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ON DOMINATION IN ZERO-DIVISOR GRAPHS OF RINGS WITH INVOLUTION

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Abstract. In this paper, we study domination in the zero-divisor graph of a $*$-ring. We first determine the domination number, the total domination number, and the connected domination number for the zero-divisor graph of the product of two $*$-rings with componentwise involution. Then, we study domination in the zero-divisor graph of a Rickart $*$-ring and relate it with the clique of the zero-divisor graph of a Rickart $*$-ring.

1. Introduction

In 1988, I. Beck [6] began to investigate the coloring of a commutative ring R by assigning a graph with vertex set as the set of all elements of R and two vertices a and b are adjacent if and only if $ab = 0$. While Beck focused on the relationship between the click number and the chromatic number of the graph, various works inspired by this structure focused on the interaction of commutative rings and their zero-divisors graphs. However, in 1999, D. F. Anderson and P. S. Livingston [4] modified and studied the zero-divisor graph of a commutative ring R whose vertex set is the set of all nonzero zero-divisor of R. Further, the zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g., [1, 3–5, 12]. The zero-divisor graph of a noncommutative ring has been introduced and studied by Akbari [2] and Redmond [13], whereas the same concept for semigroup by Demeyer et al. [8]. Moreover, the concept of dominating set of zero-divisor graph has implicitly been studied in [9, 10] and [14].

A set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. A dominating set S is said to be *connected* or *clique dominating set* if $\langle S \rangle$ is connected or $\langle S \rangle$ is complete, respectively. A dominating set S is called a *total dominating set* if every vertex in V is adjacent to some other vertex in S. The *domination number* of G,

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denoted $\lambda(G)$, is the minimum cardinality of a dominating set in G. A dominating set S of minimum cardinality in G is called λ-set of G. In a similar way, one can define the *connected domination number* $\lambda_c(G)$, the *total domination number* $\lambda_t(G)$, and the *clique domination number* $\lambda_{cl}(G)$.

This paper is based on the properties of zero-divisor graphs of rings with involution. A mapping '$*$' on an associative ring $\mathfrak{S}$ is called involution if $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in \mathfrak{S}$. A ring with involution '$*$' is called a $*$-ring. Clearly, identity mapping is an involution if and only if the ring is commutative. An element e of a $*$-ring is a projection if $e = e^2$ and $e = e^*$. For a nonempty subset B of $\mathfrak{S}$, we write $r(B) = \{a \in \mathfrak{S} : ba = 0$ for all $b \in B\}$ call a right annihilator of B in $\mathfrak{S}$. A Rickart $*$-ring is a $*$-ring in which the right annihilator of every element is generated by a projection, as a right ideal in $\mathfrak{S}$. Every Rickart $*$-ring contains unity. For each element a in a Rickart $*$-ring, there is a unique projection e such that $ae = a$ and $ax = 0$ if and only if $ex = 0$, called a right projection of a denoted by $RP(a)$. In fact, $r(\{a\}) = (1 - RP(a))A$. Similarly, the left annihilator $l(\{a\})$ and the left projection $LP(a)$ are defined for each element a in a Rickart $*$-ring $\mathfrak{S}$. The set of projections $P(\mathfrak{S})$ in a Rickart $*$-ring $\mathfrak{S}$ forms a lattice, denoted by $L(P(\mathfrak{S}))$, under the partial order '$e \leq f$ if and only if $e = fe = ef$'. In fact, $e \vee f = f + RP(e(1-f))$ and $e \wedge f = e - LP(e(1-f))$. We use to denote the set of all nontrivial projection of $\mathfrak{S}$ by $P^*(\mathfrak{S}) = P(\mathfrak{S}) \setminus \{0, 1\}$. More details about Rickart $*$-rings can be found in Berberian [7].

In [11], A. Patil and B. N. Waphare give the concept of zero-divisor graph of $*$-ring $\mathfrak{S}$. Let $\mathfrak{S}$ be a $*$-ring. The zero-divisor graph of $\mathfrak{S}$, denoted by $\Gamma^*(\mathfrak{S})$, is an undirected graph with vertex set as the set of all nonzero left zero-divisors of $\mathfrak{S}$ and two vertices a and b are adjacent if and only if $ab^* = 0$.

