Bull. Korean Math. Soc. **58** (2021), No. 6, pp. 1539–1561
https://doi.org/10.4134/BKMS.b210087
pISSN: 1015-8634 / eISSN: 2234-3016

PERELMAN TYPE ENTROPY FORMULAE AND DIFFERENTIAL HARNACK ESTIMATES FOR WEIGHTED DOUBLY NONLINEAR DIFFUSION EQUATIONS UNDER CURVATURE DIMENSION CONDITION

Yu-Zhao Wang

Abstract. We prove Perelman type $\mathcal{W}$-entropy formulae and differential Harnack estimates for positive solutions to weighed doubly nonlinear diffusion equation on weighted Riemannian manifolds with $CD(-K,m)$ condition for some $K \geq 0$ and $m \geq n$, which are also new for the non-weighted case. As applications, we derive some Harnack inequalities.

1. Introduction and main results

A weighted Riemannian manifold $(M, g, d\mu)$ is an n-dimensional Riemannian manifold (M, g) with a smooth measure $d\mu := e^{-f}\, dV$, where f is a smooth function on M, dV is the volume measure of M. The weighted Riemannian manifold carries a natural analogous Ricci curvature, that is, the m-Bakry-Émery Ricci curvature, which is defined by

$$\mathrm{Ric}_f^m := \mathrm{Ric} + \nabla\nabla f - \frac{\nabla f \otimes \nabla f}{m-n}. \tag{1.1}$$

In particular, when $m = \infty$, $\mathrm{Ric}_f^\infty = \mathrm{Ric}_f := \mathrm{Ric} + \nabla\nabla f$ is the classical Bakry-Émery Ricci curvature, which is introduced in the study of diffusion processes and functional inequalities (see [1] and also [2] for a comprehensive introduction), and then it is extensively investigated in the theory of the Ricci flow and optimal transport theory, when $m = n$ if and only if f is a constant function. There is also a natural analogous Laplacian, namely, the so-called weighted Laplacian (also called the f-Laplacian, drifting Laplacian or Witten Laplacian in the literature), denoted by $\Delta_f = \Delta - \nabla f \cdot \nabla$, which is a self-adjoint operator in $L^2(M, d\mu)$.

Received January 28, 2021; Accepted July 6, 2021.
2010 *Mathematics Subject Classification.* Primary 58J35, 35K92, 35K55.
Key words and phrases. Weighted doubly nonlinear diffusion equations, Perelman type entropy formula, differential Harnack estimates, Bakry-Émery Ricci curvature, curvature dimension condition.
This work was financially supported by NSFC No. 11701347.

There is an enhanced Bochner formula with respect to Δ_f (see p. 383 in Villani's book [27]):

$$\begin{aligned}
&\frac{1}{2}\Delta_f|\nabla\psi|^2 - \nabla\psi\cdot\nabla\Delta_f\psi\\
&= |\nabla\nabla\psi|^2 + \mathrm{Ric}_f(\nabla\psi,\nabla\psi)\\
&= \frac{(\Delta_f\psi)^2}{m} + \mathrm{Ric}_f^m(\nabla\psi,\nabla\psi) + \left|\nabla\nabla\psi - \Big(\frac{\Delta\psi}{n}\Big)g\right|^2\\
&\quad + \left(\frac{1}{n}-\frac{1}{m}\right)\left(\Delta\psi + \frac{n}{m-n}\nabla f\cdot\nabla\psi\right)^2.
\end{aligned} \tag{1.2}$$

For convenience, one can reformulate the Bochner formula in terms of the Bakry-Émery's Γ_2 formalism. For a given operator Δ_f, define the associated Γ operator by

$$\Gamma(\varphi,\psi) := \frac{1}{2}[\Delta_f(\varphi\psi) - \varphi\Delta_f\psi - \psi\Delta_f\varphi] = \nabla\varphi\cdot\nabla\psi.$$

The Γ_2 operator is defined by

$$\Gamma_2(\varphi,\psi) := \frac{1}{2}[\Delta_f\Gamma(\varphi,\psi) - \Gamma(\varphi,\Delta_f\psi) - \Gamma(\psi,\Delta_f\varphi)].$$

In particular,

$$\Gamma_2(\psi) := \Gamma_2(\psi,\psi) = \frac{1}{2}\Delta_f|\nabla\psi|^2 - \nabla\psi\cdot\nabla\Delta_f\psi.$$

By (1.2), when $\mathrm{Ric}_f^m \geq -Kg$ and $m > n$ or $m < 0$, we have

$$\Gamma_2(\psi) \geq \frac{(\Delta_f\psi)^2}{m} - K|\nabla\psi|^2. \tag{1.3}$$

If (1.3) is valid, we say that $(M, g, d\mu)$ satisfies the curvature-dimensional condition $CD(-K, m)$, which is equivalent to the m-Bakry-Emery Ricci curvature bounded below by $-K$.

In recent years, people study geometric analysis problems on the weighted Riemannian manifolds, for instance, gradient estimates and Liouville theorems for symmetric diffusion operators Δ_f [10], some comparison geometry for the Bakry-Émery Ricci tensor [32], eigenvalue estimates [5] and splitting theorems [4] etc.. In his 2002 seminal paper [24], Perelman introduced the $\mathcal{W}$-entropy

$$\mathcal{W}(g,f,\tau) := \int_M \Big(\tau(R+|\nabla f|^2) + f - n\Big)u\,dV$$

and proved its monotonicity

$$\frac{d}{dt}\mathcal{W}(g,f,\tau) = 2\tau\int_M \left|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\right|^2 u\,dV \geq 0, \tag{1.4}$$

where $u = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ satisfies the conjugate heat equation coupled with Ricci flow,

$$\partial_t g = -2\mathrm{Ric}, \quad \partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \quad \partial_t\tau = -1.$$

Later L. Ni [22,23] obtained the $\mathcal{W}$-entropy monotonicity formula for the linear heat equation on Riemannian manifolds with nonnegative Ricci curvature.

$$\frac{d}{dt}\mathcal{W}(f,t) = -2t\int_M \left(\left|\nabla_i\nabla_j f - \frac{1}{2t}g_{ij}\right|^2 + R_{ij}f_if_j\right)u\,dV, \tag{1.5}$$

where $u = (4\pi t)^{-\frac{n}{2}}e^{-f}$ is a positive solution to the heat equation $\partial_t u = \Delta u$ with $\int_M u\,dV = 1$ and $\mathcal{W}(f,t)$ is defined by

$$\mathcal{W}(f,t) := \int_M \Big(t|\nabla f|^2 + f - n\Big)u\,dV.$$

In [11], X.-D. Li established a Perelman type $\mathcal{W}$-entropy formula for the weighted linear heat equation on the weighted Riemannian manifolds with $CD(0,m)$ condition.

Theorem A (Li [11])**.** *Let $(M^n, g, d\mu)$ be a weighted Riemannian manifold and u be a positive solution to the weighted heat equation $\partial_t u = \Delta_f u$ and $\int_M u d\mu = 1$. Define the weighted $\mathcal{W}$-entorpy*

$$\mathcal{W}_f(v,t) := \int_M \Big(t|\nabla v|^2 + v - m\Big)u\ d\mu, \quad u = \frac{e^{-v}}{(4\pi t)^{m/2}},$$

then we have

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_f(v,t) = &\ -2t\int_M \left(\left|\nabla_i\nabla_j v - \frac{1}{2t}g_{ij}\right|^2 + \mathrm{Ric}_f^m(\nabla v,\nabla v)\right)u\,d\mu\\ &- \frac{2t}{m-n}\int_M \left(\nabla f\cdot\nabla v + \frac{m-n}{2t}\right)^2 u\,d\mu.\end{aligned} \tag{1.6}$$

In particular, if $CD(0,m)$ condtion holds, then $\mathcal{W}_f(v,t)$ is monotone decreasing along the weighed heat equation. When $m = n$, $f = const.$, (1.6) *reduces to* (1.5)*.*

In [12, 13], when $n \le m \in \mathbb{N}$, S. Li and X.-D. Li gave a direct proof and natural geometric interpretation of the $\mathcal{W}$-entropy formula (1.6) by using the warped product approach. Moveover, they extend the $\mathcal{W}$-entropy formula to the weighted heat equation on the weighted compact Riemannian manifolds with time dependent metrics and potentials. More recently, in [18], they introduced Perelman's $\mathcal{W}$-entropy along geodesic flow on the Wasserstein space over Riemannian manifolds. For further related study, see [14–17].

It is natural to study the entropy formulae for nonlinear equations, the authors obtained the Perelman type entropy formulae for p-heat equation [9] and porous medium equation [21] on Riemannian manifold with nonnegative Ricci curvature. Combining the analogous methods in [9], [21] and [11], Wang-Yang-Chen [31] and Huang-Li [8] proved the entropy formulae for the weighted p-Laplacian heat equation and weighted porous medium equation with $CD(0,m)$ condition, respectively. In [30], the authors got the $\mathcal{W}$-entropy formula for positive solutions to the doubly nonlinear diffusion equation on the closed Riemannian manifold with nonnegative Ricci curvature.

Theorem B (Wang-Chen [30])**.** *Let (M^n, g) be a closed Riemannian manifold and u be a positive solution to the doubly nonlinear diffusion equation*

$$\partial_t u = \Delta_p(u^\gamma). \tag{1.7}$$

Set $v = \frac{\gamma}{b}u^b$ and define Perelman-type $\mathcal{W}$-entropy

$$\mathcal{W}_p(v,t) := t^{a+1}\int_M \left((b+1)\frac{|\nabla v|^p}{v} - \frac{a+1}{t}\right) vu\, dV.$$

Then we have

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_p(v,t) = & -pbt^{a+1}\int_M \left(\left|w^{\frac{p}{2}-1}\nabla_i\nabla_j v + \frac{a}{nbt}a_{ij}\right|_A^2 + w^{p-2}R_{ij}v_iv_j\right) vu\, dV \\ & - pt^{a+1}\int_M \left(b\Delta_p v + \frac{a}{t}\right)^2 vu\, dV,\end{aligned} \tag{1.8}$$

where $b = \gamma - \frac{1}{p-1}$, $a = \frac{nb}{nb(p-1)+p}$, $w = |\nabla v|^2$, $A^{ij} = g^{ij} + (p-2)\frac{v^iv^j}{w}$ and a_{ij} is the inverse of A^{ij}.

