been huge literatures on the study of the compressible MHD problem (1) by many physicists and mathematicians, see for example, [3, 4, 11–26] and the references therein.

Now, we briefly recall some results concerned with the multi-dimensional compressible MHD equations in the absence of vacuum, which are more relatively with our problem. Kawashima [17] established the local and global well-posedness of the solutions to the compressible MHD equations as the initial density is strictly positive, see also Strohmer [21] and Vol'pert–Khudiaev [23] for the local existence results. To catch the scaling invariance property, Danchin firstly introduced in his series papers [5–9] the "Critical Besov Spaces" which were inspired by those efforts on the incompressible Navier-Stokes. Li–Mu–Wang [19] obtained the local well-posedness of the system (1) in the framework of critical Besov spaces under the assumption that the initial density is bounded away from 0. Danchin and Mucha [10] proved that the compressible Navier-Stokes system convergence to the homogeneous incompressible case for the large volume viscosity. Motivated by these works, our main goal of the present paper is devoted to extend the compressible Navier-Stokes system to the MHD system. Since there is no global well-posedness theory for general initial data, we will prove the global existence of strong solutions to (1) for a class of large initial data.

We notice that if κ tends to $+\infty$, then velocity field and magnetic field shall satisfy the incompressible MHD system:

$$\begin{cases} \partial_t U + U \cdot \nabla U - \mu \Delta U + \nabla \Pi - B \cdot \nabla B - \frac{1}{2}\nabla(|B|^2) = 0, \\ \partial_t B + U \cdot \nabla B - B \cdot \nabla U - \nu \Delta B = 0, \\ \mathrm{div} U = \mathrm{div} B = 0, \\ (U, B)|_{t=0} = (\mathcal{P} u_0, b_0), \end{cases} \tag{4}$$

where $\mathcal{P}$ is the Leray project operator.

Our main result can be stated as follows:

Theorem 1.1. *Let* $d \geq 2$, $(u_0, b_0) \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d) \times \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$ *and* $a_0 := \rho_0 - 1 \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d) \cap \dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$. *Suppose that* (4) *generates a unique global solution* $(U, B) \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d))$. *Let* C *be a large universal constant and denote*

$$M := \|U, B\|_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{2}-1}_{2,1})},$$

$$D_0 := C\exp\big(C(1+\mu^{-1}+\nu^{-1})(M+1)^2\big)\big(\|a_0, \mathcal{Q}v_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \kappa\|a_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + 1\big),$$

$$\delta_0 := C\exp\big(2C(1+\mu^{-2}+\nu^{-2})(M+1)^2\big)\big(\kappa^{-1}D_0^2 + \kappa^{-\frac{1}{2}}D_0\big).$$

Assume that κ *is large enough and* $\|a_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}}$ *is small enough such that*

$$\kappa^{-1}D_0 \ll 1 \quad \text{and} \quad \delta_0\big(\mu^{-1}+\nu^{-1}+1\big) \leq \frac{1}{2}.$$

Then the system (1)–(3) *has a unique global-in-time solution* (ρ, u, b) *which satisfies*

$$u, b \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}),$$
$$a := \rho - 1 \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{d}{2}-1}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}) \cap L^2(\mathbb{R}^+; \dot{B}^{\frac{d}{2}}_{2,1}).$$

Remark 1.2. For $d = 2$, according to Proposition 2.3, we can set

$$M := C\|U_0, B_0\|_{\dot{B}^0_{2,1}} \exp\Big(C\big(\mu^{-4}+\nu^{-4}\big)\|U_0, B_0\|^4_{L^2}\Big).$$

From Theorem 1.1, we deduce that the system (1)–(3) has a unique global-in-time solution without any smallness condition on the initial velocity and magnetic: the volume viscosity λ just has to be sufficiently large.

2. Preliminaries

In this section, we recall basic notations and some useful properties involved the homogeneous Besov spaces which will be used throughout this paper.

- The symbol $\mathcal{A} \lesssim \mathcal{B}$ means that there is a uniform positive constant C independent of $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \leq C\mathcal{B}$. We write $\mathcal{A} \approx \mathcal{B}$ to denote that $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{B} \lesssim \mathcal{A}$.
- For any $k, m \in \mathbb{Z}^+$, we also use the simplified notations if there is no ambiguity

$$\|f_1, \dots, f_k\|^m_X := \|f_1\|^m_X + \cdots + \|f_k\|^m_X.$$

- The projectors $\mathcal{P}$ and $\mathcal{Q}$ are defined by

$$\mathcal{P} := \mathrm{Id} + (-\Delta)^{-1}\nabla\mathrm{div} \quad \text{and} \quad \mathcal{Q} := -(-\Delta)^{-1}\nabla\mathrm{div}.$$

Now, let us recall the definition of the homogenous Besov space (see [1]).

Definition. Let $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} = \Big\{ f \in \mathcal{S}'_h(\mathbb{R}^d) : \|f\|_{\dot{B}^s_{p,r}} := \Big\| (2^{js}\|\dot{\Delta}_j f\|_{L^p})_j \Big\|_{\ell^r} < +\infty \Big\}.$$

Next, we give the important product acts on homogenous Besov spaces by collecting some useful lemmas from [1].

Lemma 2.1. *Let $s_1, s_2 \le \frac{d}{2}$, $s_1 + s_2 > 0$ and $(f,g) \in \dot{B}^{s_1}_{2,1}(\mathbb{R}^d) \times \dot{B}^{s_2}_{2,1}(\mathbb{R}^d)$. Then we have*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}} \le C\|f\|_{\dot{B}^{s_1}_{2,1}}\|g\|_{\dot{B}^{s_2}_{2,1}}.$$

Lemma 2.2. *Let $s > 0$. Assume that $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R})$ with $F(0) = 0$. Then for any $f \in L^\infty(\mathbb{R}^d) \cap \dot{B}^s_{2,1}(\mathbb{R}^d)$, we have*

$$\|F(f)\|_{\dot{B}^s_{2,1}} \le C(\|f\|_{L^\infty})\|f\|_{\dot{B}^s_{2,1}}.$$

Proposition 2.3. *Assume that $U_0, B_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ with $\mathrm{div}U_0 = \mathrm{div}B_0 = 0$. Then there exists a unique solution to (4) such that*

$$U, B \in \mathcal{C}(\mathbb{R}^+; \dot{B}^0_{2,1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \dot{B}^2_{2,1}(\mathbb{R}^2)).$$

