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ON WEAKLY QUASI n-ABSORBING SUBMODULES

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Abstract. Let R be a commutative ring with $1 \neq 0$, n be a positive integer and M be an R-module. In this paper, we introduce the concept of weakly quasi n-absorbing submodule which is a proper generalization of quasi n-absorbing submodule. We define a proper submodule N of M to be a weakly quasi n-absorbing submodule if whenever $a \in R$ and $x \in M$ with $0 \neq a^n x \in N$, then $a^n \in (N :_R M)$ or $a^{n-1}x \in N$. We study the basic properties of this notion and establish several characterizations.

1. Introduction

Throughout the whole paper, all rings are assumed to be commutative with $1 \neq 0$, all modules are considered to be unitary and n is a positive integer. Let R be a ring with $1 \neq 0$, M be an R-module and N be a proper submodule of M. In [9], the authors introduced and investigated the concept of 2-absorbing (resp., weakly 2-absorbing) submodules. They defined a submodule N to be a 2-absorbing submodule (resp., weakly 2-absorbing submodule) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp., $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. A more general concept than 2-absorbing submodule is the concept n-absorbing submodule. From [10], a proper submodule N of M is said to be an n-absorbing (resp., strongly n-absorbing) submodule of M if whenever $a_1 \cdots a_n m \in N$ for $a_1, \ldots, a_n \in R$ and $m \in M$ (resp., $I_1 \cdots I_n L \subset N$ for ideals $I_1, \ldots, I_n$ of R and a submodule L of M), then either $a_1 \cdots a_n \in (N :_R M)$ (resp., $I_1 \cdots I_n \subset (N :_R M)$) or there are $n-1$ of a_i's (resp., I_i's) whose product with m (resp., L) is in N. Recall that a proper submodule N of M is called semiprime if whenever $r \in R$ and $m \in M$ with $r^2 m \in N$, then $rm \in N$. For more details about the concept of n-absorbing and related notions, we refer the reader to [3, 4, 6, 7, 13].

In this paper, we introduce the concept of weakly quasi n-absorbing submodule which is a proper generalization of quasi n-absorbing submodule. We define a proper submodule N of M to be a weakly quasi n-absorbing submodule if whenever $a \in R$ and $x \in M$ with $0 \neq a^n x \in N$, then $a^n \in (N :_R M)$ or

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$a^{n-1}x \in N$. We study the basic properties of this notion and establish several characterizations.

We denote by $\sqrt{I}$, the radical of an ideal I of R. Let N be a submodule of an R-module M. We denote by $(N :_R M)$, the *residual* of N by M, that is, the set of all $r \in R$ such that $rM \subseteq N$. For $x \in M$, we denote by $\mathrm{ann}(x)$, the *annihilator* of x, that is, the set of all $r \in R$ such that $rx = 0$.

2. Results

It is worthwhile recalling that a proper submodule N of an R-module M is a quasi n-absorbing submodule for some positive integer $n \geq 1$, if $a^n x \in N$ for some $a \in R$ and $x \in M$ with $a^n x \in N$, then either $a^{n-1}x \in N$ or $a^n \in (N :_R M)$. Now, we recall the concept of weakly quasi n-absorbing submodule defined in the introduction.

Definition. A proper submodule N of an R-module M is called a weakly quasi n-absorbing submodule of M if $0 \neq a^n x \in N$ for some $a \in R$ and $x \in M$, then $a^n \in (N :_R M)$ or $a^{n-1}x \in N$.

Notice that from the previous definition, every quasi n-absorbing submodule is clearly a weakly quasi n-absorbing submodule. However, a weakly quasi n-absorbing submodule need not be a quasi n-absorbing submodule, as illustrated in the next example.

Example 2.1. Let $M := \mathbb{Z}/12\mathbb{Z}$ as $\mathbb{Z}$-module and $N = \{0\}$. Clearly, N is a weakly quasi 2-absorbing submodule of M. However, N is not a quasi 2-absorbing submodule of M since $(N :_{\mathbb{Z}} M) = 12\mathbb{Z}$ and $2^2 \cdot 3 \in N$ and neither $2^2 \in (N :_{\mathbb{Z}} M)$ nor $2 \cdots 3 \in N$.