In this paper, we focus on the weighted doubly nonlinear diffusion equation (WDNE for short)

$$\partial_t u = \Delta_{p,f}(u^\gamma) := \mathrm{div}_f\left(|\nabla u^\gamma|^{p-2}\nabla u^\gamma\right), \tag{1.9}$$

where $\gamma > 0$, $p > 1$, $f \in C^\infty(M)$, $\Delta_{p,f}$ and $\mathrm{div}_f := e^f \mathrm{div}(e^{-f}\cdot)$ denote the weighted p-Laplacian operator and weighted divergence operator, respectively. WDNE has the rich physical background and appears in several models, including non-Newtonian fluids, glaciology and turbulent flows in porous media. From a mathematical point of view such as in [26], it can be viewed as a generalization of the weighted heat equation ($p = 2, \gamma = 1$), the weighted porous medium equation ($p = 2, \gamma > 1$), fast diffusion equation ($p = 2, \gamma < 1$) and the weighted parabolic p-Laplacian equation ($\gamma = 1$). Taking the pressure transform

$$v(u) := \frac{\gamma}{b}u^b, \quad b = \gamma - \frac{1}{p-1}, \tag{1.10}$$

then the equation (1.9) satisfies

$$\partial_t v = bv\Delta_{p,f}v + |\nabla v|^p. \tag{1.11}$$

Inspired by the previous work [8,9,11,21,28,30,31], the first result in this paper is the Perelman type $\mathcal{W}$-entropy formula for the weighted doubly nonlinear diffusion equation on closed weighted Riemannian manifolds with $CD(-K, m)$ condition for $K \geq 0$ and $m \geq n$.

Theorem 1.1. *Let $(M, g, d\mu)$ be a closed weighted Riemannian manifold with $CD(-K, m)$ condition for $K > 0$ and $m > n$. Let u be a positive solution to (1.9) and v satisfy (1.11). Define the weighted Perelman-type $\mathcal{W}$-entropy*

$$\mathcal{W}_K(v,t) := \sigma_K\beta_K\int_M \left[(b+1)\frac{|\nabla v|^p}{v} - \left(\frac{1}{\beta_K} + \frac{\dot\sigma_K}{\sigma_K}\right)\right] vu\, d\mu, \; b = \gamma - \frac{1}{p-1} > 0. \tag{1.12}$$

Then we have

(1.13)

$$\begin{aligned}&\frac{d}{dt}\mathcal{W}_K(v,t)\\ \le&\ -pb\sigma_K\beta_K\int_M\left[\left||\nabla v|^{p-2}\nabla_i\nabla_j v+\frac{\eta_K}{mb}a_{ij}\right|_A^2+|\nabla v|^{2p-4}(\mathrm{Ric}_f^m+Kg)(\nabla v,\nabla v)\right]vu\,d\mu\\ &-p\sigma_K\beta_K\int_M\left[(b\Delta_{p,f}v+\eta_K)^2+\frac{b}{m-n}\left(|\nabla v|^{p-2}\nabla v\cdot\nabla f-(m-n)\frac{\eta_K}{mb}\right)^2\right]vu\,d\mu,\end{aligned}$$

where $\bar{a}=\frac{mb}{mb(p-1)+p}$ *and* $A^{ij}=g^{ij}+(p-2)\frac{v^iv^j}{|\nabla v|^2}$ *is the inverse matrix of* a_{ij}, $D=\frac{\overline{K}}{b+1}$, $\overline{K}=\frac{pbK}{2}\sup_{M\times[0,T)}(|\nabla v|^{p-2}v)$, $\sigma_K=(e^{Dt}\sinh(Dt))^{\bar{a}}$, $\beta_K=\frac{\sinh(2Dt)}{2D}$ *and* $\eta_K=\frac{2\bar{a}D}{1-e^{-2Dt}}$. *Moreover, if* $CD(-K,m)$ *holds, then* $\mathcal{W}_K(v,t)$ *is monotone decreasing along WDNE* (1.9).

Corollary 1.2. *Let* $K=0$, $\sigma_0=t^{\bar{a}}$, $\beta_0=t$, $\eta_0=\frac{\bar{a}}{t}$, *and*

$$\mathcal{W}_0(v,t)=t^{\bar{a}+1}\int_M\left((b+1)\frac{|\nabla v|^p}{v}-\frac{\bar{a}+1}{t}\right)vu\,d\mu. \tag{1.14}$$

Then we get

(1.15)

$$\begin{aligned}&\frac{d}{dt}\mathcal{W}_0(v,t)\\ =&\ -pbt^{\bar{a}+1}\int_M\left[\left|\nabla v|^{p-2}\nabla_i\nabla_j v+\frac{\bar{a}}{mbt}a_{ij}\right|_A^2+|\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v,\nabla v)\right]vu\,d\mu\\ &-pt^{\bar{a}+1}\int_M\left[\left(b\Delta_{p,f}v+\frac{\bar{a}}{t}\right)^2+\frac{b}{m-n}\left(|\nabla v|^{p-2}\nabla v\cdot\nabla f-(m-n)\frac{\bar{a}}{mbt}\right)^2\right]vu\,d\mu.\end{aligned}$$

Remark 1.3. When $K>0$, $m=n$ and $f=const.$, $\mathcal{W}$-entropy formula (1.13) is new even for doubly nonlinear diffusion equation (1.7) on the closed Riemannian manifold. When $K=0$, $m=n$ and $f=const.$, $\mathcal{W}$-entropy (1.15) is just (1.8).

Theorem 1.4. *Let* (M^n,g) *be a closed n-dimensional Riemannian manifold with Ricci curvature bounded below, i.e.,* $\mathrm{Ric}\ge -Kg$, $K\ge 0$. *Let u be a smooth positive solution to* (1.7) *and* $v=\frac{\gamma}{b}u^b$. *For any* $b=\gamma-\frac{1}{p-1}>0$, *define the Perelman-type* $\mathcal{W}$*-entropy*

$$\mathcal{W}_K(v,t):=\sigma_K\beta_K\int_M\left[(b+1)\frac{|\nabla v|^p}{v}-\left(\frac{1}{\beta_K}+\frac{\dot{\sigma}_K}{\sigma_K}\right)\right]vu\,dV. \tag{1.16}$$

Then we have

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_K(v,t)\le&\ -pb\sigma_K\beta_K\int_M\left||\nabla v|^{p-2}\nabla_i\nabla_j v+\frac{\eta_K}{nb}a_{ij}\right|_A^2 vu\,dV\\ &-p\sigma_K\beta_K\int_M\left(b|\nabla v|^{2p-4}(\mathrm{Ric}+Kg)(\nabla v,\nabla v)+(b\Delta_p v+\eta_K\right)^2 vu\,dV,\end{aligned} \tag{1.17}$$

where $a = \frac{nb}{nb(p-1)+p}$, $D = \frac{\overline{K}}{b+1}$, $\overline{K} = \frac{pbK}{2}\sup_{M\times[0,T)}|\nabla v|^{p-2}v$, $\sigma_K = (e^{Dt}\sinh(Dt))^a$, $\beta_K = \frac{\sinh(2Dt)}{2D}$ *and* $\eta_K = \frac{2aD}{1-e^{-2Dt}}$. *Moreover, if* $\mathrm{Ric} \geq -K$ *for* $K \geq 0$*, then* $\mathcal{W}_K(v,t)$ *is monotone decreasing along the doubly nonlinear diffusion equation* (1.7)*.*

Remark 1.5. When $K > 0$ and $\gamma = 1$, our results are even new for the weighted parabolic p-Laplacian equation on the weighted Riemannian manifolds. See details in Corollary 3.2. When $K > 0$ and $p = 2$, $\mathcal{W}$-entropy formulae have been obtained by the author in [29].

In the second part of this paper, we study the differential Harnack inequality for WDNE on wighted Riemannian manifold. In the classic paper [20], Li-Yau proved differential Harnack inequality (Li-Yau estimate) for positive solution to the heat equation on an n-dimensional complete Riemannian manifold with $\mathrm{Ric} \geq -Kg$, where K is a positive constant, that is, for all $\alpha > 1$

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{\alpha^2}{2(\alpha-1)}nK + \alpha^2\frac{n}{2t}.$$

In 1993, Hamilton [6] derived another gradient estimate

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{u_t}{u} \leq e^{4Kt}\frac{n}{2t}.$$

In 2011, Li-Xu [19] generalized Li-Yau type estimate,

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}\right)\frac{u_t}{u} \leq \frac{nK}{2}\Big(1 + \coth(Kt)\Big).$$

Recently, B. Qian [25] extended Li-Yau and Hamilton type estimates under some proper assumptions of $\alpha(t)$ and $\varphi(t)$.

$$\frac{|\nabla u|^2}{u^2} - \alpha(t)\frac{u_t}{u} \leq \varphi(t). \tag{1.18}$$

It is natural to prove differential Harnack esimtates for nonlinear equations. In [7, 21], various differential Harnack estimates for porous medium equation on Riemannian manifolds with Ricci curvature bounded below are derived. In [30], the author obtained a sharp Li-Yau estimate for doubly nonlinear diffusion equation (1.7) on compact Riemannian manifold with nonnegative Ricci curvature,

$$\frac{|\nabla v|^p}{v} - \frac{v_t}{v} \leq \frac{a}{t}, \tag{1.19}$$

where $v = \frac{\gamma}{b}u^b$, $b = \gamma - \frac{1}{p-1} > 0$ and $a = \frac{nb}{nb(p-1)+p}$. In recent papers [3] and [33], the authors got Li-Yau type and elliptic gradient estimates for doubly nonlinear equations on Riemannian manifold with Ricci curvature bounded below, respectively.

Motivated by previous works, especially in [25] by B. Qian, we obtain various global differential Harnack estimates for WDNE on closed weighted Riemanian manifolds with curvature dimensional condition $CD(-K, m)$.

Now we make two assumptions on any positive function $\sigma(t) \in C^1(M)$ (see [25]).

(A1) For any $t > 0$, $\sigma(t) > 0$, $\sigma'(t) > 0$, $\lim_{t \to 0} \sigma(t) = 0$ and $\lim_{t \to 0} \frac{\sigma(t)}{\sigma'(t)} = 0$;

(A2) For any $T > 0$, $\frac{(\sigma')^2}{\sigma}$ is continuous and integrable on the interval $[0, T)$.