Furthermore, there exists some universal constant C, one has for all $T > 0$,

$$\begin{aligned}
&\|U, B\|_{L^\infty_T(\dot{B}^0_{2,1})} + \|U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B\|_{L^1_T(\dot{B}^0_{2,1})} \\
&\le C\|U_0, B_0\|_{\dot{B}^0_{2,1}} \exp\Big(C\big(\mu^{-4} + \nu^{-4}\big)\|U_0, B_0\|^4_{L^2}\Big).
\end{aligned}$$

Proof. For any $t \in [0,T]$, the standard energy balance reads:

$$\|U(t), B(t)\|^2_{L^2} + 2\mu\int_0^t \|\nabla U\|^2_{L^2}\mathrm{d}t + 2\nu\int_0^t \|\nabla B\|^2_{L^2}\mathrm{d}t = \|U_0\|^2_{L^2} + \|B_0\|^2_{L^2},$$

which implies for all $T > 0$,

$$\mu^{\frac14}\|U\|_{L^4_T(\dot{B}^{\frac12}_{2,1})} + \nu^{\frac14}\|B\|_{L^4_T(\dot{B}^{\frac12}_{2,1})} \le C\|U_0, B_0\|_{L^2}. \tag{5}$$

From the estimates of the Stokes system in homogeneous Besov spaces, we have

$$\begin{aligned}
&\|U, B\|_{L^\infty_T(\dot{B}^0_{2,1})} + \|U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B\|_{L^1_T(\dot{B}^0_{2,1})} \\
&\le C\big(\|U_0, B_0\|_{\dot{B}^0_{2,1}} + \|U\cdot\nabla U - B\cdot\nabla B\|_{L^1_T(\dot{B}^0_{2,1})} + \|B\cdot\nabla U - U\cdot\nabla B\|_{L^1_T(\dot{B}^0_{2,1})}\big).
\end{aligned} \tag{6}$$

In view of the interpolation inequality and Young inequality, we deduce that

$$\begin{aligned}
\|U\cdot\nabla U\|_{L^1_T(\dot{B}^0_{2,1})} &\le C\int_0^T \|U\|_{\dot{B}^{\frac12}_{2,1}}\|\nabla U\|_{\dot{B}^{\frac12}_{2,1}}\mathrm{d}t \\
&\le C\int_0^T \|U\|_{\dot{B}^{\frac12}_{2,1}}\|\nabla U\|^{\frac14}_{\dot{B}^{-1}_{2,1}}\|\nabla U\|^{\frac34}_{\dot{B}^1_{2,1}}\mathrm{d}t
\end{aligned}$$

$$\leq \frac{C}{\varepsilon^3\mu^3}\int_0^T \|U\|^4_{\dot{B}^{\frac{1}{2}}_{2,1}}\|U\|_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\mu\|\nabla^2 U\|_{L^1_T(\dot{B}^0_{2,1})}. \tag{7}$$

Similar argument as in (7), we obtain

$$\|B\cdot\nabla B\|_{L^1_T(\dot{B}^0_{2,1})} \leq \frac{C}{\varepsilon^3\nu^3}\int_0^T \|B\|^4_{\dot{B}^{\frac{1}{2}}_{2,1}}\|B\|_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\nu\|\nabla^2 B\|_{L^1_T(\dot{B}^0_{2,1})}, \tag{8}$$

$$\|U\cdot\nabla B\|_{L^1_T(\dot{B}^0_{2,1})} \leq \frac{C}{\varepsilon^3\nu^3}\int_0^T \|U\|^4_{\dot{B}^{\frac{1}{2}}_{2,1}}\|B\|_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\nu\|\nabla^2 B\|_{L^1_T(\dot{B}^0_{2,1})}, \tag{9}$$

$$\|B\cdot\nabla U\|_{L^1_T(\dot{B}^0_{2,1})} \leq \frac{C}{\varepsilon^3\mu^3}\int_0^T \|B\|^4_{\dot{B}^{\frac{1}{2}}_{2,1}}\|U\|_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\mu\|\nabla^2 U\|_{L^1_T(\dot{B}^0_{2,1})}. \tag{10}$$

Therefore, combing (6)–(10) and choosing ε small enough, we find that

$$\begin{aligned}&\|U,B\|_{L^\infty_T(\dot{B}^0_{2,1})} + \|U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B\|_{L^1_T(\dot{B}^0_{2,1})}\\ &\leq C\Big(\|U_0,B_0\|_{\dot{B}^0_{2,1}} + \big(\mu^{-3}+\nu^{-3}\big)\int_0^T \|U,B\|^4_{\dot{B}^{\frac{1}{2}}_{2,1}}\|U,B\|_{\dot{B}^0_{2,1}}\mathrm{d}t\Big).\end{aligned}$$

It follows from the Gronwall inequality and (5) that the desired result of Proposition 2.3. □

3. Global a priori estimates

The section is devoted to deduce the global a priori estimates which are crucial to prove our main result.

Setting $a=\rho-1$, we infer from (1) that

$$\begin{cases}\partial_t a + \mathrm{div}(au) + \mathrm{div}u = 0,\\ \partial_t u + u\cdot\nabla u + P'(1+a)\nabla a - b\cdot\nabla b + \frac{1}{2}\nabla(|b|^2) - \mu\Delta u\\ -\nabla((\mu+\lambda)\mathrm{div}u) = -a(u_t + u\cdot\nabla u),\\ \partial_t b + (\mathrm{div}u)b + u\cdot\nabla b - b\cdot\nabla u - \nu\Delta b = 0,\\ \mathrm{div}b = 0,\\ (u,b,a)(t,x)|_{t=0} = (u_0(x), b_0(x), a_0(x)).\end{cases} \tag{11}$$

Before we move on, we recall the following local well-posedness of system (11).