In this paper, we study about the domination in zero-divisor graph of $*$-ring and Rickart $*$-ring.

2. Preliminaries

In this section, we discusses about several graphs whose dominating sets and domination number are clear. Some of them are following, where there proofs are straightforward.

Example 2.1 ([15])**.** (1) For a complete graph K_n, $\lambda(K_n) = 1$.
(2) Let G be a complete m-partite graph ($m \geq 2$) with partition sets $U_1, U_2, \ldots, U_m$. If $|U_i| \geq 2$ for $1 \leq i \leq m$, then $\lambda(G) = 2$, because one vertex of U_1 and one vertex of U_2 dominate G. If $|U_i| = 1$ for some i, then $\lambda(G) = 1$.
(3) For a star graph $K_{1,n}$, $\lambda(K_{1,n}) = 1$.
(4) If G is a bistar graph, then domination number of G is 2 because the set containing of two centers of G is a dominating set.
(5) If C_n is a cycle and P_n is a path with n vertices, then $\lambda(C_n) = \lceil \frac{n}{3} \rceil = \lambda(P_n)$.

3. The domination number of zero-divisor graph of rings with involution

In this section, we study domination in the zero-divisor graph of a $*$-ring. We first determine the domination number, the total domination number, and the connected domination number for the zero-divisor graph of the product of two $*$-rings with componentwise involution. We then generalized these results to the zero-divisor graph of finite product of $*$-rings with componentwise involution.

We start this section with definition of zero-divisor graph of ring with involution.

Definition 3.1. Let $\mathfrak{S}$ be a $*$-ring. The zero-divisor graph of $\mathfrak{S}$, denoted by $\Gamma^*(\mathfrak{S})$, is an undirected graph with vertex set $\{a(\neq 0) \in \mathfrak{S} : ab = 0$, where b is a nonzero element of $\mathfrak{S}\}$ and two distinct vertices a and b are adjacent if and only if $ab^* = 0$.

Now, we consider some examples of zero-divisor graph of $*$-rings and their domination number.

Example 3.1. If $\mathfrak{S} = F_1 \times F_2$, where F_1 and F_2 are fields with $|F_1| \geq 3$ and $|F_2| \geq 3$. Then $\{(1,0),(0,1)\}$ is a dominating set of $\Gamma^*(\mathfrak{S})$ and hence $\lambda(\Gamma^*(\mathfrak{S})) = 2$.

Example 3.2. If $\mathfrak{S} = R \times R$, where R is a domain, with involution $(x,y)^* = (y,x)$. Then $\{(1,0),(0,1)\}$ is a dominating set of $\Gamma^*(\mathfrak{S})$ and hence $\lambda(\Gamma^*(\mathfrak{S})) = 2$.

Example 3.3. If $\mathfrak{S}$ be a Rickart $*$-ring which contains exactly 4 projections, then $\Gamma^*(\mathfrak{S})$ is complete bipartite. Hence, $\lambda(\Gamma^*(\mathfrak{S})) = 2$.

Example 3.4. If $\mathfrak{S} = \mathbb{Z}_6$ with identity mapping as an involution. Then $V(\Gamma^*(\mathfrak{S})) = \{2,3,4\}$ and the graph $\Gamma^*(\mathfrak{S})$ is shown in Fig. 1. The set $\{3\}$ is a dominating set of $\Gamma^*(\mathfrak{S})$ and hence $\lambda(\Gamma^*(\mathfrak{S})) = 1$.

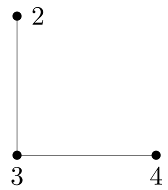

Fig. 1.

Now, we discuss about the domination number of zero-divisor graph of product of two $*$-rings with componentwise involution. We begin with the following lemma.