Now, we introduce the following definition which will be useful for studying the weakly quasi n-absorbing submodules.

Definition. Let R be a ring, M be an R-module and N be a weakly quasi n-absorbing submodule of M. An element $a \in R$ is called an unbreakable element of N if there exists an element $x \in M$ such that $a^n x = 0$ and neither $a^n \in (N :_R M)$ nor $a^{n-1}x \in N$.

It is worthwhile mentioning that if N is a weakly quasi n-absorbing submodule of M and there is no unbreakable element, then N is a quasi n-absorbing submodule of M. The next lemma gives some basic facts about unbreakable elements.

Lemma 2.2. *Let R be a ring, M be an R-module and N be a proper weakly quasi n-absorbing submodule of M. If $a \in R$ is an unbreakable element of N. Then the following statements hold*:

(1) $a^n N = 0$.
(2) *$a + s$ is an unbreakable element of N for every $s \in (N :_R M)$.*

Proof. (1) Let a be an unbreakable element of N. Then there exists $x \in M$ with $a^n x = 0$ but neither $a^n \in (N :_R M)$ nor $a^{n-1}x \in N$. Assume by the way of contradiction that $0 \neq a^n N$, then $0 \neq a^n y \in N$ for some $y \in N$. Since N is a weakly quasi n-absorbing submodule of M and $a^n \notin (N :_R M)$, then $a^{n-1}y \in N$. On the other hand, $0 \neq a^n(x+y) = a^n y \in N$ and $a^n \notin (N :_R M)$ implies that $a^{n-1}(x+y) \in N$. Thus $a^{n-1}x \in N$, which is a contradiction. Hence, $a^n N = 0$.

(2) Since a is an unbreakable element of N, then there exists $x \in M$ with $a^n x = 0$ and neither $a^n \in (N :_R M)$ nor $a^{n-1}x \in N$. Now let $s \in (N :_R M)$. Assume that $0 \neq (a+s)^n x$. We have:

$$(a+s)^n x = \sum_{j=0}^{m-1} \binom{n}{j} a^j s^{n-j} x \in N.$$

The fact that N is a weakly quasi n-absorbing submodule of M, gives either $(a+s)^{n-1}x \in N$ or $(a+s)^n \in (N :_R M)$. Two cases are then possible:

Case 1 : $(a+s)^{n-1}x \in N$. Then one can easily check that $a^{n-1}x \in N$ since for all $j = 1, \dots, n-1$, $a^j s^{n-1-j} x \in N$, the desired contradiction.

Case 2 : $(a+s)^n \in (N :_R M)$. Since $a^j s^{n-j} \in (N :_R M)$, then $a^n \in (N :_R M)$. Hence, $(a+s)^n x = 0$ and neither $(a+s)^{n-1}x \in N$ nor $(a+s)^n \in (N :_R M)$. Thus, it follows that $a+s$ is an unbreakable element of N.

Finally, $a+s$ is an unbreakable element of N, as desired. □

Theorem 2.3. *Let R be a ring, M be an R-module and N be a proper weakly quasi n-absorbing submodule which is not quasi n-absorbing submodule of M. Then $(N :_R M) \subseteq \sqrt{ann(N)}$.*

Proof. Since N is a weakly quasi n-absorbing submodule which is not quasi n-absorbing submodule of M, then there exists an unbreakable element b of N. By Lemma 2.2(2), for every $a \in (N :_R M)$, we have $(b+a)^n N = 0$. So, $a+b \in \sqrt{\text{ann}(N)}$. By Lemma 2.2(1), $b \in \sqrt{\text{ann}(N)}$ and so $a \in \sqrt{\text{ann}(N)}$. Hence, $(N :_R M) \subseteq \sqrt{\text{ann}(N)}$, as desired. □