Theorem 3.1 (See [19])**.** *Assume that the initial data* $(a_0 := \rho - 1, u_0, b_0)$ *satisfy* $\mathrm{div}b_0 = 0$ *and*

$$(a_0,u_0,b_0) \in \dot{B}^{\frac{d}{2}}_{2,1}\times\dot{B}^{\frac{d}{2}-1}_{2,1}\times\dot{B}^{\frac{d}{2}-1}_{2,1}.$$

In addition, $\inf_{x\in\mathbb{R}^d} a_0(x) > -1$*, then there exists some time* $T>0$ *such that the system* (11) *has a local unique solution* (a,u,b) *on* $[0,T]\times\mathbb{R}^d$ *which belongs to the function space*

$$E_T := \widetilde{\mathcal{C}}([0,T];\dot{B}^{\frac{d}{2}}_{2,1})\times(\widetilde{\mathcal{C}}([0,T];\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1_T\dot{B}^{\frac{d}{2}+1}_{2,1})^{2d},$$

where $\widetilde{\mathcal{C}}([0,T];\dot{B}^s_{q,1}) := \mathcal{C}([0,T];\dot{B}^s_{q,1}) \cap \widetilde{L}^\infty([0,T];\dot{B}^s_{q,1})$ *with* $s\in\mathbb{R}$ *and* $1\le q\le\infty$.

Introducing the following new quantities

$$v = u - U \quad \text{and} \quad c = b - B,$$

we shall treat the potential $\mathcal{Q}v$ and divergence-free $\mathcal{P}v$ parts of v, respectively.

Applying the operator $\mathcal{Q}$ to $(11)_2$ and noticing that $\mathcal{Q}v=\mathcal{Q}u$, one has

$$\partial_t(\mathcal{Q}v) + \mathcal{Q}((v+U)\cdot\nabla\mathcal{Q}v) - \kappa\Delta\mathcal{Q}v + \nabla a = -\mathcal{Q}(aU_t + av_t) - \mathcal{Q}R_1, \tag{12}$$

here we denote $k(a) = P'(1+a)-1$ (here we assume that $P'(1)=1$ for notational simplicity) and

$$\begin{aligned} R_1 &= (1+a)(v+U)\cdot\nabla\mathcal{P}v + (1+a)(v+U)\cdot\nabla U + a(v+U)\cdot\nabla\mathcal{Q}v \\ &\quad + k(a)\nabla a - (B+c)\cdot\nabla(B+c) + \frac{1}{2}\nabla(|B+c|^2). \end{aligned}$$

In view of equation $(11)_1$, then using $u=\mathcal{Q}v+\mathcal{P}v+U$, we find that a satisfies

$$\partial_t a + (v+U)\cdot\nabla a + \mathrm{div}\mathcal{Q}v = -a\mathrm{div}\mathcal{Q}v. \tag{13}$$

Note that $\mathcal{P}U=U$ and $\mathcal{P}(\mathcal{Q}v\cdot\nabla\mathcal{Q}v) = \mathcal{P}(a\nabla a)=0$, applying the operator $\mathcal{P}$ to $(11)_2$, we have

$$\partial_t(\mathcal{P}v) + \mathcal{P}((v+U)\cdot\nabla\mathcal{P}v) - \mu\Delta\mathcal{P}v = -\mathcal{P}(aU_t + av_t + a\nabla a) - \mathcal{P}R_2, \tag{14}$$

where

$$\begin{aligned} R_2 &= (1+a)(v+U)\cdot\nabla\mathcal{Q}v + (1+a)v\cdot\nabla U + a(v+U)\cdot\nabla\mathcal{P}v \\ &\quad + aU\cdot\nabla U - (B+c)\cdot\nabla c - c\cdot\nabla B \\ &= (1+a)\mathcal{P}v\cdot\nabla(U+\mathcal{Q}v) + (1+a)U\cdot\nabla\mathcal{Q}v + (1+a)\mathcal{Q}v\cdot\nabla U \\ &\quad + a(v+U)\cdot\nabla\mathcal{P}v + aU\cdot\nabla U + a\mathcal{Q}v\cdot\nabla\mathcal{Q}v - (B+c)\cdot\nabla c - c\cdot\nabla B. \end{aligned}$$

According to the magnetic equation of (11), we can show that c satisfies

$$\partial_t c + (v+U)\cdot\nabla c - \nu\Delta c = -R_3 \quad \text{with} \quad c|_{t=0}=0, \tag{15}$$

where

$$R_3 = (\mathrm{div}\mathcal{Q}v)B + (\mathrm{div}\mathcal{Q}v)c + v\cdot\nabla B - (B+c)\cdot\nabla v - c\cdot\nabla U.$$

In the sequel, we denote a^ℓ and a^h the low and high frequencies parts of a as

$$a^\ell = \sum_{2^j\kappa\le1}\dot\Delta_j a \quad \text{and} \quad a^h = \sum_{2^j\kappa>1}\dot\Delta_j a \tag{16}$$

and set

$$\begin{aligned} X_d(T) &= \|\mathcal{Q}v, a, \kappa\nabla a\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|\mathcal{Q}v_t+\nabla a, \kappa\nabla^2\mathcal{Q}v, \kappa\nabla^2 a^\ell, \nabla a^h\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}, \\ Y_d(T) &= Y_{d,1}(T) + Y_{d,2}(T) \\ &= \|\mathcal{P}v, c\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|\mathcal{P}v_t, c_t, \mu\nabla^2\mathcal{P}v, \nu\nabla^2 c\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}, \end{aligned}$$

$$Z_d(T) = \|U, B\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})},$$
$$X_d(0) = \|a_0, \mathcal{Q}v_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \kappa\|a_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}}.$$

It is easy to show that

$$Z_d(T) \le M \quad \text{for all } T > 0. \tag{17}$$

We concentrate our attention on the proof of global in time a priori estimates, as the local well-posedness issue has been ensured by Theorem 3.1. We claim that if κ is large enough then one may find some (large) D and (small) δ so that there holds for all $T < T^*$,

$$\begin{cases} X_d(T) \le D, \quad Y_d(T) \le \delta, \quad \kappa^{-1}D \ll 1, \\ \delta\big(\mu^{-1} + \nu^{-1} + 1\big) \le 1, \quad D \ge (M+1), \quad \|a(t,\cdot)\|_{L^\infty} \le \frac{1}{2}. \end{cases} \tag{18}$$

Next, our aim is to establish the global a priori estimates. We will divide it into three steps.