Lemma 3.1. *Let $\mathfrak{S}_1$, $\mathfrak{S}_2$ be two $*$-rings and $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$ with componentwise involution. Then (a,b) is a left zero-divisor if and only if at least one of a and b is a left zero-divisor.*

Proposition 3.1. *Let R be a $*$-ring and $\mathfrak{S} = \mathbb{Z}_2 \times R$ with componentwise involution. If R is a domain, then $\lambda(\Gamma^*(\mathfrak{S})) = 1$.*

Proof. Notice that $\Gamma^*(\mathfrak{S})$ is a star graph and any star graph has domination number one. □

As a consequence, if R is a domain, then $\lambda_c(\Gamma^*(\mathfrak{S})) = 1$ and $\lambda_t(\Gamma^*(\mathfrak{S})) = 2$.

Definition 3.2. Let $\mathfrak{S}$ be a $*$-ring with unity and $V(\Gamma^*(\mathfrak{S})) \neq \emptyset$. A subset $S \subseteq V(\Gamma^*(\mathfrak{S}))$ is called a semi-total dominating set in $\Gamma^*(\mathfrak{S})$ if S is a dominating set for $\Gamma^*(\mathfrak{S})$ and for any $a \in S$ there is a vertex $b \in S$ (not necessarily distinct) such that $ab^* = 0$.

The semi-total domination number $\lambda_{st}(\Gamma^*(\mathfrak{S}))$ of $\Gamma^*(\mathfrak{S})$ is the minimum cardinality of a semi-total dominating set in $\Gamma^*(\mathfrak{S})$.

We refer to a semi-total dominating set of $\Gamma^*(\mathfrak{S})$ of minimum cardinality as a $\lambda_{st}(\Gamma^*(\mathfrak{S}))$-set.

Proposition 3.2. *Let R be a $*$-ring and $\mathfrak{S} = \mathbb{Z}_2 \times R$ with componentwise involution. If R is not a domain, then $\lambda(\Gamma^*(\mathfrak{S})) = \lambda_{st}(\Gamma^*(R)) + 1$.*

Proof. Let D be a $\lambda_{st}(\Gamma^*(R))$-set. Then the set $\{(0,x) : x \in D\} \cup \{(1,0)\}$ is a dominating set for $\Gamma^*(\mathfrak{S})$. So $\lambda(\Gamma^*(\mathfrak{S})) \leq \lambda_{st}(\Gamma^*(R)) + 1$. Now, let T be a $\lambda(\Gamma^*(\mathfrak{S}))$-set. Consider $T_1 = \{x : (0,x) \in T\}$. We claim that T_1 is a dominating set for $\Gamma^*(R)$. Let $d \in V(\Gamma^*(R))$. Then $(1,d) \in V(\Gamma^*(\mathfrak{S}))$. So, there exists $(a,b) \in T$ such that $(a,b)(1,d)^* = (0,0)$. This implies that $a = 0$ and $b \in T_1$ such that $db^* = 0$. Thus, T_1 is a dominating set for $\Gamma^*(R)$. On the other hand, for any $x \in D_1$, $(1,x) \in V(\Gamma^*(\mathfrak{S}))$, and so there exists $(a,b) \in T$ such that $(a,b)(1,x)^* = (0,0)$. Then $a = bx^* = 0$. So $b \in T_1$ and $bx^* = 0$. Thus, T_1 is a $\lambda_{st}(\Gamma^*(R))$-set.

We next show that $|T| > |T_1|$. Suppose to the contrary that $|T| = |T_1|$. Then $(0,1)$ is not dominated by T, a contradiction. Thus $|T| > |T_1|$. We conclude that $\lambda(\Gamma^*(\mathfrak{S})) = |T| \geq |T_1| + 1 \geq \lambda_{st}(\Gamma^*(R)) + 1$. □

We next assign a parameter $a^*(\mathfrak{S})$ to a $*$-ring $\mathfrak{S}$. For a $*$-ring $\mathfrak{S}$ with unity, we let

$$a^*(\mathfrak{S}) = \begin{cases} 1, & \text{if } V(\Gamma^*(\mathfrak{S})) = \emptyset; \\ \lambda_{st}(\Gamma^*(\mathfrak{S})), & \text{if } V(\Gamma^*(\mathfrak{S})) \neq \emptyset. \end{cases}$$

Theorem 3.3. *Let $n \geq 3$ be a fixed integer and $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$, where $\mathfrak{S}_i$ is a $*$-ring with unity for each $i = 1, 2, \ldots, n$ and $\mathfrak{S}$ is a $*$-ring with componentwise involution. Then,*

$$\lambda(\Gamma^*(\mathfrak{S})) = a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2) + \cdots + a^*(\mathfrak{S}_n).$$

Proof. We consider the following cases.