Let R be a ring and M be an R-module. Recall that M is called a multiplication module if for each submodule N of $M, N = IM$ for some ideal I of R. In this case, we can take $I = (N :_R M)$ [11]. Also, recall that for a submodule N of M, if $N = IM$ for some ideal I of R, then I is called a presentation ideal of N. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1 M$ and $K = I_2 M$ for some ideals I_1 and I_2 of R, the product N and K denoted by NK is defined by $NK = I_1 I_2 M$. From [1, Theorem 3.4], the product of N and K is independent of presentation of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule and $NK \subseteq N \cap K$ [1]. A submodule N of an R module M is called nilpotent if $(N :_R M)^k N = 0$ for some positive integer k [2]. The next corollary is a consequence of Theorem 2.3.

Corollary 2.4. *Let R be a Noetherian ring and M be an R-module. If N is a proper weakly quasi n-absorbing submodule which is not quasi n-absorbing submodule of M. Then*:

(1) *N is nilpotent.*
(2) *If M is a faithful multiplication module, then $N^p = 0$ for some positive integer p.*

Proof. (1) By Theorem 2.3, we have $(N :_R M) \subseteq \sqrt{\text{ann}(N)}$. Since R is Noetherian, then there exists a positive integer $k \geq 1$ such that $(N :_R M)^k \subseteq \text{ann}(N)$. So, $(N :_R M)^k N = 0$. Hence, N is a nilpotent submodule of M.

(2) By assertion (1) above, we have $(N :_R M)^k N = 0$ for some positive integer $k \geq 1$. It follows that $(N :_R M)^{k+1} \subseteq ((N :_R M)^k N :_R M) = (0 :_R M) = 0$, as M is faithful. Therefore, $(N :_R M)^{k+1} = 0$. Thus, $N^{k+1} = 0$. □

Let N be a proper submodule of a nonzero R-module M. Then the M-radical of N, denoted here by $M - \sqrt{N}$ is defined in [12] to be the intersection of all prime submodules of M containing N. It is shown in [11, Theorem 2.12] that if N is a proper submodule of M, then $M - \sqrt{N} = M - \sqrt{(N :_R M)M}$. The next corollary is an application of Theorem 2.3.

Corollary 2.5. *Let R be a ring, M be a multiplication R-module and N be a proper faithful weakly quasi n-absorbing submodule which is not quasi n-absorbing submodule of M. Then $N \subseteq M - \sqrt{0}$.*

Proof. Since M is a multiplication module, then $N = (N :_R M)M$. So, by Theorem 2.3, it follows that $N = (N :_R M)M \subseteq \sqrt{0}M = M - \sqrt{0}$, as N is faithful. □

Recall that a ring is called von Neumann regular if, for every $x \in R$ there exists $y \in R$ such that $x^2 y = x$. It is well known that a commutative ring is von Neumann regular if and only if every proper ideal is radical. The next corollary is another consequence of Theorem 2.3.

Corollary 2.6. *Let R be a von Neumann regular ring, M be an R-module and N be a proper weakly quasi n-absorbing submodule which is not quasi n-absorbing submodule of M. Then $(N :_R M)N = 0$.*

Proof. Assume that R is a von Neumann regular ring. Since N is a weakly quasi n-absorbing submodule which is not quasi n-absorbing submodule of M, then by Theorem 2.3, $(N :_R M) \subseteq \sqrt{ann(N)}$. Using the fact that R is a von Neumann regular ring, then $\sqrt{ann(N)} = ann(N)$. Thus, it follows that $(N :_R M)N = 0$. □

The next corollary is another application of Corollary 2.6.

Corollary 2.7. *Let R be a von Neumann regular ring, M be a faithful R-module and N be a proper weakly quasi n-absorbing submodule which is not quasi n-absorbing submodule of M. Then $(N :_R M)^2 = 0$.*

Proof. By Corollary 2.6, we have $(N :_R M)N = 0$. So, $(N :_R M)^2 \subseteq ((N :_R M)N :_R M) = (0 :_R M) = \text{ann}(M) = 0$ as M is faithful and so $(N :_R M)^2 = 0$, as desired. □

In the following theorem, we establish that for a ring R in which 2 is unit of R and M be an R-module, we have $(N :_R M)^2 N = 0$ for every weakly quasi 2-absorbing submodule N which is not quasi 2-absorbing submodule of M.