Step 1. Estimate on the terms $\mathcal{P}v$ and c.

We first consider the estimates for $\mathcal{P}v$. Applying $\dot{\Delta}_j$ to (14), taking the L^2 inner product with $\dot{\Delta}_j\mathcal{P}v$ then using that $\mathcal{P}^2 = \mathcal{P}$, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j\mathcal{P}v\|^2_{L^2} + \mu\|\nabla\dot{\Delta}_j\mathcal{P}v\|^2_{L^2} \\ &\quad= \int_{\mathbb{R}^d} ([v+U, \dot{\Delta}_j]\cdot\nabla\mathcal{P}v)\cdot\dot{\Delta}_j\mathcal{P}v\mathrm{d}x \\ &\qquad- \int_{\mathbb{R}^d} \dot{\Delta}_j(aU_t + av_t + a\nabla a + R_2)\cdot\dot{\Delta}_j\mathcal{P}v\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\dot{\Delta}_j\mathcal{P}v|^2\mathrm{div}v\mathrm{d}x. \end{aligned} \tag{19}$$

According to the commutator estimates of Lemma 2.100 in [1], we have

$$2^{j(\frac{d}{2}-1)}\|[v+U, \dot{\Delta}_j]\cdot\nabla\mathcal{P}v\|_{L^2} \le Cc_j\|\nabla(v+U)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\mathcal{P}v\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}, \tag{20}$$

where $\|c_j\|_{\ell^1} = 1$.

Multiplying both sides of (19) by $2^{j(\frac{d}{2}-1)}$ and summing up over $j \in \mathbb{Z}$, then using Lemma 2.1 and (20), we obtain that

$$\begin{aligned} &\|\mathcal{P}v\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \mu\|\nabla^2\mathcal{P}v\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\ &\lesssim \int_0^T \|\nabla(v+U)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\mathcal{P}v\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t \\ &\quad+ \|a(U_t + \mathcal{P}v_t + (\mathcal{Q}v_t + \nabla a))\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|R_2\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}. \end{aligned} \tag{21}$$

In order to bound $\|\mathcal{P}v_t\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}$, we infer from (14) and (21) that

$$\|\mathcal{P}v\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|\mathcal{P}v_t, \mu\nabla^2\mathcal{P}v\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \tag{22}$$

$$\lesssim \int_0^T \|\nabla(v+U)\|_{\dot B^{\frac d2}_{2,1}}\|\mathcal{P}v\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t+\int_0^T \|(v+U)\cdot\nabla\mathcal{P}v\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t$$
$$+\int_0^T \|a(U_t+\mathcal{P}v_t+(\mathcal{Q}v_t+\nabla a))\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t+\int_0^T \|R_2\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t.$$

For the second term of RHS for (22), we can infer from Lemma 2.1 that

$$\|(v+U)\cdot\nabla\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})} \tag{23}$$
$$\lesssim \int_0^T \|\mathcal{P}v\|_{\dot B^{\frac d2-1}_{2,1}}\|\mathcal{P}v\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+\int_0^T \|\mathcal{Q}v+U\|_{\dot B^{\frac d2}_{2,1}}\|\mathcal{P}v\|_{\dot B^{\frac d2}_{2,1}}\mathrm{d}t$$
$$\lesssim \int_0^T \|\mathcal{P}v\|_{\dot B^{\frac d2-1}_{2,1}}\|\mathcal{P}v\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+\frac{1}{\varepsilon\mu}\int_0^T \|(\mathcal{Q}v,U)\|^2_{\dot B^{\frac d2}_{2,1}}\|\mathcal{P}v\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t$$
$$+\varepsilon\|\mu\nabla^2\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}.$$

For the third term of RHS for (22), from Lemma 2.1, one has

$$\int_0^T \|a(U_t+\mathcal{P}v_t+(\mathcal{Q}v_t+\nabla a))\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t \tag{24}$$
$$\lesssim \|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})}\Big(\|U_t\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}$$
$$+\|\mathcal{P}v_t\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}+\|\mathcal{Q}v_t+\nabla a\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\Big)$$
$$\lesssim \kappa^{-1}X_d(T)\big(X_d(T)+Y_d(T)+Z_d(T)\big).$$

For the last term of RHS for (22), by Lemma 2.1, we can estimate them as follows

$$\|(1+a)\mathcal{P}v\cdot\nabla(U+\mathcal{Q}v)\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})} \tag{25}$$
$$\lesssim \int_0^T (1+\|a\|_{\dot B^{\frac d2}_{2,1}})\|U+\mathcal{Q}v\|_{\dot B^{\frac d2+1}_{2,1}}\|\mathcal{P}v\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t,$$

$$\|(1+a)(\mathcal{Q}v\cdot\nabla U+U\cdot\nabla\mathcal{Q}v)\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})} \tag{26}$$
$$\lesssim (1+\|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})})\|\mathcal{Q}v\|_{L^2_T(\dot B^{\frac d2}_{2,1})}\|U\|_{L^2_T(\dot B^{\frac d2}_{2,1})}$$
$$\lesssim \big(1+\kappa^{-1}X_d(T)\big)\kappa^{-\frac12}\mu^{-\frac12}X_d(T)Z_d(T),$$

$$\|a(v+U)\cdot\nabla\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})} \tag{27}$$
$$\lesssim \|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})}\|v+U\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}$$
$$\lesssim \kappa^{-1}\mu^{-1}X_d(T)Y_d(T)\big(X_d(T)+Y_d(T)+Z_d(T)\big),$$

$$\begin{aligned}(28)\quad &\|aU\cdot\nabla U + a\mathcal{Q}v\cdot\nabla\mathcal{Q}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\ &\lesssim \|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})}\Big(\|U\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|U\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}\\ &\quad+\|\mathcal{Q}v\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|\mathcal{Q}v\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}\Big)\\ &\lesssim \kappa^{-1}X_d(T)\big(\mu^{-1}Z_d^2(T)+\kappa^{-1}X_d^2(T)\big),\end{aligned}$$

$$\begin{aligned}(29)\quad &\|(B+c)\cdot\nabla c + c\cdot\nabla B\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\ &\lesssim \int_0^T\|c\|_{\dot B^{\frac d2-1}_{2,1}}\|c\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+\int_0^T\|B\|_{\dot B^{\frac d2}_{2,1}}\|c\|_{\dot B^{\frac d2}_{2,1}}\mathrm{d}t\\ &\lesssim \int_0^T\|c\|_{\dot B^{\frac d2-1}_{2,1}}\|c\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+\frac{1}{\varepsilon\nu}\int_0^T\|B\|^2_{\dot B^{\frac d2}_{2,1}}\|c\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t\\ &\quad+\varepsilon\|\nu\nabla^2 c\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}.\end{aligned}$$