Case 1: $V(\Gamma^*(\mathfrak{S}_i)) = \emptyset$ for each $1 \leq i \leq n$. Clearly, the set $S = \{a_1, a_2, \ldots, a_n\}$, where $a_1 = (x_1, 0, 0, \ldots, 0), a_2 = (0, x_2, 0, \ldots, 0), \ldots, a_3 = (0, 0, \ldots, x_n)$ and x_i is a fixed nonzero element of $\mathfrak{S}_i$ for each $1 \leq i \leq n$, is a dominating set for

$\Gamma^*(\mathfrak{S})$. A nonzero element $(a_1, a_2, \ldots, a_n)$ is a vertex in $\Gamma^*(\mathfrak{S})$ if and only if at least one of its components is zero. We need to show that for any $T \subset S$, T is not a dominating set of $\Gamma^*(\mathfrak{S})$. Suppose to the contrary that there exists $T \subset S$ such that T is a dominating set of $\Gamma^*(\mathfrak{S})$. Let $T = \{v_1, v_2, \ldots, v_{n-1}\}$, where the i^{th} component of v_i is nonzero. Consider $u = (1, 1, \ldots, 1, 0)$. Then u is a vertex of $\Gamma^*(\mathfrak{S})$ such that no element of T is adjacent to it. Thus,

$$\lambda(\Gamma^*(\mathfrak{S})) = n.$$

Case 2: $V(\Gamma^*(\mathfrak{S}_i)) \neq \emptyset$ for some $i = 1, 2, \ldots, n$. Without loss of generality, we assume that $V(\Gamma^*(\mathfrak{S}_j)) \neq \emptyset$ for $1 \leq j \leq m$ and $V(\Gamma^*(\mathfrak{S}_k)) = \emptyset$ for $m + 1 \leq k \leq n$. First, let S_j be a $\lambda_{st}(\Gamma^*(\mathfrak{S}_j))$-set, for $1 \leq \mathrm{j} \leq m$. Define $D_j = \{(0, 0, \ldots, a_j, 0, \ldots, 0) : a_j \in S_j\}$ for each $1 \leq j \leq m$, $D'_k = \{(0, 0, \ldots, 1, 0, \ldots, 0)\}$, with k^{th} component 1 for each $m + 1 \leq k \leq n$, and $D = (\bigcup_{j=1}^{m} D_j) \cup (\bigcup_{k=m+1}^{n} D'_k)$. We have to show that D is a dominating set for $\Gamma^*(\mathfrak{S})$. Observe that, a nonzero element $(x_1, x_2, \ldots, x_n)$ is a vertex of $\Gamma^*(\mathfrak{S})$ if and only if at least one of $x_j \in V(\Gamma^*(\mathfrak{S}_j))$, $1 \leq j \leq m$, or at least one of x_k is zero for $1 \leq k \leq n$.

Let $(x_1, x_2, \ldots, x_n)$ be a vertex of $\Gamma^*(\mathfrak{S})$ such that $x_j \in V(\Gamma^*(\mathfrak{S}_j))$ for some $1 \leq j \leq m$. Since S_j is a dominating set of $\Gamma^*(\mathfrak{S}_j)$, there exists $y_j \in S_j$ such that $x_j y_j^* = 0$. Observe that, $(0, 0, \ldots, y_j, 0, \ldots, 0) \in D_j$ such that $(x_1, x_2, \ldots, x_n)(0, 0, \ldots, y_j, 0, \ldots, 0)^* = (0, 0, \ldots, 0)$. Similarly, we can show that other vertices of $\Gamma^*(\mathfrak{S})$ are dominated by some element of D. Thus, D is a dominating set for $\Gamma^*(\mathfrak{S})$. Hence,

$$\lambda(\Gamma^*(\mathfrak{S})) \leq a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2) + \cdots + a^*(\mathfrak{S}_m) + (n - m).$$