Theorem 2.8. *Let R be a ring with 2 is unit in R and M be an R-module. If N is a weakly quasi 2-absorbing submodule which not a quasi 2-absorbing submodule, then $(N :_R M)^2 N = 0$.*

Proof. By Lemma 2.2, for every $s \in (N :_R M), (a + s)^2 N = (a - s)^2 N = 0$ where a is an unbreakable element of N. Thus $2(a^2 + s^2)N = 2s^2 N = 0$. Since 2 is unit, then $s^2 N = 0$ for every $s \in (N :_R M)$. Now let $s, t \in (N :_R M)$, we have $2stN = ((s + t)^2 - s^2 - t^2)N = 0$, so $stN = 0$ as 2 is unit. We conclude that $(N :_R M)^2 N = 0$. □

Let M be an R-module and N be a proper submodule of M. We say that N is a weakly strongly quasi n-absorbing submodule of M if whenever $0 \neq I^n L \subseteq N$ for some proper ideal I of R and a proper submodule of M, then either $I^n \subseteq (N :_R M)$ or $I^{n-1} L \subseteq N$. It is clear that a weakly strongly quasi n-absorbing submodule is a weakly quasi n-absorbing submodule. In the next theorem, we show that the notions weakly strongly quasi n-absorbing submodule and weakly quasi n-absorbing submodule collapse in the case the ring R is a principal domain.

Theorem 2.9. *Let R be a principal domain and N be a proper submodule of an R-module M. Then the following assertions are equivalent*:

(1) *N is a weakly quasi n-absorbing submodule of M.*
(2) *N is a weakly strongly quasi n-absorbing submodule of M.*

Proof. (1) $\Rightarrow$ (2) Let $0 \neq I^n L \subseteq N$ for some proper ideal I of R and a proper submodule L of M. Since R is a principal domain, then there exists an element $a \in R$ such that $I = Ra$. So, $0 \neq a^n L \subseteq N$. Assume that $a^n \notin (N :_R M)$. we claim that $a^{n-1} L \subseteq N$. Indeed, let $x \in L$. If $0 \neq a^n x$, then $a^{n-1} x \in N$ since N is a weakly quasi n-absorbing submodule and $a^n \notin (N :_R M)$. Now assume that $a^n x = 0$. Since $a^n L \neq 0$, then $0 \neq a^n y = a^n(x + y) \in N$ for some $y \in N$. Consequently, $a^{n-1}(x + y) \in N$ and so $a^{n-1} x \in N$ as $a^{n-1} y \in N$ which is a weakly quasi n-absorbing submodule. Therefore, $a^{n-1} L \subseteq N$. Hence, $I^{n-1} L \subseteq N$.

(2) $\Rightarrow$ (1) Straightforward. □

Proposition 2.10. *Let N be a proper submodule of M. Then the following statements are equivalent*:

(1) *If $0 \neq I^n L \subseteq N$ for some ideal I of R and submodule L of M, then either $I^n \subseteq (N :_R M)$ or $I^{n-1} L \subseteq N$.*

(2) *If* $0 \neq I^n x \subseteq N$ *for some ideal* I *of* R *and* $x \in M$, *then* $I^n \subseteq (N :_R M)$ *or* $I^{n-1}x \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (1) Suppose that $0 \neq I^n L \subseteq N$ for some ideal I of R and submodule L of M. Assume that $I^n \nsubseteq (N :_R M)$ and we show that $I^{n-1}L \subseteq N$. By the way of contradiction, suppose $I^{n-1}L \nsubseteq N$. Then there exists an element x of L with $I^{n-1}x \nsubseteq N$. Two cases are then possible:

Case 1 : If $0 \neq I^n x \subseteq N$. Since $I^n \nsubseteq (N :_R M)$, from assumption it follows that $I^{n-1}x \subseteq N$, which is a contradiction.