Therefore, summing up (22)–(29), we obtain

$$\begin{aligned}(30)\quad &\|\mathcal{P}v\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}+\|\mathcal{P}v_t,\mu\nabla^2\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\ &\lesssim \kappa^{-1}X_d(T)\big(X_d(T)+Y_d(T)+Z_d(T)\big)\\ &\quad+\big(1+\kappa^{-1}X_d(T)\big)\kappa^{-\frac12}\mu^{-\frac12}X_d(T)Z_d(T)\\ &\quad+\kappa^{-1}X_d(T)\big(\mu^{-1}Z_d^2(T)+\kappa^{-1}X_d^2(T)\big)\\ &\quad+\kappa^{-1}\mu^{-1}X_d(T)Y_d(T)\big(X_d(T)+Y_d(T)+Z_d(T)\big)\\ &\quad+\int_0^T\Big((1+\|a\|_{\dot B^{\frac d2}_{2,1}})\|\mathcal{P}v,\mathcal{Q}v,U,c\|_{\dot B^{\frac d2+1}_{2,1}}+\tfrac{1}{\varepsilon\mu}\|\mathcal{Q}v,U\|^2_{\dot B^{\frac d2}_{2,1}}+\tfrac{1}{\varepsilon\nu}\|B\|^2_{\dot B^{\frac d2}_{2,1}}\Big)Y_{d,1}\mathrm{d}t\\ &\quad+\varepsilon Y_d(T).\end{aligned}$$

Similar argument as in (22) and (23), we infer from (15) that

$$\begin{aligned}(31)\quad &\|c\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}+\|c_t,\nu\nabla^2 c\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\ &\lesssim \int_0^T\|\mathcal{P}v,\mathcal{Q}v,U\|_{\dot B^{\frac d2+1}_{2,1}}\|c\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t+\|(v+U)\cdot\nabla c\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}+\|R_3\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}.\end{aligned}$$

For the last two terms of RHS for (31), due to Lemma 2.1, we can tackle with them as follows:

$$\begin{aligned}(32)\quad &\|(v+U)\cdot\nabla c\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\ &\lesssim \int_0^T\|\mathcal{P}v\|_{\dot B^{\frac d2-1}_{2,1}}\|c\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+\int_0^T\|\mathcal{Q}v+U\|_{\dot B^{\frac d2}_{2,1}}\|c\|_{\dot B^{\frac d2}_{2,1}}\mathrm{d}t\end{aligned}$$

$$\lesssim \int_0^T \|\mathcal{P}v\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \|c\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t + \frac{1}{\varepsilon\nu}\int_0^T \|\mathcal{Q}v, U\|^2_{\dot{B}^{\frac{d}{2}}_{2,1}} \|c\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \mathrm{d}t + \varepsilon\|\nu\nabla^2 c\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})},$$

$$\|(\mathrm{div}\mathcal{Q}v)c - c\cdot\nabla v - c\cdot\nabla U\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \lesssim \int_0^T \|U, \mathcal{P}v, \mathcal{Q}v\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \|c\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \mathrm{d}t, \tag{33}$$

$$\begin{aligned} &\|(\mathrm{div}\mathcal{Q}v)B + v\cdot\nabla B - B\cdot\nabla v\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\ &\lesssim \int_0^T \|\mathcal{Q}v\|_{B^{\frac{d}{2}}_{2,1}} \|B\|_{B^{\frac{d}{2}}_{2,1}} \mathrm{d}t + \int_0^T \|\mathcal{P}v\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|B\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathrm{d}t \\ &\lesssim \int_0^T \|\mathcal{Q}v\|_{B^{\frac{d}{2}}_{2,1}} \|B\|_{B^{\frac{d}{2}}_{2,1}} \mathrm{d}t + \frac{1}{\varepsilon\mu}\int_0^T \|\mathcal{P}v\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \|B\|^2_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathrm{d}t \\ &\quad + \varepsilon\|\mu\nabla^2\mathcal{P}v\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\ &\lesssim \kappa^{-\frac12}\nu^{-\frac12} X_d(T) Z_d(T) + \frac{1}{\varepsilon\mu}\int_0^T \|\mathcal{P}v\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \|B\|^2_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathrm{d}t \\ &\quad + \varepsilon\|\mu\nabla^2\mathcal{P}v\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}. \end{aligned} \tag{34}$$

Hence, collecting the estimates (31)–(34), we get

$$\begin{aligned} &\|c\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|c_t, \nu\nabla^2 c\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\ &\lesssim \nu^{-1} Y_d^2(T) + \varepsilon Y_d(T) + \kappa^{-\frac12}\nu^{-\frac12} X_d(T) Z_d(T) \\ &\quad + \int_0^T \Big(\|\mathcal{P}v, \mathcal{Q}v, U, c\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \varepsilon^{-1}\big(\mu^{-1}+\nu^{-1}\big)\|\mathcal{Q}v, U, B\|^2_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big) Y_d \mathrm{d}t. \end{aligned} \tag{35}$$