Theorem 1.6. *Let $(M^n, g, d\mu)$ be a closed weighted Riemannian manifold with $CD(-K, m)$ condition for $K \geq 0$. Let $v(x,t)$ be a positive solution to equation* (1.11)*. For any $b > 0$, we have*

$$\frac{|\nabla v|^p}{v} - \alpha(t) \frac{v_t}{v} \leq \varphi(t). \tag{1.20}$$

Here

$$\begin{aligned} \alpha(t) &= 1 + \frac{2\overline{K}}{\sigma} \int_0^t \sigma(s) ds, \\ \varphi(t) &= \overline{K}\bar{a} + \frac{\overline{K}^2 \bar{a}}{\sigma} \int_0^t \sigma(s) ds + \frac{\bar{a}}{4\sigma} \int_0^t \frac{(\sigma'(s))^2}{\sigma(s)} ds, \end{aligned} \tag{1.21}$$

and $\sigma(t)$ is any function satisfying the assumptions (A1) *and* (A2)*, $\bar{a} = \frac{mb}{mb(p-1)+p}$, $\overline{K} = \frac{pbK}{2} \sup_{M \times (0,T]} (v|\nabla v|^{p-2})$.*

By choosing different $\sigma(t)$, we can obtain various differential Harnack inequalities, which are also new for the weighted doubly nonlinear diffusion equation.

Corollary 1.7. *Let $(M, g, d\mu)$ be a closed weighted Riemannian manifold with $CD(-K, m)$ condition. Let v be a smooth positive solution to* (1.11)*. Then we have the following estimates*:

(1) *Linearized Li-Xu-Qian type*: $\sigma(t) = t^\beta$ *for* $\beta > 1$,

$$\frac{|\nabla v|^p}{v} - \left(1 + \frac{2\overline{K}t}{\beta + 1}\right) \frac{v_t}{v} \leq \bar{a}\Big(\frac{\beta^2}{4(\beta - 1)} \frac{1}{t} + \overline{K} + \frac{\overline{K}^2 t}{\beta + 1}\Big). \tag{1.22}$$

In particular, $\beta = 2$, $\sigma(t) = t^2$,

$$\frac{|\nabla v|^p}{v} - \left(1 + \frac{2}{3}\overline{K}t\right) \frac{v_t}{v} \leq \bar{a}\Big(\frac{1}{t} + \overline{K} + \frac{1}{3}\overline{K}^2 t\Big). \tag{1.23}$$

(2) *Li-Xu type*: $\sigma(t) = \sinh^2(\overline{K}t)$

$$\frac{|\nabla v|^p}{v} - \left(1 + \frac{\sinh(\overline{K}t)\cosh(\overline{K}t) - \overline{K}t}{\sinh^2(\overline{K}t)}\right) \frac{v_t}{v} \leq \bar{a}\overline{K}\Big(1 + \coth(\overline{K}t)\Big). \tag{1.24}$$

(3) *Baudoin-Vatamanelu-Qian type*: $\sigma(t) = e^{-\frac{2\overline{K}t}{\beta+1}}(1-e^{-\frac{2\overline{K}t}{\beta+1}})^{\beta}$ *with* $\beta > 1$,

$$\frac{|\nabla v|^p}{v} - e^{\frac{2\overline{K}t}{\beta+1}}\frac{v_t}{v} \leq \frac{\beta^2\overline{K}\bar{a}}{2(\beta^2-1)}\frac{e^{\frac{4\overline{K}t}{\beta+1}}}{e^{\frac{2\overline{K}t}{\beta+1}}-1}. \tag{1.25}$$

In particular, $\beta = 2$, $\sigma(t) = e^{-\frac{2\overline{K}t}{3}}(1-e^{-\frac{2\overline{K}t}{3}})^2$

$$\frac{|\nabla v|^p}{v} - e^{\frac{2\overline{K}t}{3}}\frac{v_t}{v} \leq \frac{2\overline{K}\bar{a}}{3}\frac{e^{\frac{4\overline{K}t}{3}}}{e^{\frac{2\overline{K}t}{3}}-1}. \tag{1.26}$$

Remark 1.8. (1) When $p = 2$, $\gamma = 1$ and $f = const.$, the results in Theorem 1.6 and Corollary 1.7 reduce to the linear case in [19, 25].

(2) When $p = 2$ and $\gamma > 1$, the estimate (1.20) reduces the case of the weighted porous medium equation, which has been proved in [29] by the author.

(3) Theorem 1.6 is new for non-weighted case. In [3], the authors obtained Li-Xu type estimates for the doubly nonlinear diffusion equation, but when $K = 0$, their estimates are not optimal.

Theorem 1.9 (Hamilton type estimate)**.** *Let* $(M^n, g, d\mu)$ *be a closed weighted Riemannian manifold satisfying* $CD(-K, m)$ *condition for* $K > 0$*. Let* $v(x,t)$ *be a positive solution to equation* (1.11)*, for any* $b > 0$*, we have*

$$\frac{|\nabla v|^p}{v} - e^{2\overline{K}t}\frac{v_t}{v} \leq e^{4\overline{K}t}\frac{\bar{a}}{t}, \tag{1.27}$$

where $\overline{K} = \frac{pbK}{2}\sup_{M\times[0,T)}(v|\nabla v|^{p-2})$.

Integrating on minimizing path for estimates in Theorem 1.6 and Corollary 1.7, we can prove the Harnack inequalities for positive solutions to WDNE.

Corollary 1.10. *For any* (x_1, t_1) *and* (x_2, t_2) *with* $0 < t_1 \leq t_2 < T$*, we have*

$$\begin{aligned} &v(x_1,t_1) - v(x_2,t_2) \\ &\leq v_{\max}\int_{t_1}^{t_2}\frac{\varphi(t)}{\alpha(t)}dt + \frac{p-1}{p^{p^*}}\frac{d(x_2,x_1)^{p^*}}{(t_2-t_1)^{p^*}}\int_{t_1}^{t_2}\alpha^{\frac{1}{p-1}}(t)dt \end{aligned} \tag{1.28}$$

and

$$\begin{aligned} &\frac{v(x_1,t_1)}{v(x_2,t_2)} \\ &\leq \exp\left(\int_{t_1}^{t_2}\frac{\varphi(t)}{\alpha(t)}dt + \frac{p-1}{p^{p^*}}\frac{1}{v_{\max}}\frac{d(x_2,x_1)^{p^*}}{(t_2-t_1)^{p^*}}\int_{t_1}^{t_2}\alpha^{\frac{1}{p-1}}(t)dt\right), \end{aligned} \tag{1.29}$$

where $p^* = \frac{p}{p-1}$ *and* $v_{\max} = \sup_{M\times[0,T)} v$.

When $(M, g, d\mu)$ has nonnegative m-Bakry-Emery Ricci curvature, i.e., $CD(0, m)$-condition, we can prove an optimal Li-Yau type estimates, which is a generalization for the case $m = n$ in [30].

Theorem 1.11 (Optimal Li-Yau type estimate). *Let $(M, g, d\mu)$ be a closed weighted Riemannian manifold with $CD(0,m)$-condition. Let v be a smooth solution to* (1.11). *Then we have*

(1) *for any $(p-1)\gamma > 1$,*

$$\frac{|\nabla v|^p}{v} - \frac{v_t}{v} \le \frac{\bar{a}}{t}; \tag{1.30}$$

(2) *for any $1 - \frac{p}{m} < (p-1)\gamma < 1$,*

$$\frac{|\nabla v|^p}{v} - \frac{v_t}{v} \ge \frac{\bar{a}}{t}. \tag{1.31}$$

Moreover, these estimates are optimal, i.e., when u is a fundamental solution to (1.9) *on $\mathbb{R}^m$, equality holds in* (1.30) *and* (1.31).

This paper is organized as follows. In Section 2, we derive some useful evolution equations by the weighted p-Bochner formula. In Section 3 we prove the $\mathcal{W}$-entropy monotonicity formula, i.e., Theorem 1.1. In Section 4, we obtain Qian type, Hamilton type and optimal Li-Yau type estimates, i.e., Theorem 1.6, Theorem 1.9 and Theorem 1.11. Finally, Harnack inequalities are derived as applications.

2. Nonlinear Bochner formulae and evolution equations

Let $(M, g, d\mu)$ be a closed weighted Riemannian manifold. Suppose u is a smooth positive solution to (1.9) and $v = \frac{\gamma}{b}u^b$ satisfies (1.11). Assume that $w := |\nabla v|^2 > 0$ on a region of M and define the linearized operator of weighted p-Laplacian at point v

$$\mathcal{L}_f(\psi) := \mathrm{div}_f(w^{\frac{p}{2}-1}A(\nabla\psi)) \tag{2.1}$$

and its parabolic operator

$$\Box_f := \partial_t - bv\mathcal{L}_f, \tag{2.2}$$

where A is a tensor and defined by

$$A = g + (p-2)\frac{\nabla v \otimes \nabla v}{w}.$$

Thus,

$$\Box_f v = |\nabla v|^p - (p-2)bv\Delta_{p,f}v. \tag{2.3}$$

Lemma 2.1. *Let β be a constant. Then we have the following evolution equations*:

$$\Box_f v_t = bv_t\Delta_{p,f}v + pw^{\frac{p}{2}-1}\langle\nabla v, \nabla v_t\rangle, \tag{2.4}$$

$$\Box_f v^\beta = \beta\big(1 - b(p-1)(\beta-1)\big)v^{\beta-1}w^{\frac{p}{2}} - \beta b(p-2)v^\beta\Delta_{p,f}v, \tag{2.5}$$

$$\Box_f w = pw^{\frac{p}{2}-1}\langle\nabla v, \nabla w\rangle + 2bw\Delta_{p,f}v - (\frac{p}{2}-1)bvw^{\frac{p}{2}-2}|\nabla w|^2 \tag{2.6}$$

$$- 2bvw^{\frac{p}{2}-1}\Big(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big),$$

$$\Box_f(w^{\frac{p}{2}}) = pw^{\frac{p}{2}-1}\langle\nabla v, \nabla w^{\frac{p}{2}}\rangle - pbvw^{p-2}\Big(|\nabla\nabla v|^2_A + \mathrm{Ric}_f(\nabla v, \nabla v)\Big) + pbw^{\frac{p}{2}}\Delta_{p,f}v. \tag{2.7}$$