Case 2 : If $I^n x = 0$. The fact that $0 \neq I^n L \subseteq N$, there exists an element y of L with $0 \neq I^n y \subseteq N$. Now $0 \neq I^n(x + y) = I^n y \subseteq N$. Since $I^n \nsubseteq (N :_R M)$, then it follows that $I^{n-1}y \subseteq N$ and $I^{n-1}(x + y) \subseteq N$. Hence, $I^{n-1}x \subseteq N$, which is a contradiction again.

Finally, $I^{n-1}L \subseteq N$. □

In the next proposition, we study the stability of homomorphic image of a weakly quasi n-absorbing submodule.

Proposition 2.11. *Let* N, L *be submodules of an* R*-module* M *with* $L \subseteq N$. *If* N *is a weakly quasi* n*-absorbing submodule of* M, *then* N/L *is a weakly quasi* n*-absorbing submodule of* M/L. *The converse holds if* L *is a weakly quasi* n*-absorbing submodule of* M.

Proof. Assume that N is a weakly quasi n-absorbing submodule of M. Let $a \in R$ and $x + L \in M/L$ with $0_{M/L} \neq a^n(x + L) \in N/L$. If $a^n \in (N :_R M)$, then we are done. We may assume that $a^n \notin (N :_R M)$. The fact that $0_{M/L} \neq a^n(x + L)$ implies that $a^n x \in N$ and $a^n x \notin L$. So, $0 \neq a^n x \in N$. Since N is a weakly quasi n-absorbing submodule of M and $a^n \notin (N :_R M)$, then $a^{n-1}x \in N$. Therefore, $a^{n-1}(x + L) \in N/L$ and so N/L is a weakly quasi n-absorbing submodule of M/N. Conversely, assume that L is a weakly quasi n-absorbing submodule of M and N/L is a weakly quasi n-absorbing submodule of M/L. Let $a \in R$ and $x \in M$ with $0 \neq a^n x \in N$. Then $a^n(x + L) \in N/L$. If $a^n(x + L) = 0_{M/L}$, then $0 \neq a^n x \in L$. Using the fact that L is a weakly quasi n-absorbing submodule of M, then either $a^{n-1}x \in L \subseteq N$ or $a^n \in (L :_R M) \subseteq (N :_R M)$. If $a^n(x + L) \neq 0_{M/L}$. Then either $a^n \in (N/L :_R M/L)$ or $a^{n-1}(x + L) \in N/L$. Hence, $a^n \in (N :_R M)$ or $a^{n-1}x \in N$. Finally, N is a weakly quasi n-absorbing submodule of M, as desired. □

Recall that from [8, Definition 2.20(2)], a submodule N of an R-module M is said to be a strongly (m, n)-closed submodule if whenever I is an ideal and L is a submodule of M with $I^m L \subseteq N$ implies that $I^n \subseteq (N :_R M)$ or $I^{n-1}L \subseteq N$.

Theorem 2.12. *Let* N *be a proper submodule of an* R*-module* M. *Then the following statements are equivalent*:

(1) *If* $0 \neq I^n K \subseteq N$ *for some ideal* I *of* R *and submodule* K *of* M, *then either* $I^n \subseteq (N :_R M)$ *or* $I^{n-1}K \subseteq N$.

(2) *For any ideal I of R and $N \subseteq L$ a submodule of M with $0 \neq I^m L \subseteq N$ implies $I^n \subseteq (N :_R M)$ or $I^{n-1} L \subseteq N$.*

Proof. (1) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (1) Let I be an ideal of R and K be a submodule of M with $0 \neq I^m K \subseteq N$. Then $0 \neq I^m(K+N) \subseteq N$. Since N is a strongly (m,n)-closed submodule of M and $L := K + N \supseteq N$, then $I^n \subseteq (N :_R M)$ or $I^{n-1}L = I^{n-1}(K+N) \subseteq N$ from the hypothesis (2). Thus $I^n \subseteq (N :_R M)$ or $I^{n-1}K \subseteq N$. $\square$

In the next theorem we show the relationship between a weakly quasi n-absorbing submodule N and the ideal $(N :_R x)$ of R, where $x \in M \setminus N$. Recall from [5] that an ideal I of a ring R is a weakly semi n-absorbing ideal of R if $0 \neq x^{n+1} \in I$ implies $x^n \in I$.