Now, let T be a $\lambda(\Gamma^*(\mathfrak{S}))$-set, and define $T_j = \{x_j : (0, 0, \ldots, 0, x_j, 0, \ldots, 0) \in D\}$ for $1 \leq j \leq m$. We show that T_j is a semi-total dominating set for $\Gamma^*(\mathfrak{S}_j)$ for $1 \leq j \leq m$. For any $a_j \in V(\Gamma^*(\mathfrak{S}_j)$, $(1, 1, \ldots, 1, a_j, 1, \ldots, 1) \in V(\Gamma^*(\mathfrak{S}))$. So, there is $(x_1, x_2, \ldots, x_n) \in T$ such that

$$(1, 1, \ldots, 1, a_j, 1, \ldots, 1)(x_1, x_2, \ldots, x_n)^* = (0, 0, \ldots, 0).$$

This implies that $x_1 = x_2 = \cdots = x_{j-1} = a_j x_j^* = x_{j+1} = \cdots = x_n = 0$. So $x_j \in T_j$, and a_j is dominated by an element of T_j. We deduce that T_j is a dominating set for $\Gamma^*(\mathfrak{S}_j)$, for each $1 \leq j \leq m$. On the other hand, for any $b_j \in T_j$, $(1, 1, \ldots, 1, b_j, 1, \ldots, 1) \in V(\Gamma^*(\mathfrak{S}))$, and so is dominated by an element $(z_1, z_2, \ldots, z_n)$ of T. We obtain that $(z_1, z_2, \ldots, z_n)(1, 1, \ldots, 1, b_j, 1, \ldots, 1)^* = (0, 0, \ldots, 0)$. This gives that $z_1 = z_2 = \cdots = z_{j-1} = b_j z_j^* = z_{j+1} = \cdots = z_n = 0$. So $z_j \in T_j$ such that $b_j z_j^* = 0$. Hence, T_j is a semi-total dominating set for $\Gamma^*(\mathfrak{S}_j)$ for each $1 \leq j \leq m$. This shows that

$$|T| \geq |T_1| + |T_2| + \cdots + |T_m| \geq a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2) + \cdots + a^*(\mathfrak{S}_m).$$

If,

$$|T| = a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2) + \cdots + a^*(\mathfrak{S}_m),$$

then the vertices $v_i = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$ with i^{th} components zero for each $m+1 \leq i \leq n$, are not dominated by any vertex of T, which is a contradiction. So,

$$|T| \geq a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2) + \cdots + a^*(\mathfrak{S}_m) + (n - m).$$

Case 3: $V(\Gamma^*(\mathfrak{S}_i)) \neq \emptyset$ for each $1 \leq i \leq n$. Let S be a $\lambda(\Gamma^*(\mathfrak{S}))$-set. Define $S_i = \{x_i : (0, 0, \ldots, 0, x_i, 0, \ldots, 0) \in D\}$ for each $1 \leq i \leq n$. Similar to Case 2, we obtain that S_i is a semi-total dominating set for $\Gamma^*(\mathfrak{S}_i)$, for each $1 \leq i \leq n$. So

$$|D| \geq a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2) + \cdots + a^*(\mathfrak{S}_n).$$

On the other hand, let D_i be a $\lambda(\Gamma^*(\mathfrak{S}_i))$-set, for each $1 \leq i \leq n$. Define $T_i = \{(0, 0, \ldots, 0, x_i, 0, \ldots, 0) : x_i \in D_i\}$ for each $1 \leq i \leq n$. Since any vertex of $\Gamma^*(\mathfrak{S})$ is of the form $(x_1, x_2, \ldots, x_n)$, where at least on of $x_i \in V(\Gamma^*(\mathfrak{S}_i))$, for each $1 \leq \mathrm{i} \leq n$, we obtain that $T = \bigcup_{i=1}^{n} T_i$ is a dominating set for $\Gamma^*(\mathfrak{S})$. Hence,

$$\lambda(\Gamma^*(\mathfrak{S})) = a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2) + \cdots + a^*(\mathfrak{S}_n). \qquad \square$$

In the consequences of Theorem 3.3, we obtain the following interested corollary.