Combining (30) and (35), choosing ε small enough and using Gronwall's inequality, we have

$$\begin{aligned} &Y_d(T) \\ &\le C\exp\Big(C\|\mathcal{P}v, \mathcal{Q}v, U, c\|_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} + C\big(\mu^{-1}+\nu^{-1}\big)\|\mathcal{Q}v, U, B\|^2_{L^1_T(\dot{B}^{\frac{d}{2}}_{2,1})}\Big) \\ &\quad \times\Big\{\kappa^{-1} X_d(T)\big(X_d(T) + Y_d(T) + Z_d(T)\big) \\ &\qquad + \big(1 + \kappa^{-1} X_d(T)\big)\kappa^{-\frac12}\mu^{-\frac12} X_d(T) Z_d(T) \\ &\qquad + \kappa^{-1} X_d(T)\big(\mu^{-1} Z_d^2(T) + \kappa^{-1} X_d^2(T)\big) + \kappa^{-\frac12}\nu^{-\frac12} X_d(T) Z_d(T) \\ &\qquad + \kappa^{-1}\mu^{-1} X_d(T) Y_d(T)\big(X_d(T) + Y_d(T) + Z_d(T)\big)\Big\}. \end{aligned} \tag{36}$$

Step 2. Estimate on the terms $\mathcal{Q}v$ and a.

Now, applying $\dot{\Delta}_j$ to (11) and (12) yields that

$$\begin{cases}\partial_t a_j + (v+U)\cdot\nabla a_j + \mathrm{div}\mathcal{Q}v_j = g_j,\\ \partial_t\mathcal{Q}v_j + \mathcal{Q}((v+U)\cdot\nabla\mathcal{Q}v_j) - \kappa\Delta\mathcal{Q}v_j + \nabla a_j = f_j,\end{cases} \tag{37}$$

where

$$a_j = \dot{\Delta}_j a, \quad \mathcal{Q}v_j = \dot{\Delta}_j\mathcal{Q}v, \quad g_j = -\dot{\Delta}_j(a\mathrm{div}\mathcal{Q}v) - [\dot{\Delta}_j,(v+U)]\cdot\nabla a,$$
$$f_j = -\dot{\Delta}_j\mathcal{Q}(aU_t + av_t) - \dot{\Delta}_j\mathcal{Q}R_1 - \mathcal{Q}[\dot{\Delta}_j,(v+U)]\cdot\nabla\mathcal{Q}v.$$

Taking the L^2 inner product of $(37)_1$ with a_j and $(37)_2$ with $\mathcal{Q}v_j$ yields

$$\begin{cases}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|a_j\|_{L^2}^2 + (a_j,\mathrm{div}\mathcal{Q}v_j) = \frac{1}{2}(\mathrm{div}v, a_j^2) + (g_j,a_j),\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{Q}v_j\|_{L^2}^2 + \kappa\|\nabla\mathcal{Q}v_j\|_{L^2}^2 - (a_j,\mathrm{div}\mathcal{Q}v_j)\\ = \frac{1}{2}(\mathrm{div}v, |\mathcal{Q}v_j|^2) + (f_j,\mathcal{Q}v_j).\end{cases} \tag{38}$$

We next want to estimate for $\|\nabla a_j\|_{L^2}^2$. From $(37)_1$, we have

$$\partial_t\nabla a_j + (v+U)\cdot\nabla\nabla a_j + \nabla\mathrm{div}\mathcal{Q}v_j = \nabla g_j - \nabla(v+U)\cdot\nabla a_j. \tag{39}$$

From (39) and $(37)_2$, by simple calculations, we obtain

$$\begin{cases}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla a_j\|_{L^2}^2 + ((v+U)\cdot\nabla\nabla a_j,\nabla a_j) + (\nabla\mathrm{div}\mathcal{Q}v_j,\nabla a_j)\\ = (\nabla g_j - \nabla(v+U)\cdot\nabla a_j,\nabla a_j),\\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{Q}v_j,\nabla a_j) + (v+U,\nabla(\mathcal{Q}v_j\cdot\nabla a_j)) - \kappa(\Delta\mathcal{Q}v_j,\nabla a_j) + \|\nabla a_j\|_{L^2}^2\\ +(\nabla\mathrm{div}\mathcal{Q}v_j,\mathcal{Q}v_j) = (\nabla g_j - \nabla(v+U)\cdot\nabla a_j,\mathcal{Q}v_j) + (f_j,\nabla a_j).\end{cases} \tag{40}$$

Notice that $(\nabla\mathrm{div}\mathcal{Q}v_j,\nabla a_j) = (\Delta\mathcal{Q}v_j,\nabla a_j)$ and $\Delta\mathcal{Q}v_j = \nabla\mathrm{div}\mathcal{Q}v_j$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\kappa\|\nabla a_j\|_{L^2}^2 + 2(\mathcal{Q}v_j\cdot\nabla a_j)) + (\|\nabla a_j\|_{L^2}^2 - \|\nabla\mathcal{Q}v_j\|_{L^2}^2)\\ &= (\frac{1}{2}\kappa|\nabla a_j|^2 + \mathcal{Q}v_j\cdot\nabla a_j,\mathrm{div}v) + \kappa(\nabla g_j - \nabla(v+U)\cdot\nabla a_j,\nabla a_j)\\ &\quad + (\nabla g_j - \nabla(v+U)\cdot\nabla a_j,\mathcal{Q}v_j) + (f_j,\nabla a_j).\end{aligned} \tag{41}$$

Multiplying (41) by κ and adding up twice (38) yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_j^2 + \kappa(\|\nabla\mathcal{Q}v_j\|_{L^2}^2 + \|\nabla a_j\|_{L^2}^2)\\ &= \int_{\mathbb{R}^d}(2g_ja_j + 2f_j\cdot\mathcal{Q}v_j + \kappa^2\nabla g_j\cdot\nabla a_j + \kappa\nabla g_j\cdot\mathcal{Q}v_j + \kappa f_j\cdot\nabla a_j)\mathrm{d}x\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d}(2a_j^2 + 2|\mathcal{Q}v_j|^2 + 2\kappa\mathcal{Q}v_j\cdot\nabla a_j + |\kappa\nabla a_j|^2)\mathrm{div}v\mathrm{d}x\\ &\quad - \kappa\int_{\mathbb{R}^d}(\nabla(v+U)\cdot\nabla a_j)\cdot(\kappa\nabla a_j + \mathcal{Q}v_j)\mathrm{d}x \qquad \text{with}\end{aligned} \tag{42}$$

$$\mathcal{L}_j^2 = \int_{\mathbb{R}^d}(2a_j^2 + 2|\mathcal{Q}v_j|^2 + 2\kappa\mathcal{Q}v_j\cdot\nabla a_j + |\kappa\nabla a_j|^2)\mathrm{d}x \tag{43}$$

$$= \int_{\mathbb{R}^d} (2a_j^2 + |\mathcal{Q}v_j|^2 + |\mathcal{Q}v_j + \kappa\nabla a_j|^2)\mathrm{d}x \approx \|(\mathcal{Q}v_j, a_j, \kappa\nabla a_j)\|_{L^2}^2.$$