Theorem 2.13. *Let M be an R-module and N be a proper submodule of M.*

(1) *If $(N :_R x)$ is a weakly semi n-absorbing ideal of R for every $x \in M \setminus N$, then N is a weakly quasi n-absorbing submodule of M.*

(2) *Assume that N is a weakly quasi n-absorbing submodule of M. Let x be an element of $M \setminus N$ such that $ann(x)$ is a quasi n-absorbing ideal of R. Then $(N :_R x)$ is a weakly quasi n-absorbing ideal of R for each $x \in M \setminus N$.*

Proof. (1) Let $0 \neq a^n y \in N$ for some $a \in R$ and $y \in M$. If $y \in N$, then we are done. We may assume $y \in M \setminus N$. If $a^n \in (N :_R M)$, then we are done. So, we may assume $a^n \notin (N :_R M)$ and so $0 \neq a^n$. Since $a^n \in (N :_R y)$ which is a weakly semi n-absorbing ideal of R, then $a^{n-1} \in (N :_R y)$ and so $a^{n-1}y \in N$. Hence, N is a weakly quasi n-absorbing submodule of M.

(2) Let $x \in M \setminus N$. Suppose that $0 \neq a^n y \in (N :_R x)$ and $a^n \notin (N :_R x)$ for some $a \in R$ and $y \in M$. If $0 \neq a^n yx \in N$. Since N is a weakly quasi n-absorbing submodule of M and $a^n \notin (N :_R M)$, then $a^{n-1}yx \in N$. Hence, $a^{n-1}y \in (N :_R x)$. Now, suppose that $a^n yx = 0$. From assumption, it follows that $a^{n-1}y \in \operatorname{ann}(x)$, which implies that $a^{n-1}y \in (N :_R x)$. Consequently, $(N :_R x)$ is a weakly quasi n-absorbing ideal of R, as desired. $\square$

Theorem 2.14. *Let M be a faithful R-module and N be a proper submodule of M. If N is a weakly quasi n-absorbing submodule of R, then $(N :_R M)$ is a weakly quasi n-absorbing ideal of R. The converse holds if M is a cyclic faithful R-module.*

The proof of the previous theorem requires the following lemma.

Lemma 2.15. *Let N be a proper submodule of an R-module M. Then the following statements are equivalent*:

(1) *N is a weakly quasi n-absorbing submodule of M.*

(2) *For every $a \in R$ and L a submodule of M with $0 \neq a^n L \subset N$, then $a^{n-1}L \subset N$ or $a^n \in (N :_R M)$.*

Proof. (1) ⇒ (2) Assume that N is a weakly quasi n-absobing submodule of M. Let $a \in R$ and L be a submodule of M such that $0 \neq a^n L \subset N$ and $a^n \notin (N :_R M)$. Let $x \in L$. If $0 \neq a^n x$, then $a^{n-1}x \in N$ (as N is a weakly quasi n-absorbing submodule of R). We may assume that $a^n x = 0$. The fact that $0 \neq a^n L \subset N$ gives $0 \neq a^n y \in N$ for some $y \in L$. Since $a^n \notin (N :_R M)$, it follows that $a^{n-1}y \in N$. Set $z = y + x \in L$. So, $a^n z \neq 0$ and with similar argument as above, we get $a^{n-1}z \in L$. Therefore, $a^{n-1}x \in N$. Hence, for every $x \in L$, $a^{n-1}x \in N$. Finally, $a^{n-1}L \subset N$.