Proof. By the definition of $\mathcal{L}_f$ and $\Box_f$ in (2.1) and (2.2), a direct calculation implies that

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_{p,f}v) &= \mathrm{div}_f\Big(\big(w^{\frac{p}{2}-1}\nabla v_t + (\frac{p}{2}-1)w^{\frac{p}{2}-2}w_t\nabla v\big)\Big)\\ &= \mathrm{div}_f\Big(w^{\frac{p}{2}-1}\big(\nabla v_t + (p-2)\frac{\langle\nabla v, \nabla v_t\rangle}{w}\nabla v\big)\Big)\\ &= \mathrm{div}_f(|\nabla v|^{p-2}A(\nabla v_t)) = \mathcal{L}_f(v_t),\end{aligned}$$

then

$$\begin{aligned}\Box_f v_t &= \partial_t v_t - bv\mathcal{L}_f(v_t)\\ &= \partial_t v_t - \partial_t(bv\Delta_{p,f}v) + bv_t\Delta_{p,f}v\\ &= pw^{\frac{p}{2}-1}\langle\nabla v, \nabla v_t\rangle + bv_t\Delta_{p,f}v.\end{aligned}$$

There exists a nonlinear Bochner formula for $\mathcal{L}_f$ (see [31]),

$$\mathcal{L}_f w = 2w^{\frac{p}{2}-1}\big(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\big) + 2\langle\nabla v, \nabla\Delta_{p,f}v\rangle + (\frac{p}{2}-1)w^{\frac{p}{2}-2}|\nabla w|^2. \tag{2.8}$$

Hence,

$$\begin{aligned}\Box_f w &= w_t - bv\mathcal{L}_f(w)\\ &= 2\nabla v\cdot\nabla\Big(bv\Delta_{p,f}v + w^{\frac{p}{2}}\Big) - 2bvw^{\frac{p}{2}-1}(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v))\\ &\quad - bv\Big(2\langle\nabla v, \nabla\Delta_{p,f}v\rangle + (\frac{p}{2}-1)|\nabla w|^2 w^{\frac{p}{2}-2}\Big)\\ &= 2bw\Delta_{p,f}v + pw^{\frac{p}{2}-1}\langle\nabla v, \nabla w\rangle - (\frac{p}{2}-1)bv|\nabla w|^2 w^{\frac{p}{2}-2}\\ &\quad - 2bvw^{\frac{p}{2}-1}\Big(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big)\end{aligned}$$

and

$$\begin{aligned}\Box_f(w^{\frac{p}{2}}) &= \partial_t w^{\frac{p}{2}} - bv\mathcal{L}_f(w^{\frac{p}{2}})\\ &= \frac{p}{2}w^{\frac{p}{2}-1}w_t - \frac{p}{2}bve^f\mathrm{div}\Big(e^{-f}(w^{\frac{p}{2}-1})w^{\frac{p}{2}-1}A(\nabla w)\Big)\\ &= \frac{p}{2}w^{\frac{p}{2}-1}\Box_f w - \frac{p}{2}(\frac{p}{2}-1)bvw^{p-3}\nabla w\cdot A(\nabla w)\\ &= \frac{p}{2}w^{\frac{p}{2}-1}\Big(2bw\Delta_{p,f}v + pw^{\frac{p}{2}-1}\langle\nabla v, \nabla w\rangle - (\frac{p}{2}-1)bv|\nabla w|^2 w^{\frac{p}{2}-2}\Big)\\ &\quad - pbvw^{p-2}\Big(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big)\end{aligned}$$

$$
\begin{aligned}
&- \frac{p}{2}(\frac{p}{2}-1)bvw^{p-3}\Big(|\nabla w|^2 + (p-2)\frac{|\nabla v\cdot\nabla w|^2}{w}\Big)\\
&= pbw^{\frac{p}{2}}\Delta_{p,f}v + pw^{\frac{p}{2}-1}\langle\nabla v, \nabla w^{\frac{p}{2}}\rangle\\
&\quad - pbvw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big),
\end{aligned}
$$

where $|\nabla\nabla v|_A^2 = |\nabla\nabla v|^2 + \frac{p-2}{2}\frac{|\nabla w|^2}{w} + \frac{(p-2)^2}{4}\frac{|\nabla v\cdot\nabla w|^2}{w^2}$. □

Proposition 2.2. *For a constant α, let $y = \frac{|\nabla v|^p}{v}$, $z = \frac{v_t}{v}$ and define*

$$F_\alpha := \alpha\frac{v_t}{v} - \frac{|\nabla v|^p}{v} = \alpha z - y.$$

Then we have

$$
\begin{aligned}
\Box_f F_\alpha &= \delta w^{\frac{p}{2}-1}\langle\nabla v, \nabla F_\alpha\rangle + pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\
&\quad + (p-1)\Big(F_1^2 + (\alpha-1)\Big(\frac{v_t}{v}\Big)^2\Big),
\end{aligned}
\tag{2.9}
$$

where $\delta = 2\gamma(p-1)+(p-2)$. In particular, when $\alpha = 1$,

$$F_1 = \frac{v_t}{v} - \frac{|\nabla v|^p}{v} = b\Delta_{p,f}v$$

and

$$
\begin{aligned}
\Box_f(F_1) &= \delta w^{\frac{p}{2}-1}\langle\nabla v,\nabla F_1\rangle + pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\
&\quad + (p-1)F_1^2.
\end{aligned}
\tag{2.10}
$$

Proof. Following the proof in [30], a useful formula for operator $\Box_f$ is

$$\Box_f\left(\frac{h}{g}\right) = \frac{\Box_f h}{g} - \frac{h\Box_f g}{g^2} + 2bvw^{\frac{p}{2}-1}\left\langle A\Big(\nabla\Big(\frac{f}{g}\Big)\Big), \nabla\log g\right\rangle. \tag{2.11}$$

Applying (2.11) and Lemma 2.1, we have

$$
\begin{aligned}
&\Box_f\left(\frac{|\nabla v|^p}{v}\right)\\
&= \frac{1}{v}\Big((2p-2)w^{\frac{p}{2}}F_1 + pw^{\frac{p}{2}-1}\langle\nabla v,\nabla w^{\frac{p}{2}}\rangle\Big)\\
&\quad - pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\
&\quad - \frac{w^p}{v^2} + 2bvw^{\frac{p}{2}-1}\Big\langle A\Big(\nabla\Big(\frac{w^{\frac{p}{2}}}{v}\Big)\Big), \nabla\log v\Big\rangle,
\end{aligned}
\tag{2.12}
$$

and

$$
\begin{aligned}
\Box_f\Big(\frac{v_t}{v}\Big) &= \frac{1}{v}\Big((p-1)F_1 v_t + pw^{\frac{p}{2}-1}\langle\nabla v,\nabla v_t\rangle\Big) - \frac{v_t}{v}\frac{w^{p/2}}{v}\\
&\quad + 2bvw^{\frac{p}{2}-1}\left\langle A\Big(\nabla\Big(\frac{v_t}{v}\Big)\Big), \nabla\log v\right\rangle.
\end{aligned}
\tag{2.13}
$$

Combining (2.12) and (2.13), we get

$$\begin{aligned}\Box_f F_\alpha &= 2bvw^{\frac{p}{2}-1}\left\langle \nabla \log v, A(\nabla F_\alpha)\right\rangle + pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\ &\quad + (p-1)F_1\Big(\alpha\frac{v_t}{v} - \frac{pw^{\frac{p}{2}}}{v}\Big) + \frac{w^p}{v^2} - \alpha\frac{v_t}{v}\frac{w^{p/2}}{v}\\ &\quad + pw^{\frac{p}{2}-1}\Big\langle \nabla\log v, \nabla(\alpha v_t - w^{\frac{p}{2}})\Big\rangle.\end{aligned}$$

A direct calculation implies that

$$\langle \nabla \log v, \nabla(vF_\alpha)\rangle = \langle \nabla v, \nabla F_\alpha\rangle + F_\alpha\frac{|\nabla v|^2}{v}$$

then we have

$$\begin{aligned}\Box_f F_\alpha &= w^{\frac{p}{2}-1}\Big\langle \nabla v, \Big(2bA(\nabla F_\alpha) + p\nabla F_\alpha\Big)\Big\rangle\\ &\quad + pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\ &\quad + (p-1)F_1\Big(\alpha\frac{v_t}{v} - \frac{2w^{\frac{p}{2}}}{v}\Big) + \frac{w^p}{v^2} - \alpha\frac{v_t}{v}\frac{w^{\frac{p}{2}}}{v} + \frac{pw^{\frac{p}{2}}}{v}F_\alpha.\end{aligned}$$

Now we rewrite the last five terms as

$$(p-1)(z-y)(\alpha z-2y) + y^2 - \alpha yz + py(\alpha z - y) = (p-1)\big((y-z)^2 + (\alpha-1)z^2\big).$$

Which gives the desired formula (2.9). When $\alpha = 1$, (2.10) is a direct result of (2.9). □

3. Entropy formulae

Applying the weighted nonlinear Bochner-type formula in Lemma 2.1, we get the following integral formulae.