By (43), we obtain

$$\kappa\big(\|\nabla\mathcal{Q}v_j\|_{L^2}^2 + \|\nabla a_j\|_{L^2}^2\big) \geq c\min\big(\kappa 2^{2j}, \kappa^{-1}\big)\mathcal{L}_j^2,$$

which along with (42) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_j^2 + c\min(\kappa 2^{2j}, \kappa^{-1})\mathcal{L}_j^2 \leq \Big(\frac{1}{2}\|\mathrm{div}v\|_{L^\infty} + C\|\nabla(v+U)\|_{L^\infty}\Big)\mathcal{L}_j^2 + C\|g_j, f_j, \kappa\nabla g_j\|_{L^2}\mathcal{L}_j. \tag{44}$$

Multiplying both sides of (44) by $2^{j(\frac{d}{2}-1)}$ and summing up over $j \in \mathbb{Z}$, we have

$$\begin{aligned}&\|a, \kappa\nabla a, \mathcal{Q}v\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\kappa\nabla^2\mathcal{Q}v, \kappa\nabla^2 a^\ell, \nabla a^h\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}\\ &\lesssim \|(a, \kappa\nabla a, \mathcal{Q}v)(0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \int_0^T \|v, U\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\|a, \kappa\nabla a, \mathcal{Q}v\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}\mathrm{d}t\\ &\quad + \int_0^T \sum_{j\in\mathbb{Z}} 2^{j(\frac{d}{2}-1)}\|g_j, f_j, \kappa\nabla g_j\|_{L^2}\mathrm{d}t,\end{aligned} \tag{45}$$

where the notations a^ℓ and a^h have been defined in (16).

Combining the estimates

$$\|a\mathrm{div}\mathcal{Q}v, \kappa\nabla(a\mathrm{div}\mathcal{Q}v)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \lesssim \int_0^T \|\mathrm{div}\mathcal{Q}v\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|a, \kappa\nabla a\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}\mathrm{d}t$$

and

$$\begin{aligned}&\int_0^T \sum_{j\in\mathbb{Z}} 2^{j(\frac{d}{2}-1)}\|[\dot{\Delta}_j, (v+U)]\nabla a, \kappa\nabla([\dot{\Delta}_j, (v+U)]\nabla a)\|_{L^2}\mathrm{d}t\\ &\lesssim \int_0^T \|\nabla(v+U)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|a, \kappa\nabla a\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}\mathrm{d}t,\end{aligned}$$

we have

$$\int_0^T \sum_{j\in\mathbb{Z}} 2^{j(\frac{d}{2}-1)}\|g_j, \kappa\nabla g_j\|_{L^2}\mathrm{d}t \lesssim \int_0^T \|v, U\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\|a, \kappa\nabla a\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}\mathrm{d}t. \tag{46}$$

Next, we will estimate the last term $\int_0^T \sum_{j\in\mathbb{Z}} 2^{j(\frac{d}{2}-1)}\|f_j\|_{L^2}\mathrm{d}t$.

According to Lemmas 2.1-2.2 and the commutator estimates of Lemma 2.100 in [1], we have

$$\begin{aligned}&\|(1+a)(v+U)\cdot\nabla(\mathcal{P}v+U)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}\\ &\lesssim \big(1 + \|a\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}\big)\|\mathcal{P}v, U\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}\|\nabla\mathcal{P}v, \nabla U\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}})}\end{aligned} \tag{47}$$

$$
\begin{aligned}
&+\int_0^T (1+\|a\|_{\dot B^{\frac d2}_{2,1}})\|\nabla\mathcal{P}v,\nabla U\|_{\dot B^{\frac d2}_{2,1}}\|\mathcal{Q}v\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t\\
&\lesssim \big(1+\kappa^{-1}X_d(T)\big)\mu^{-1}\big(Y_d^2(T)+Z_d^2(T)\big)\\
&+\int_0^T (1+\|a\|_{\dot B^{\frac d2}_{2,1}})\|\nabla\mathcal{P}v,\nabla U\|_{\dot B^{\frac d2}_{2,1}}\|\mathcal{Q}v\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
&\|a(v+U)\cdot\nabla\mathcal{Q}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\lesssim \|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})}\|v,U\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|\nabla\mathcal{Q}v\|_{L^1_T(\dot B^{\frac d2}_{2,1})}\\
&\lesssim \kappa^{-2}X_d^2(T)\big(X_d(T)+Y_d(T)+Z_d(T)\big),
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
\|k(a)\nabla a\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\lesssim \|a\|^2_{L^2_T(\dot B^{\frac d2}_{2,1})}&\lesssim \int_0^T \|a^\ell,a^h\|^2_{\dot B^{\frac d2}_{2,1}}\mathrm{d}t\\
&\lesssim \int_0^T\Big(\|a^\ell\|_{\dot B^{\frac d2-1}_{2,1}}\|a^\ell\|_{\dot B^{\frac d2+1}_{2,1}}+\|a^h\|^2_{\dot B^{\frac d2}_{2,1}}\Big)\mathrm{d}t\\
&\lesssim \kappa^{-1}X_d^2(T),
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&\int_0^T\sum_{j\in\mathbb{Z}}2^{j(\frac d2-1)}\|[\dot\Delta_j,v+U]\nabla\mathcal{Q}v\|_{L^2}\mathrm{d}t\\
&\lesssim \int_0^T\|\nabla(v+U)\|_{\dot B^{\frac d2}_{2,1}}\|\mathcal{Q}v\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t,
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&\|aU_t\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}+\|av_t\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\lesssim \|U_t,\mathcal{P}v_t,\mathcal{Q}v_t+\nabla a\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})}+\|a\|^2_{L^2_T(\dot B^{\frac d2}_{2,1})}\\
&\lesssim \kappa^{-1}X_d(T)\big(X_d(T)+Y_d(T)+Z_d(T)\big),
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
&\Big\|\frac12\nabla(|B+c|^2)-(B+c)\cdot\nabla(B+c)\Big\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\lesssim \|B,c\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|B,c\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}\\
&\lesssim \nu^{-1}\big(Z_d^2(T)+Y_d^2(T)\big).
\end{aligned}
\tag{52}
$$