Corollary 3.1. *Let $n \geq 3$ be a fixed integer and $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$, where $\mathfrak{S}_i$ is a $*$-ring with unity for each $i = 1, 2, \ldots, n$ and $\mathfrak{S}$ is a $*$-ring with componentwise involution. Then,*

$$\lambda(\Gamma^*(\mathfrak{S})) = \lambda_{st}(\Gamma^*(\mathfrak{S})).$$

The minimum dominating sets for $\Gamma^*(\mathfrak{S})$ in proof of Theorem 3.3 are connected. This leads to the following.

Corollary 3.2. *Let $n \geq 3$ be a fixed integer and $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$, where $\mathfrak{S}_i$ is a $*$-ring with unity for each $i = 1, 2, \ldots, n$ and $\mathfrak{S}$ is a $*$-ring with componentwise involution. Then,*

$$\lambda(\Gamma^*(\mathfrak{S})) = \lambda_t(\Gamma^*(\mathfrak{S})) = \lambda_c(\Gamma^*(\mathfrak{S})).$$

Theorem 3.4. *Let $\mathfrak{S}_1$, $\mathfrak{S}_2$ be $*$-rings with unity and $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$ with componentwise involution. If $\min\{|\mathfrak{S}_1|, |\mathfrak{S}_2|\} \geq 3$, then*

$$\lambda(\Gamma^*(\mathfrak{S})) = a^*(\mathfrak{S}_1) + a^*(\mathfrak{S}_2).$$

Proof. The proof of this theorem is same as Theorem 3.3. $\square$

In Theorem 3.4, if $|\mathfrak{S}_1| = 2$ or $|\mathfrak{S}_2| = 2$, then $\lambda(\mathbb{Z}_2 \times R) = 2$, where R is a domain, which contradict Proposition 3.1.

Corollary 3.3. *Let $\mathfrak{S}$ be a $*$-ring with unity. If $\mathfrak{S}$ contains a nontrivial central projection, then*

$$\lambda(\Gamma^*(\mathfrak{S})) = \lambda_{st}(\Gamma^*(\mathfrak{S})) = \lambda_t(\Gamma^*(\mathfrak{S})) = \lambda_c(\Gamma^*(\mathfrak{S})).$$

Proof. Suppose $\mathfrak{S}$ contains a nontrivial central projection, say h. Then $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$, where $\mathfrak{S}_1 = h\mathfrak{S}$, $\mathfrak{S}_2 = (1-h)\mathfrak{S}$ are $*$-rings with unity. Since $|\mathfrak{S}_1| \geq 3$ and $|\mathfrak{S}_2| \geq 3$, the result follows from Theorem 3.4. $\square$

4. The domination number of zero-divisor graph of Rickart $*$-rings

In this section, we study domination in the zero-divisor graph of a Rickart $*$-ring. We begin our discussion with following known results given by A. Patil and B. N. Waphare [11].

Theorem 4.1 ([11, Lemma 3.1]). *Let $\mathfrak{S}$ be a Rickart $*$-ring. Then $a \in V(\Gamma^*(\mathfrak{S}))$ if and only if $RP(a)$ is nontrivial.*

Theorem 4.2 ([11, Proposition 3.2]). *Let $\mathfrak{S}$ be a Rickart $*$-ring. Then the vertices a and b are adjacent in $\Gamma^*(\mathfrak{S})$ if and only if $RP(a)RP(b) = 0$. Moreover, a vertex x is adjacent to a if and only if x is adjacent to $RP(a)$.*

Example 4.1. Let $\mathfrak{S} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with identity mapping as an involution. Observe that $\mathfrak{S}$ is a Rickart $*$-ring, with $V(\Gamma^*(\mathfrak{S})) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, where $a_1 = (1,0,0)$, $a_2 = (0,1,0)$, $a_3 = (0,0,1)$, $a_4 = (1,1,0)$, $a_5 = (1,0,1)$, $a_6 = (0,1,1)$ and set of projections $P(\mathfrak{S}) = \{0, 1, a_1, a_2, a_3, a_4, a_5, a_6\}$. The graph $\Gamma^*(\mathfrak{S})$ and the lattice of projection $L(P(\mathfrak{S}))$ is shown in Fig. 2. The set $T = \{a_1, a_2, a_3\}$ is a dominating set of $\Gamma^*(\mathfrak{S})$ and hence $\lambda(\Gamma^*(\mathfrak{S})) = 3$.