Lemma 3.1. *Let u and v be positive solutions to* (1.9) *and* (1.11)*. Then we have*

$$\frac{d}{dt}\int_M vu\, d\mu = \int_M bv(\Delta_{p,f}v)u\, d\mu = -(b+1)\int_M |\nabla v|^p u\, d\mu \tag{3.1}$$

and

$$\frac{d^2}{dt^2}\int_M vu\, d\mu = p\int_M\Big(bw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big) + (b\Delta_{p,f}v)^2\Big)vu\, d\mu. \tag{3.2}$$

Proof. Note that $\nabla v = \gamma u^{b-1}\nabla u$. Then

$$\nabla u^\gamma = \nabla v u^{\frac{1}{p-1}}, \quad u\nabla v = bv\nabla u. \tag{3.3}$$

Combining (3.3), (1.9) and (1.11), we have

$$\begin{aligned}\frac{d}{dt}\int_M vu\,d\mu &= \int_M [(bv(\Delta_{p,f}v) + |\nabla v|^p)u + v(\Delta_{p,f}u^\gamma)]\,d\mu\\ &= \int_M bv(\Delta_{p,f}v)u\,d\mu + \int_M |\nabla v|^p u\,d\mu - \int_M \nabla v\cdot\nabla u^\gamma|\nabla u^\gamma|^{p-2}\,d\mu\\ &= \int_M bv(\Delta_{p,f}v)u\,d\mu.\end{aligned}$$

Integration by parts yields that

$$\begin{aligned}\int_M bv(\Delta_{p,f}v)u\,d\mu = \gamma\int_M(\Delta_{p,f}v)u^{b+1}\,d\mu &= -\gamma\int_M \nabla v\cdot\nabla u^{b+1}|\nabla v|^{p-2}\,d\mu\\ &= -(b+1)\int_M |\nabla v|^p u\,d\mu.\end{aligned}$$

Applying (1.9), (1.11) and $\partial_t = \Box_f + bv\mathcal{L}_f$, we have

$$\begin{aligned}\frac{d}{dt}\int_M bv(\Delta_{p,f}v)u\,d\mu =& \int_M \frac{\partial}{\partial t}\Big(b\Delta_{p,f}v\Big)vu\,d\mu + \int_M \frac{\partial}{\partial t}(vu)b\Delta_{p,f}v\,d\mu\\ =& \int_M \Big(\Box_f(b\Delta_{p,f}v) + bv\mathcal{L}_f(b\Delta_{p,f}v)\Big)vu\,d\mu\\ &+ \int_M b\Delta_{p,f}v\Big((\Delta_{p,f}u^\gamma)v + (bv\Delta_{p,f}v + |\nabla v|^p)u\Big)\,d\mu.\end{aligned}$$

By using (2.10), we have

$$\begin{aligned}&\frac{d}{dt}\int_M bv(\Delta_{p,f}v)u\,d\mu\\ =& \int_M p\left[bw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big) + (b\Delta_{p,f}v)^2\right]vu\,d\mu\\ &+ (2\gamma(p-1) + (p-2))b\int_M w^{\frac{p}{2}-1}\left\langle\nabla v,\nabla\Delta_{p,f}v\right\rangle vu\,d\mu\\ &+ b^2\int_M \mathcal{L}_f(\Delta_{p,f}v)v^2u\,d\mu + b\int_M \Delta_{p,f}v\Big((\Delta_{p,f}u^\gamma)v + |\nabla v|^p u\Big)\,d\mu.\end{aligned}$$

Note that $b\nabla(v^2u) = (2b+1)uv\,\nabla v$ and integration by parts,

$$\begin{aligned}b^2\int_M \mathcal{L}_f(\Delta_{p,f}v)v^2u\,d\mu =& -b^2\int_M\langle\nabla(v^2u), A(\nabla\Delta_{p,f}v)\rangle|\nabla v|^{p-2}\,d\mu\\ =& -(2b+1)(p-1)b\int_M\langle\nabla v,\nabla\Delta_{p,f}v\rangle|\nabla v|^{p-2}vu\,d\mu.\end{aligned}$$

Finally, (3.3) and integration by parts again imply that

$$\begin{aligned} & b\int_M v\Delta_{p,f}v(\Delta_{p,f}u^\gamma)\,d\mu \\ = & -b\int_M \langle\nabla(v\Delta_{p,f}v),\nabla u^\gamma\rangle|\nabla u^\gamma|^{p-2}\,d\mu \\ = & -b\int_M \Delta_{p,f}v|\nabla v|^p u\,d\mu - b\int_M\langle\nabla v,\nabla\Delta_{p,f}v\rangle|\nabla v|^{p-2}vu\,d\mu. \end{aligned}$$

Putting these equalities together, we get the desired formula (3.2). □

Proof of Theorem 1.1. Define the Shannon type entropy

$$\mathcal{N}_K(t) := -\sigma_K(t)\int_M vu\,d\mu,$$

where $\sigma_K(t)$ is a function of t, then by the integral formula (3.1), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{N}_K(t) &= -\dot\sigma_K\int_M vu\,d\mu - \sigma_K\int_M b(\Delta_{p,f}v)vu\,d\mu \\ &= -\sigma_K\int_M (b\Delta_{p,f}v + (\log\sigma_K)')\,vu\,d\mu, \end{aligned}\tag{3.4}$$

where $\cdot$ and $'$ denote the time derivative.

By the formulae (3.2), (3.4) and the definition of

$$\overline{K} = \frac{pbK}{2}\sup_{M\times[0,T)}|\nabla v|^{p-2}v,$$

we have

$$\begin{aligned} \frac{d^2}{dt^2}\mathcal{N}_K(t) = & -\sigma_K\frac{d^2}{dt^2}\int_M vu\,d\mu - 2\dot\sigma_K\frac{d}{dt}\int_M vu\,d\mu - \ddot\sigma_K\int_M vu\,d\mu \\ = & -\sigma_K\int_M p\left(bw^{p-2}\left(|\nabla\nabla v|^2_A + \mathrm{Ric}_f(\nabla v,\nabla v)\right) + (b\Delta_{p,f}v)^2\right)vu\,d\mu \\ & + \frac{2\dot\sigma_K}{\sigma_K}\frac{d}{dt}\mathcal{N}_K + \left(\frac{\ddot\sigma_K}{\sigma_K} - \frac{2\dot\sigma_K^2}{\sigma_K^2}\right)\mathcal{N}_K. \end{aligned}\tag{3.5}$$

When $b>0$, $v>0$, we have $\overline{K}>0$, then

$$\begin{aligned} & \frac{d^2}{dt^2}\mathcal{N}_K(t) \\ \le & -\sigma_K\int_M p\left(bw^{p-2}\left(|\nabla\nabla v|^2_A + (\mathrm{Ric}_f + Kg)(\nabla v,\nabla v)\right) + (b\Delta_{p,f}v)^2\right)vu\,d\mu \\ & + 2\left(\frac{\dot\sigma_K}{\sigma_K} + \frac{\overline{K}}{b+1}\right)\frac{d}{dt}\mathcal{N}_K + \left(\frac{\ddot\sigma_K}{\sigma_K} - \frac{2\dot\sigma_K^2}{\sigma_K^2} - \frac{2\overline{K}}{b+1}\frac{\dot\sigma_K}{\sigma_K}\right)\mathcal{N}_K. \end{aligned}\tag{3.6}$$

Define the Perelman type $\mathcal{W}$-entropy

$$\mathcal{W}_K(t) := \frac{1}{\dot\alpha_K(t)}\frac{d}{dt}(\alpha_K(t)\mathcal{N}_K(t)) = \mathcal{N}_K + \beta_K(t)\frac{d}{dt}\mathcal{N}_K \tag{3.7}$$

$$= -\sigma_K \int_M \big(\beta_K(b\Delta_{p,f}v) + \big(1 + (\log\sigma_K)'\beta_K\big)\big)\, vu\, d\mu,$$

$$= \sigma_K\beta_K \int_M \left[(b+1)\frac{|\nabla v|^p}{v} - \left(\frac{1}{\beta_K} + \frac{\dot\sigma_K}{\sigma_K}\right)\right] vu\, d\mu,$$

where $\beta_K(t) = \frac{\alpha_K}{\dot\alpha_K}$, then

$$\frac{d}{dt}\mathcal{W}_K(t) = \beta_K\left(\frac{d^2}{dt^2}\mathcal{N}_K + \frac{1+\dot\beta_K}{\beta_K}\frac{d}{dt}\mathcal{N}_K\right).$$

Combining (3.4) and (3.6), we have

$$\begin{aligned}(3.8)\quad & \frac{d}{dt}\mathcal{W}_K(t)\\ \le\ & -\sigma_K\beta_K\int_M p\left(bw^{p-2}\Big(|\nabla\nabla v|_A^2 + (\mathrm{Ric}_f + Kg)(\nabla v, \nabla v)\Big) + (b\Delta_{p,f}v)^2\right) vu\, d\mu\\ & +2\beta_K\left(\frac{\dot\sigma_K}{\sigma_K} + \frac{1+\dot\beta_K}{2\beta_K} + \frac{\overline{K}}{b+1}\right)\frac{d}{dt}\mathcal{N}_K + \beta_K\left(\frac{\ddot\sigma_K}{\sigma_K} - \frac{2\dot\sigma_K^2}{\sigma_K^2} - \frac{2\overline{K}}{b+1}\frac{\dot\sigma_K}{\sigma_K}\right)\mathcal{N}_K.\end{aligned}$$

Using the identity,

$$\begin{aligned}(3.9)\qquad & b\left|w^{\frac{p}{2}-1}\nabla_i\nabla_j v + \frac{\eta_K(t)}{mb}a_{ij}\right|_A^2\\ & = bw^{p-2}|\nabla\nabla v|_A^2 + \frac{2\eta_K}{m}w^{\frac{p}{2}-1}\mathrm{tr}_A(\nabla\nabla v) + \frac{n\eta_K^2}{m^2 b}\\ & = bw^{p-2}|\nabla\nabla v|_A^2 + \frac{2\eta_K}{m}\Big(\Delta_{p,f}v + w^{\frac{p}{2}-1}\langle\nabla f, \nabla v\rangle\Big) + \frac{n\eta_K^2}{m^2 b}\end{aligned}$$

and putting (3.9) into (3.8), we get

(3.10)

$$\begin{aligned}&\frac{d}{dt}\mathcal{W}_K(t)\\ \le\ & -\sigma_K\beta_K\int_M pb\left(\left|w^{\frac{p}{2}-1}\nabla_i\nabla_j v + \frac{\eta_K}{mb}a_{ij}\right|_A^2 + w^{p-2}(\mathrm{Ric}_f^m + Kg)(\nabla v, \nabla v)\right) vu\, d\mu\\ & -\sigma_K\beta_K\int_M\left(p(b\Delta_{p,f}v)^2 + 2\left((\log\sigma_K)' + \frac{1+\dot\beta_K}{2\beta_K} + \frac{\overline{K}}{b+1} - \frac{p\eta_K}{mb}\right)(b\Delta_{p,f}v)\right) vu\, d\mu\\ & -\sigma_K\beta_K\int_M\left((\log\sigma_K)'' + \frac{1+\dot\beta_K}{\beta_K}(\log\sigma_K)' + ((\log\sigma_K)')^2 - \frac{p\eta_K^2}{mb}\right) vu\, d\mu\\ & -\sigma_K\beta_K\int_M\frac{pb}{m-n}\left(w^{\frac{p}{2}-1}\langle\nabla v, \nabla f\rangle - \frac{\eta_K}{mb}(m-n)\right)^2 vu\, d\mu.\end{aligned}$$