Therefore, collecting (45)–(52), we have

$$
\begin{aligned}
X_d(T)\le C\exp\Big(&C(1+\|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})})\|\mathcal{P}v,\mathcal{Q}v,U\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}\Big)\\
&\times\Big\{X_d(0)+C\big(1+\kappa^{-1}X_d(T)\big)(\mu^{-1}+\nu^{-1})\big(Y_d^2(T)+Z_d^2(T)\big)\\
&+C\big(\kappa^{-2}X_d^2(T)+\kappa^{-1}X_d(T)\big)\big(X_d(T)+Y_d(T)+Z_d(T)\big)\Big\}.
\end{aligned}
\tag{53}
$$

Step 3. Closure of the a priori estimates.

By (17) and (18), we can deduce that

$$
\begin{aligned}
(54)\qquad &(1+\|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})})\|\mathcal{P}v,\mathcal{Q}v,U,c\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}\\
&\le (1+\kappa^{-1}D)(\kappa^{-1}D+\mu^{-1}M+\mu^{-1}\delta+\nu^{-1}\delta)\\
&\le 2(1+\mu^{-1}+\nu^{-1})(M+1)
\end{aligned}
$$

and

$$
\begin{aligned}
(55)\qquad &(\mu^{-1}+\nu^{-1})\|\mathcal{Q}v,U,B\|^2_{L^1_T(\dot B^{\frac d2}_{2,1})}\\
&\le (\mu^{-1}+\nu^{-1})\big(\kappa^{-1}D^2+(\mu^{-1}+\nu^{-1})M^2\big)\\
&\le (1+\mu^{-2}+\nu^{-2})(M+1)^2.
\end{aligned}
$$

Using (18), (36), (53)–(55), for a suitable large (universal) constant C, we have

$$
\begin{aligned}
(56)\quad Y_d(T)&\le C\exp\big(C(1+\mu^{-2}+\nu^{-2})(M+1)^2\big)\\
&\quad\times\big(\kappa^{-1}D^2+\kappa^{-\frac12}(\mu^{-\frac12}+\nu^{-\frac12})DM+\kappa^{-1}\mu^{-1}DM^2+\kappa^{-1}\mu^{-1}D^2\delta\big)\\
&\le C\exp\big(C(1+\mu^{-2}+\nu^{-2})(M+1)^2\big)\big(\kappa^{-1}D^2+\kappa^{-\frac12}D\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(57)\qquad X_d(T)&\le C\exp\big(C(1+\mu^{-1}+\nu^{-1})(M+1)\big)\\
&\quad\times\big(X_d(0)+(\mu^{-1}+\nu^{-1})(1+M^2)+M+1\big)\\
&\le C\exp\big(C(1+\mu^{-1}+\nu^{-1})(M+1)^2\big)\big(X_d(0)+1\big).
\end{aligned}
$$

So it is natural to take first

$$
(58)\qquad D:=C\exp\big(C(1+\mu^{-2}+\nu^{-2})(M+1)^2\big)\big(X_d(0)+1\big)
$$

and then to set

$$
(59)\qquad \delta:=C\exp\big(2C(1+\mu^{-2}+\nu^{-2})(M+1)^2\big)\big(\kappa^{-1}D^2+\kappa^{-\frac12}D\big)
$$

for a suitable large (universal) constant C.

It is easy to prove that $\|a(t,\cdot)\|_{L^\infty}\le C\kappa^{-1}D$.

Therefore, if we make the assumption that κ is large enough such that

$$
\kappa^{-1}D\ll 1\quad\text{and}\quad \delta\big(\mu^{-1}+\nu^{-1}+1\big)\le\frac12,
$$

then we deduce from (56)–(59) that the desired result (18).

4. Proof of Theorem 1.1

Theorem 3.1 implies that there exists a unique maximal solution (a,u,b) to (11) which belongs to $\widetilde{\mathcal{C}}([0,T];\dot B^{\frac d2}_{2,1})\times(\widetilde{\mathcal{C}}([0,T];\dot B^{\frac d2-1}_{2,1})\cap L^1_T\dot B^{\frac d2+1}_{2,1})^{2d}$ on some time interval $[0,T^*)$. With the a priori estimates (17) and (18) at our hand, one can conclude that $T^* = +\infty$. In fact, let us assume (by contradiction) that $T^* < \infty$. Applying (17) and (18) for all $t < T^*$ yields

$$\|a\|_{L^\infty_{T^*}(\dot{B}^{\frac{d}{2}}_{2,1})} + \|u, b\|_{L^\infty_{T^*}(\dot{B}^{\frac{d}{2}-1}_{2,1})} \leq C < \infty. \tag{60}$$

Then, for all $t_0 \in [0, T^*)$, one can solve (11) starting with data (a_0, u_0, b_0) at time $t = t_0$ and get a solution according to Theorem 3.1 on the interval $[t_0, T + t_0]$ with T independent of t_0. Choosing $t_0 > T^* - T$ thus shows that the solution can be continued beyond T^*, a contradiction.

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Jinlu Li
School of Mathematics and Computer Sciences
Gannan Normal University
Ganzhou 341000, P. R. China
Email address: lijinlu@gnnu.edu.cn

Yanghai Yu
School of Mathematics and Statistics
Anhui Normal University
Wuhu, Anhui 241002, P. R. China
Email address: yuyanghai214@sina.com

Weipeng Zhu
School of Mathematics and Big Data
Foshan University
Foshan, Guangdong 528000, P. R. China
Email address: mathzwp2010@163.com