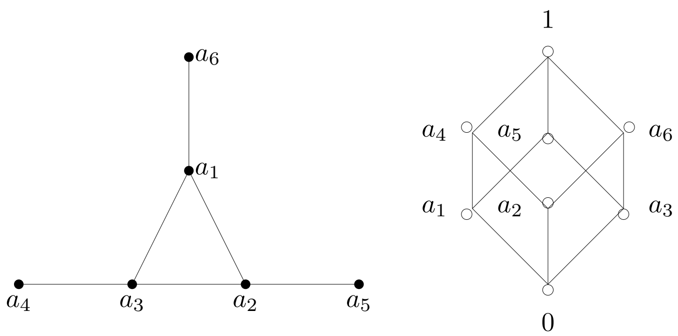

Fig. 2.

To characterize the domination in zero-divisor graph of Rickart $*$-ring, we needed the following results.

Proposition 4.1. *For any Rickart $*$-ring $\mathfrak{S}$, the set of all nontrivial projection $P^*(\mathfrak{S})$ is a dominating set of $\Gamma^*(\mathfrak{S})$.*

Proof. Let a be any vertex of $\Gamma^*(\mathfrak{S})$. Then, there exists a nontrivial projection e such that $RP(a) = e$. Since e is adjacent to $1 - e$, therefore by Theorem 4.2, a is also adjacent with $1 - e$. This proved that $P^*(\mathfrak{S})$ is a dominating set of $\Gamma^*(\mathfrak{S})$. □

"An element e in a lattice is an atom if $0 \leq f \leq e$ implies either $f = 0$ or $f = e$".

Theorem 4.3 ([11, Lemma 3.4]). *Let $\mathfrak{S}$ be a Rickart $*$-ring such that $\Gamma^*(\mathfrak{S})$ does not contain an infinite clique. Then the lattice $L(P(\mathfrak{S}))$ of projections of $\mathfrak{S}$ satisfies DCC (Descending Chain Condition).*

Theorem 4.4 ([11, Corollary 3.4])**.** *Let $\mathfrak{S}$ be a Rickart $*$-ring such that $\Gamma^*(\mathfrak{S})$ does not contain an infinite clique. Then every nonzero projection in $\mathfrak{S}$ contains an atom in $P(\mathfrak{S})$.*

Theorem 4.5. *Let $\mathfrak{S}$ be a Rickart $*$-ring such that $\Gamma^*(\mathfrak{S})$ does not contain an infinite clique. Then $\lambda(\Gamma^*(\mathfrak{S})) \leq n$, where n is the cardinality of maximal set of atoms in $P(\mathfrak{S})$.*

Proof. If $\Gamma^*(\mathfrak{S})$ does not contain an infinite clique, then by Theorem 4.3 and Theorem 4.4, $P(\mathfrak{S})$ contains finite number of atoms. Let $A = \{f_1, f_2, \ldots, f_n\}$ be the maximal set of atoms. We show that A is a dominating set for $\Gamma^*(\mathfrak{S})$. Let a be any vertex of $\Gamma^*(\mathfrak{S})$. Then there exists a nontrivial projection f such that $RP(a) = f$. We claim that f is adjacent with f_i for some $i = 1, 2, \ldots, n$. If $1 - f = f_i$ for some i, then we are done. If $1 - f \neq f_i$ for all i, then by Theorem 4.4, there exists an element $f_m \in A$ such that $f_m < (1 - f)$. This gives $ff_m = f_m f = 0$. Thus, $ff_i = 0$ for some $i = 1, 2, \ldots, n$. This shows that a is adjacent with f_i for some $i = 1, 2, \ldots, n$. Hence, A is a dominating set of $\Gamma^*(\mathfrak{S})$. This implies that $\lambda(\Gamma^*(\mathfrak{S})) \leq n$. □

With the help of the above theorem we can directly find the domination number of $\Gamma^*(\mathfrak{S})$, if we know all of its atoms in $P(\mathfrak{S})$. Consider the following example.