In order to get a complete square formula in the second and third line in (3.10), we choose a proper function $\eta_K(t)$ such that

$$(3.11)\qquad \begin{cases} p\eta_K = \lambda + \frac{1+\dot\beta_K}{2\beta_K} + \frac{\overline{K}}{b+1} - \frac{p}{mb}\eta_K,\\ p\eta_K^2 = \lambda' + \lambda^2 + \frac{1+\dot\beta_K}{\beta_K}\lambda - \frac{p}{mb}\eta_K^2,\end{cases}$$

where $\lambda = (\log \sigma_K)'$, which is equivalent to

$$\begin{aligned} 0 &= \eta_K^2 - 2\lambda\eta_K + \frac{\bar{a}}{\bar{a}+1}\left(\lambda^2 - \lambda' + 2D\lambda\right) \\ &= (\eta_K - \lambda)^2 - \frac{1}{\bar{a}+1}\left(\lambda^2 + \bar{a}\left(\lambda' - 2D\lambda\right)\right), \end{aligned} \tag{3.12}$$

where $D = \frac{\overline{K}}{b+1}$. Thus, a special solution of the equation (3.12) is

$$\eta_K = \lambda = \frac{2\bar{a}D}{1 - e^{-2Dt}}.$$

Inserting this back to system (3.11), we get

$$\beta_K = \frac{\sinh(2Dt)}{2D},\ \alpha_K = D\tanh(Dt),\ \sigma_K = (e^{Dt}\sinh(Dt))^{\bar{a}} = \left(\frac{e^{2Dt}-1}{2}\right)^{\bar{a}}.$$

Thus, from (3.10), we get the Perelma type W-entropy formula,

$$\begin{aligned} &\frac{d}{dt}\mathcal{W}_K(t) \\ \le & -\sigma_K\beta_K \int_M pb\left(\left|w^{\frac{p}{2}-1}\nabla_i\nabla_j v + \frac{\eta_K}{mb}a_{ij}\right|_A^2 + w^{p-2}(\mathrm{Ric}_f^m + Kg)(\nabla v, \nabla v)\right) vud\mu \\ & -\sigma_K\beta_K \int_M p\left((b\Delta_{p,f}v + \eta_K)^2 + \frac{b}{m-n}\left(w^{\frac{p}{2}-1}\langle\nabla v, \nabla f\rangle - (m-n)\frac{\eta_K}{mb}\right)^2\right) vud\mu. \end{aligned} \tag{3.13}$$

Therefore, W-entropy (3.7) is monotone decreasing along the weighted doubly nonlinear diffusion equation with $CD(-K, m)$ condition. □

Proof of Corollary 1.2. In particular, when $K = 0$, $\sigma_0(t) = t^{\bar{a}}$, $\beta_0(t) = t$ and $\eta_0(t) = \frac{\bar{a}}{t}$, all of the inequalities become equalities in the proof of Theorem 1.1. Then we have entropy monotonicity formula for WDNE with $CD(0, m)$ condition,

$$\frac{d}{dt}\mathcal{N}_{p,f}(u,t) = -t^{\bar{a}} \int_M \left(b\Delta_{p,f}v + \frac{\bar{a}}{t}\right) vu\, d\mu \tag{3.14}$$

and

$$\begin{aligned} &\frac{d}{dt}\mathcal{W}_{p,f}(v,t) \\ = & -pbt^{\bar{a}+1}\int_M \left[\left|w^{\frac{p}{2}-1}\nabla_i\nabla_j v + \frac{\bar{a}}{mbt}a_{ij}\right|_A^2 + w^{p-2}\mathrm{Ric}_f^m(\nabla v, \nabla v)\right] vu\, d\mu \\ & -pt^{\bar{a}+1}\int_M \left[\frac{b}{m-n}\left(w^{\frac{p}{2}-1}\nabla v\cdot\nabla f - \frac{\bar{a}(m-n)}{mbt}\right)^2 + \left(b\Delta_{p,f}v + \frac{\bar{a}}{t}\right)^2\right] vu\, d\mu, \end{aligned} \tag{3.15}$$

where the Shannon-type entropy and Perelman-type entropy are defined by

$$\mathcal{N}_0(t) = \mathcal{N}_{p,f}(v,t) := -t^{\bar{a}} \int_M vu\, d\mu$$

and

$$\mathcal{W}_0(t) = \mathcal{W}_{p,f}(v,t) := t^{\bar{a}+1}\int_M \Big((b+1)\frac{|\nabla v|^p}{v} - \frac{\bar{a}+1}{t}\Big) vu\, d\mu. \tag{3.16}$$ □

Corollary 3.2. *Let $(M, g, d\mu)$ be a weighted Riemannian manifold and u be a smooth positive solution to the weighted parabolic p-Laplacian equation*

$$\partial_t u = \Delta_{p,f}(u) := \mathrm{div}_f\big(|\nabla u|^{p-2}\nabla u\big). \tag{3.17}$$

$\mathcal{W}$-entropy is defined by

$$\mathcal{W}_K(v,t) := \sigma_K\beta_K \int_M \left[\frac{p-2}{p-1}\frac{|\nabla v|^p}{v} - \left(\frac{1}{\beta_K} + \frac{\dot{\sigma}_K}{\sigma_K}\right)\right] vu\, d\mu.$$

Then we have

(3.18)

$$\begin{aligned}
&\frac{d}{dt}\mathcal{W}_K(v,t)\\
\le& -\sigma_K\beta_K \int_M pb\left(\Big||\nabla v|^{p-2}\nabla_i\nabla_j v + \frac{\eta_K}{mb}a_{ij}\Big|_A^2 + |\nabla v|^{2p-4}(\mathrm{Ric}_f^m + Kg)(\nabla v,\nabla v)\right) vu\, d\mu\\
&-\sigma_K\beta_K \int_M p\left(\Big(b\Delta_{p,f}v + \eta_K\Big)^2 + \frac{b}{m-n}\left(|\nabla v|^{p-2}\nabla v\cdot\nabla f - (m-n)\frac{\eta_K}{mb}\right)^2\right) vu\, d\mu,
\end{aligned}$$

where $\bar{a} = \frac{(p-2)m}{(p-1)((p-2)m+p)}$, $v = \frac{p-1}{p-2}u^{\frac{p-2}{p-1}}$ is the pressure, and

$D = \frac{p(p-2)}{2p-3}K \sup\limits_{M\times[0,T)} v$, $\sigma_K = (e^{Dt}\sinh(Dt))^{\bar{a}}$, $\beta_K = \frac{\sinh(2Dt)}{2D}$, $\eta_K = \frac{2\bar{a}D}{1-e^{-2Dt}}$.

Moreover, if $CD(-K,m)$ holds for $K \ge 0$, then $\mathcal{W}_K(v,t)$ is monotone decreasing along the weighted parabolic p-Laplacian equation (3.17).

4. Differential Harnack estimates and applications

In this section, we prove various differential Harnack estimates for WDNE on the weighed manifolds with $CD(-K,m)$ condtion, including sharp Li-Yau type estimate, Hamilton type estimate and Li-Xu type estimate, etc.. As applications, Harnack inequalities are derived.

Proposition 4.1. *Let u be a smooth positive solution to* (1.9) *and v satisfies* (1.11). *Define*

$$F := \alpha(t)\frac{v_t}{v} - \frac{|\nabla v|^p}{v} + \varphi(t),$$

where $\alpha(t), \varphi(t)$ are defined in (1.21). *If $\sigma(t)$ is a function of t and satisfies the assumption* (A1) *and* (A2), *then we have*

$$\begin{aligned}
\Box_f F \ge{}& \delta w^{\frac{p}{2}-1}\langle\nabla v,\nabla F\rangle + \bar{a}\left(\frac{b^2}{\bar{a}}\Delta_{p,f}v + \frac{\sigma'}{2\sigma} + \overline{K}\right)^2 - \frac{\sigma'}{\sigma}F\\
&+ (p-1)(\alpha-1)\Big(\frac{v_t}{v}\Big)^2,
\end{aligned} \tag{4.1}$$

and

$$\Box_f(\sigma F) \geq \delta\sigma w^{\frac{p}{2}-1}\langle \nabla v, \nabla F\rangle + \bar{a}\sigma\left(\frac{b^2}{\bar{a}}\Delta_{p,f}v + \frac{\sigma'}{2\sigma} + \overline{K}\right)^2 + (p-1)(\alpha-1)\sigma\Big(\frac{v_t}{v}\Big)^2, \tag{4.2}$$

where $b = \gamma - \frac{1}{p-1} > 0$, $\bar{a} = \frac{mb}{mb(p-1)+p}$, $\delta = 2\gamma(p-1)+(p-2)$, $w = |\nabla v|^2$ *and* $\overline{K} = \frac{pbK}{2}\sup\limits_{M\times(0,T]}(v|\nabla v|^{p-2})$.