Example 4.2. Let $\mathfrak{S} = M_2(\mathbb{Z}_3)$ with transposition as an involution. By Berberian [7], $\mathfrak{S}$ is a Rickart $*$-ring and the set of projections in $\mathfrak{S}$ is $\{0, 1, e, 1 - e, f, 1 - f\}$, where $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $f = \left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$. The lattice of projection $P(\mathfrak{S})$ is shown in Fig. 3. The set of atoms of $P(\mathfrak{S})$ is $\{e, 1 - e, f, 1 - f\}$, which dominate $\Gamma^*(\mathfrak{S})$.

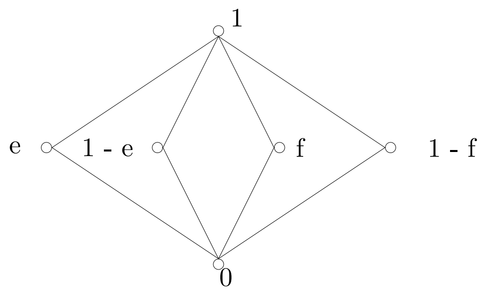

Fig. 3.

Lemma 4.1 ([7, Theorem 6])**.** *A $*$-ring $\mathfrak{S}$ with finitely many elements is a Rickart $*$-ring if and only if $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_r$, where $\mathfrak{S}_i$ is a field or $\mathfrak{S}_i$ is a 2×2-matrix over a finite field $\mathbb{F}(p^\alpha)$ with α an odd positive integer and p is a prime of the form $4k + 3$.*

Theorem 4.6 ([11, Theorem 3.7])**.** *Let $\mathfrak{S}$ be a Rickart $*$-ring such that $\Gamma^*(\mathfrak{S})$ does not contain an infinite clique. Then $\chi(\Gamma^*(\mathfrak{S})) = \omega(\Gamma^*(\mathfrak{S})) = n$, where n is the cardinality of maximal set of pairwise orthogonal atoms in $P(\mathfrak{S})$.*

Theorem 4.7. *Let $\mathfrak{S}$ is a Rickart $*$-ring and $n \geq 2$ be a fixed positive integer. If $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$, where $\mathfrak{S}_i$ is a field for each $i = 1, 2, \ldots, n$. Then*:

(a) $\lambda(\Gamma^*(\mathfrak{S})) = n = \omega(\Gamma^*(\mathfrak{S}))$ *if* $n \geq 3$;
(b) $\lambda(\Gamma^*(\mathfrak{S})) = 2 = \omega(\Gamma^*(\mathfrak{S}))$ *if* $n = 2$ *and* $\min\{|\mathfrak{S}_1|, |\mathfrak{S}_2|\} \geq 3$; *and*
(c) $\lambda(\Gamma^*(\mathfrak{S})) = 1 < \omega(\Gamma^*(\mathfrak{S}))$ *if* $n = 2$ *and* $\min\{|\mathfrak{S}_1|, |\mathfrak{S}_2|\} = 2$.

Proof. Clearly, the set $S = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$ is a dominating set of $\Gamma^*(\mathfrak{S})$. Also, S is the maximal set of pairwise orthogonal atoms in $P(\mathfrak{S})$. Thus, by Theorem 4.6, $\lambda(\Gamma^*(\mathfrak{S})) = n = \omega(\Gamma^*(\mathfrak{S}))$ if $n \geq 3$.

(b) and (c) follows from the fact that for $n = 2$, $\Gamma^*(\mathfrak{S})$ is a complete bipartite graph. □

Corollary 4.1. *For any given positive integer k, there exists a Rickart $*$-ring $\mathfrak{S}$ whose zero-divisor graph contains a maximal clique dominating set of size k, and obviously its domination number is equal to k. Moreover, for $k \geq 2$, $\lambda(\Gamma^*(\mathfrak{S})) = \omega(\Gamma^*(\mathfrak{S})) = k$.*

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