Proof. Applying (2.9), we have

$$\Box_f F = \delta w^{\frac{p}{2}-1}\langle \nabla v, \nabla F\rangle + pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big) + (p-1)\Big((b\Delta_{p,f}v)^2 + (\alpha-1)\Big(\frac{v_t}{v}\Big)^2\Big) + \alpha'\Big(\frac{v_t}{v}\Big) + \varphi'. \tag{4.3}$$

The elementary inequality

$$(a+b)^2 \geq \frac{a^2}{1+\epsilon} - \frac{b^2}{\epsilon}, \quad \forall\, \epsilon > 0$$

implies that

$$\begin{aligned} & w^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big) \\ &\geq \frac{1}{n}\left(w^{\frac{p}{2}-1}\mathrm{tr}_A(\nabla\nabla v)\right)^2 + w^{p-2}\mathrm{Ric}_f(\nabla v, \nabla v) \\ &= \frac{1}{n}(\Delta_{p,f}v + w^{\frac{p}{2}-1}\langle\nabla v, \nabla f\rangle)^2 + w^{p-2}\left(\mathrm{Ric}_f^m(\nabla v, \nabla v) + \frac{\langle\nabla v, \nabla f\rangle^2}{m-n}\right) \\ &\geq \frac{1}{m}(\Delta_{p,f}v)^2 + w^{p-2}\mathrm{Ric}_f^m(\nabla v, \nabla v), \end{aligned} \tag{4.4}$$

where $\epsilon = \frac{m}{n} - 1$. Combining (4.3), (4.4) and $\mathrm{Ric}_f^m \geq -K$, we get

$$\Box_f F \geq \delta w^{\frac{p}{2}-1}\langle\nabla v, \nabla F\rangle + \frac{1}{\bar{a}}(b\Delta_{p,f}v)^2 - pbKw^{p-1} + (p-1)(\alpha-1)\Big(\frac{v_t}{v}\Big)^2 + \alpha'\Big(\frac{v_t}{v}\Big) + \varphi'. \tag{4.5}$$

Note that $\mathrm{tr}_A(w^{\frac{p}{2}-1}v_{ij}) = \Delta_p v$ and $\frac{pbK}{2}(v|\nabla v|^{p-2}) \leq \overline{K}$, one has

$$\begin{aligned} & \Box_f F - \delta w^{\frac{p}{2}-1}\langle\nabla v, \nabla F\rangle \\ &\geq \frac{b^2}{\bar{a}}(\Delta_{p,f}v + \eta)^2 - \frac{2b^2}{\bar{a}}\eta\Delta_{p,f}v - \frac{b^2}{\bar{a}}\eta^2 - pbKw^{p-1} \\ &\quad + (p-1)(\alpha-1)\Big(\frac{v_t}{v}\Big)^2 + \alpha'\Big(\frac{v_t}{v}\Big) + \varphi' \\ &\geq \frac{b^2}{\bar{a}}(\Delta_{p,f}v + \eta)^2 + \Big(\alpha' - \frac{2b}{\bar{a}}\eta\Big)\frac{v_t}{v} - \Big(2\overline{K} - \frac{2b}{\bar{a}}\eta\Big)\frac{w^{\frac{p}{2}}}{v} - \frac{b^2}{\bar{a}}\eta^2 \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&\quad + (p-1)(\alpha-1)\Big(\frac{v_t}{v}\Big)^2 + \varphi' \\
&= \frac{b^2}{\bar a}(\Delta_{p,f}v+\eta)^2 + 2\Big(\overline{K} - \frac{b}{\bar a}\eta\Big)\left(\frac{\alpha' - \frac{2b}{\bar a}\eta}{2\overline{K} - \frac{2b}{\bar a}\eta}\frac{v_t}{v} - \frac{w^{\frac p2}}{v} + \varphi\right) \\
&\quad + (p-1)(\alpha-1)\Big(\frac{v_t}{v}\Big)^2 + \varphi' - 2\Big(\overline{K} - \frac{b}{\bar a}\eta\Big)\varphi - \frac{b^2}{\bar a}\eta^2.
\end{aligned}
$$

Now we choose the proper functions $\sigma(t)$ and $\eta(t)$ such that $\alpha(t)$ and $\varphi(t)$ satisfy the following system

$$
\left\{
\begin{aligned}
\tfrac{\sigma'}{\sigma} &= 2(\tfrac{b}{\bar a}\eta - \overline{K}),\\
\alpha &= \tfrac{\alpha' - \frac{2b}{\bar a}\eta}{2(\overline{K} - \frac{b}{\bar a}\eta)},\\
\eta^2 &= \tfrac{\bar a}{b^2}\left(\varphi' - 2\Big(\overline{K} - \tfrac{b}{\bar a}\eta\Big)\varphi\right).
\end{aligned}
\right. \tag{4.7}
$$

Plugging (4.7) into (4.6), we have

$$
\Box_f F \geq \delta w^{\frac p2 - 1}\langle \nabla v, \nabla F\rangle + \frac{b^2}{\bar a}(\Delta_{p,f}v+\eta)^2 - \frac{\sigma'}{\sigma}F + (p-1)(\alpha-1)\Big(\frac{v_t}{v}\Big)^2. \tag{4.8}
$$

By (4.1) and

$$
\Box_f G = \Box_f(\sigma F) = \sigma \Box_f F + \sigma' F,
$$

we can easily get (4.2). In fact, the first equation in (4.7) is equivalent to

$$
\eta(t) = \frac{\bar a}{2b}\Big(\frac{\sigma'}{\sigma} + 2\overline{K}\Big).
$$

Inserting this into the last two equations in (4.7), we have

$$
(\sigma\alpha)' = \sigma' + 2\overline{K}\sigma \quad \text{and} \quad (\sigma\varphi)' = \frac{\bar a \sigma}{4}\Big(\frac{\sigma'}{\sigma} + 2\overline{K}\Big)^2.
$$

Integrating above identities on $[0,t]$, we can obtain the exact expressions of $\alpha(t)$ and $\varphi(t)$ in (1.21). □

Proof of Theorem 1.6. Since $b > 0$, $\bar a > 0$, $p > 1$, applying the parabolic maximum principle in (4.2), it is easy to get $F \geq 0$, that is (1.20) in Theorem 1.6. □

Remark 4.2. Integrating the differential Harnack estimate (1.20) yields

$$
\begin{aligned}
0 \leq \int_M F v u\, d\mu &= \int_M \Big(\alpha(t)\frac{v_t}{v} - \frac{|\nabla v|^p}{v} + \varphi(t)\Big) v u\, d\mu \\
&= \int_M \Big(\alpha(b\Delta_{p,f}v) + (\alpha-1)\frac{|\nabla v|^p}{v} + \varphi(t)\Big) v u\, d\mu \\
&= \int_M \Big(\frac{\alpha b + 1}{b+1}(b\Delta_{p,f}v) + \varphi(t)\Big) v u\, d\mu.
\end{aligned} \tag{4.9}
$$

Set

$$
(\log \sigma_K)' = \frac{b+1}{\alpha b + 1}\varphi(t), \tag{4.10}
$$

we have

$$\frac{d}{dt}\mathcal{N}_K(t) \le 0, \tag{4.11}$$

that is Shannon type entropy $\mathcal{N}_K(t)$ is monotone decreasing along WDNE. In particular, when $\alpha(t) = 1 + \frac{2}{3}\overline{K}t$ and $\varphi(t) = \bar{a}\Big(\frac{1}{t} + \overline{K} + \frac{1}{3}\overline{K}^2 t\Big)$, by solving ODE (4.10), we have

$$\sigma_K(t) = e^{\frac{(b+1)\bar{a}\overline{K}t}{2b}} \left(1 + \frac{2}{3}\frac{b}{b+1}\overline{K}t\right)^{-\left(\frac{1}{4}+\frac{3}{4b^2}\right)\bar{a}} t^{\bar{a}}. \tag{4.12}$$

Proof of Theorem 1.9. Define

$$G := \alpha^{-1}F = \frac{v_t}{v} - \alpha^{-1}\frac{|\nabla v|^p}{v} + \alpha^{-1}\varphi.$$

By (4.5) and $\overline{K} = \frac{pbK}{2}\sup_{M\times[0,T)} v|\nabla v|^{p-2}$, we have

$$\begin{aligned}
\Box_f G &= (\alpha^{-1})'F + \alpha^{-1}\Box_f F \\
&\ge \delta w^{\frac{p}{2}-1}\langle \nabla v, \nabla G\rangle + \frac{\alpha^{-1}}{\bar{a}}(b\Delta_{p,f}v)^2 + (p-1)(1-\alpha^{-1})\Big(\frac{v_t}{v}\Big)^2 \\
&\quad + \alpha'\alpha^{-1}\frac{v_t}{v} - 2\overline{K}\alpha^{-1}\frac{w^{\frac{p}{2}}}{v} + \alpha^{-1}\varphi' + (\alpha^{-1})'F \\
&= \delta w^{\frac{p}{2}-1}\langle \nabla v, \nabla G\rangle + \frac{\alpha^{-1}}{\bar{a}}(b\Delta_{p,f}v)^2 + (p-1)(1-\alpha^{-1})\Big(\frac{v_t}{v}\Big)^2 \\
&\quad + (\log\alpha)'\Big(\frac{v_t}{v} - 2\overline{K}[(\log\alpha)']^{-1}\alpha^{-1}\frac{w^{\frac{p}{2}}}{v} + \alpha^{-1}\varphi\Big) \\
&\quad + (\alpha^{-1})'F - \alpha'\alpha^{-2}\varphi + \alpha^{-1}\varphi' \\
&\ge \delta w^{\frac{p}{2}-1}\langle \nabla v, \nabla G\rangle + \frac{e^{-2\overline{K}t}}{\bar{a}}(b\Delta_{p,f}v)^2 + (p-1)(1-\alpha^{-1})\Big(\frac{v_t}{v}\Big)^2 \\
&\quad + \frac{2\overline{K}\bar{a}e^{2\overline{K}t}}{t} - \frac{\bar{a}e^{2\overline{K}t}}{t^2} \\
&\ge \delta w^{\frac{p}{2}-1}\langle \nabla v, \nabla G\rangle + \frac{e^{-2\overline{K}t}}{\bar{a}}(b\Delta_{p,f}v)^2 - \frac{\bar{a}e^{2\overline{K}t}}{t^2},
\end{aligned} \tag{4.13}$$

where $\alpha(t) = e^{2\overline{K}t}$ and $\varphi(t) = \frac{\bar{a}}{t}e^{4\overline{K}t}$. Now we show that $G \ge 0$ on $M \times [0,T)$. Since M is closed, assume that G attains its minimum at point (x_0, t_0) and $G(x_0,t_0) < 0$. Then at (x_0, t_0),

$$\Box_f G \le 0, \quad \nabla G = 0.$$

On the other hand, at this point,

$$0 \le \frac{\bar{a}e^{2\overline{K}t}}{t} < e^{-2\overline{K}t}\frac{|\nabla v|^p}{v} - \frac{v_t}{v} \le \frac{|\nabla v|^p}{v} - \frac{v_t}{v} = -b\Delta_{p,f}v.$$

Put this inequality into (4.13), we have

$$\Box_f G > 0.$$

This is a contradiction. Thus we have finished the proof of Theorem 1.9. □

Proof of Corollary 1.10. Let $\varsigma(t)$ be a constant speed geodesic connected $\varsigma(t_1) = x_1$ and $\varsigma(t_2) = x_2$ with $|\dot{\varsigma}(t)| = \frac{d(x_2,x_1)}{t_2-t_1}$. Applying the differential Harnack estimate (1.20) and Young's inequality, we have

$$\begin{aligned} & v(x_2,t_2) - v(x_1,t_1) \\ = & \int_{t_1}^{t_2} v_t + \langle \nabla v, \dot{\varsigma}(t)\rangle dt \\ \geq & \int_{t_1}^{t_2} \left(\frac{1}{\alpha(t)}|\nabla v|^p - \frac{\varphi(t)}{\alpha(t)}v - \frac{1}{\alpha(t)}|\nabla v|^p - \frac{p-1}{p^{p^*}}\alpha^{\frac{1}{p-1}}(t)|\dot{\varsigma}(t)|^{p^*}\right) dt \\ \geq & -v_{max}\int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)}dt - \frac{p-1}{p^{p^*}}\frac{d(x_2,x_1)^{p^*}}{(t_2-t_1)^{p^*}}\int_{t_1}^{t_2} \alpha^{\frac{1}{p-1}}(t)dt \end{aligned}$$

and

$$\begin{aligned} & \log\frac{v(x_2,t_2)}{v(x_1,t_1)} \\ = & \int_{t_1}^{t_2} \left(\frac{d}{dt}\log v(x,t) + \nabla \log v \cdot \dot{\varsigma}(t)\right) dt \\ \geq & \int_{t_1}^{t_2} \left(\frac{1}{\alpha(t)}\Big(|\nabla v|^p - \varphi(t)\Big) - \frac{1}{\alpha(t)}|\nabla v|^p - \frac{p-1}{p^{p^*}}\frac{|\dot{\varsigma}(t)|^{p^*}}{v_{max}}\alpha^{\frac{1}{p-1}}(t)\right) dt \\ \geq & -\int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)}dt - \frac{p-1}{p^{p^*}}\frac{1}{v_{max}}\frac{d(x_2,x_1)^{p^*}}{(t_2-t_1)^{p^*}}\int_{t_1}^{t_2} \alpha^{\frac{1}{p-1}}(t)dt. \end{aligned}$$

Here $p^* = \frac{p}{p-1}$. This finishes the proof of Corollary 1.10. □

Proof of Theorem 1.11. Estimate (1.30) is a direct result when we take $K = 0$ in (1.22), (1.24) or (1.27). In fact, we can also give a direct proof for (1.30). Using (2.10) and (4.4), for any $b > 0$ and $\mathrm{Ric}_f^m \geq 0$, we get

$$\begin{aligned} \Box_f(F_1) &\geq \delta w^{\frac{p}{2}-1}\langle \nabla v, \nabla F_1\rangle + \frac{pb}{m}(\Delta_{p,f}v)^2 + (p-1)F_1^2 \\ &= \delta w^{\frac{p}{2}-1}\langle \nabla v, \nabla F_1\rangle + \frac{1}{\bar{a}}F_1^2, \end{aligned}$$

where $\bar{a} = \frac{mb}{(p-1)mb+p}$. Since M is closed, the standard parabolic maximum principle implies the estimate (1.30). Moreover, if we take the Barenblatt-type solution of WDNE (1.9) on $\mathbb{R}^m$,

$$u(x,t) = t^{-\frac{\bar{a}}{b}}F(t^{-\frac{\bar{a}}{mb}}x), \quad F(\xi) = (C - \kappa|\xi|^{\frac{p}{p-1}})_+^{\frac{1}{b}}, \tag{4.14}$$

where $\bar{a} = \frac{mb}{(p-1)mb+p}$, $\kappa = \frac{(p-1)b}{p}(\frac{\bar{a}}{mb})^{\frac{1}{p-1}}$, C is any positive constant, then equality holds in (1.30), i.e., the differential Harnack estimate (1.30) is sharp. The similar argument holds for (1.31). □

References

[1] D. Bakry and M. Émery, *Diffusions hypercontractives*, in *Séminaire de probabilités, XIX, 1983/84*, 177–206, Lecture Notes in Math., 1123, Springer, Berlin. https://doi.org/10.1007/BFb0075847

[2] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften, 348, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-00227-9

[3] D. Chen and C. Xiong, *Gradient estimates for doubly nonlinear diffusion equations*, Nonlinear Anal. **112** (2015), 156–164. https://doi.org/10.1016/j.na.2014.08.017

[4] F. Fang, X.-D. Li, and Z. Zhang, *Two generalizations of Cheeger-Gromoll splitting theorem via Bakry-Emery Ricci curvature*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 2, 563–573.

[5] A. Futaki, H. Li, and X.-D. Li, *On the first eigenvalue of the Witten-Laplacian and the diameter of compact shrinking solitons*, Ann. Global Anal. Geom. **44** (2013), no. 2, 105–114. https://doi.org/10.1007/s10455-012-9358-5

[6] R. S. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993), no. 1, 113–126. https://doi.org/10.4310/CAG.1993.v1.n1.a6

[7] G. Huang, Z. Huang, and H. Li, *Gradient estimates for the porous medium equations on Riemannian manifolds*, J. Geom. Anal. **23** (2013), no. 4, 1851–1875. https://doi.org/10.1007/s12220-012-9310-8

[8] G. Huang and H. Li, *Gradient estimates and entropy formulae of porous medium and fast diffusion equations for the Witten Laplacian*, Pacific J. Math. **268** (2014), no. 1, 47–78. https://doi.org/10.2140/pjm.2014.268.47

[9] B. Kotschwar and L. Ni, *Local gradient estimates of p-harmonic functions, $1/H$-flow, and an entropy formula*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 1, 1–36. https://doi.org/10.24033/asens.2089

[10] X.-D. Li, *Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds*, J. Math. Pures Appl. (9) **84** (2005), no. 10, 1295–1361. https://doi.org/10.1016/j.matpur.2005.04.002

[11] ———, *Perelman's entropy formula for the Witten Laplacian on Riemannian manifolds via Bakry-Emery Ricci curvature*, Math. Ann. **353** (2012), no. 2, 403–437. https://doi.org/10.1007/s00208-011-0691-y

[12] S. Li and X.-D. Li, *The W-entropy formula for the Witten Laplacian on manifolds with time dependent metrics and potentials*, Pacific J. Math. **278** (2015), no. 1, 173–199. https://doi.org/10.2140/pjm.2015.278.173

[13] ———, *Harnack inequalities and W-entropy formula for Witten Laplacian on manifolds with the K-super Perelman Ricci flow*, arXiv:1412.7034v1.

[14] ———, *W-entropy formulas on super Ricci flows and Langevin deformation on Wasserstein space over Riemannian manifolds*, Sci. China Math. **61** (2018), no. 8, 1385–1406. https://doi.org/10.1007/s11425-017-9227-7

[15] ———, *Hamilton differential Harnack inequality and W-entropy for Witten Laplacian on Riemannian manifolds*, J. Funct. Anal. **274** (2018), no. 11, 3263–3290. https://doi.org/10.1016/j.jfa.2017.09.017

[16] ———, *On Harnack inequalities for Witten Laplacian on Riemannian manifolds with super Ricci flows*, Asian J. Math. **22** (2018), no. 3, 577–597. https://doi.org/10.4310/AJM.2018.v22.n3.a10

[17] ———, *W-entropy, super Perelman Ricci flows, and (K,m)-Ricci solitons*, J. Geom. Anal. **30** (2020), no. 3, 3149–3180. https://doi.org/10.1007/s12220-019-00193-4

[18] ———, *W-entropy formula and Langevin deformation of flows on Wasserstein space over Riemannian manifolds*, arXiv:1604.02596.

[19] J. Li and X. Xu, *Differential Harnack inequalities on Riemannian manifolds I: linear heat equation*, Adv. Math. **226** (2011), no. 5, 4456–4491. https://doi.org/10.1016/j.aim.2010.12.009

[20] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201. https://doi.org/10.1007/BF02399203

[21] P. Lu, L. Ni, J. Vázquez, and C. Villani, *Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds*, J. Math. Pures Appl. (9) **91** (2009), no. 1, 1–19. https://doi.org/10.1016/j.matpur.2008.09.001

[22] L. Ni, *The entropy formula for linear heat equation*, J. Geom. Anal. **14** (2004), no. 1, 87–100. https://doi.org/10.1007/BF02921867

[23] ———, *Addenda to*: *"The entropy formula for linear heat equation* , J. Geom. Anal. **14** (2004), no. 2, 369–374. https://doi.org/10.1007/BF02922078

[24] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv.org/abs/maths0211159.

[25] B. Qian, *Remarks on differential Harnack inequalities*, J. Math. Anal. Appl. **409** (2014), no. 1, 556–566. https://doi.org/10.1016/j.jmaa.2013.07.043

[26] J. L. Vázquez, *The porous medium equation*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007.

[27] C. Villani, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften, 338, Springer-Verlag, Berlin, 2009. https://doi.org/10.1007/978-3-540-71050-9

[28] Y.-Z. Wang, *Differential Harnack estimates and entropy formulae for weighted p-heat equations*, Results Math. **71** (2017), no. 3-4, 1499–1520. https://doi.org/10.1007/s00025-017-0675-7

[29] ———, $\mathcal{W}$-entropy formulae and differential Harnack estimates for porous medium equations on Riemannian manifolds, Commun. Pure Appl. Anal. **17** (2018), no. 6, 2441–2454. https://doi.org/10.3934/cpaa.2018116

[30] Y. Wang and W. Chen, *Gradient estimates and entropy monotonicity formula for doubly nonlinear diffusion equations on Riemannian manifolds*, Math. Methods Appl. Sci. **37** (2014), no. 17, 2772–2781. https://doi.org/10.1002/mma.3016

[31] Y. Wang, J. Yang, and W. Chen, *Gradient estimates and entropy formulae for weighted p-heat equations on smooth metric measure spaces*, Acta Math. Sci. Ser. B (Engl. Ed.) **33** (2013), no. 4, 963–974. https://doi.org/10.1016/S0252-9602(13)60055-7

[32] G. Wei and W. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405. https://doi.org/10.4310/jdg/1261495336

[33] S. Yan and L. F. Wang, *Elliptic gradient estimates for the doubly nonlinear diffusion equation*, Nonlinear Anal. **176** (2018), 20–35. https://doi.org/10.1016/j.na.2018.06.004

Yu-Zhao Wang
School of Mathematical Sciences
Shanxi University
Taiyuan, 030006, Shanxi, China
Email address: wangyuzhao@sxu.edu.cn