(2) ⇒ (1) Assume that for every $a \in R$ and L a submodule of M with $0 \neq a^n L \subset N$, then $a^{n-1}L \subset N$ or $a^n \in (N :_R M)$. Let $0 \neq a^n x \in N$ for some $a \in R$ and $x \in M$. Set $L = Rx$. Then $0 \neq a^n L \subset N$. From assumption, we get $a^n \in (N :_R M)$ or $a^{n-1}L \subset N$ and so $a^{n-1}x \in N$ or $a^n \in (N :_R M)$. Hence, N is weakly quasi n-absorbing submodule of M, as desired. □

Proof of Theorem 2.14. Let $0 \neq a^n b \in (N :_R M)$ for some $a, b \in R$. Since M is a faithful R-module, then $0 \neq a^n bM = a^n(bM) \subset N$. By Lemma 2.15, $a^{n-1}(bM) = a^{n-1}bM \subset N$ or $a^n \in (N :_R M)$. Hence, $(N :_R M)$ is a weakly quasi n-absorbing ideal of R. Conversely, assume that $(N :_R M)$ is a weakly quasi n-absorbing ideal of R and $M = Rm$ is a cyclic faithful R-module. Let $a \in R$ and $x \in M$ such that $0 \neq a^n x \in N$. Then there exists $b \in R$ such that $x = bm$. So, $0 \neq a^n bm \in N$. Therefore, $0 \neq a^n b \in (N :_R m) = (N :_R M)$. The fact that $(N :_R M)$ is a weakly quasi n-absorbing ideal of R, gives either $a^n \in (N :_R M)$ or $a^{n-1}b \in (N :_R M)$. Hence, $a^n \in (N :_R M)$ or $a^{n-1}bm = a^{n-1}x \in N$, making N, a weakly quasi n-absorbing submodule of M. □

It is worth to mention that in Theorem 2.14 the condition "M is a faithful R-module" is necessary. Otherwise, if N is a weakly quasi n-absorbing submodule of M, then $(N :_R M)$ need not be a weakly quasi n-absorbing ideal of R, as shown in the next example.

Example 2.16. Consider the $\mathbb{Z}$-module $M := \mathbb{Z}/16\mathbb{Z}$ and $N = \{0\}$. Observe that $\text{ann}(M) = 16\mathbb{Z}$. So, M is not faithful. On the other hand, N is a weakly quasi 2-absorbing submodule and $(N :_{\mathbb{Z}} M) = 16\mathbb{Z}$ is not a weakly quasi 2-absorbing ideal of $\mathbb{Z}$ since $2^2.4 \in (N :_{\mathbb{Z}} M)$ but neither $2.4 = 8 \in (N :_{\mathbb{Z}} M) = 16\mathbb{Z}$ nor $2^2 \in (N :_{\mathbb{Z}} M)$.

Let R be a ring. It is well known that a proper submodule N of an R-module M is said to be a weakly semiprime submodule of M if $0 \neq r^2 x \in N$ for some $r \in R$ and $x \in M$, then $rx \in N$. In the next theorem, we show that the class of weakly semiprime submodules is contained in the class of weakly quasi n-absorbing submodules for every positive integer $n \geq 2$.

Theorem 2.17. *Let R be a ring, M be an R-module and N be a proper submodule of M. If N is a weakly semiprime submodule of M, then N is a weakly quasi n-absorbing submodule of M for every positive integer $n \geq 2$.*

Proof. Let $0 \neq a^n x \in N$ for some $a \in R$, $x \in M$ and for some positive integer $n \geq 2$. Then $0 \neq a^2(a^{n-2}x) \in N$. Since N is a weakly semiprime submodule of M, we get $0 \neq a^{n-1}x \in N$. Hence, N is a weakly quasi n-absorbing submodule of M, as desired. □

The following theorem shows that the intersection of a family of weakly semiprime submodules is a weakly quasi-$n$absorbing submodule.

Theorem 2.18. *Let R be a ring, M be an R-module. Let $(N_i)_{i\in I}$ be a family of weakly semiprime submodules of M. Then $\bigcap_{i\in I} N_i$ is a weakly quasi n-absorbing submodule of M for all positive integer $n \geq 2$.*

Proof. Suppose that $0 \neq a^n x \in N := \bigcap_{i\in I} N_i$ for some $a \in R$ and $x \in M$. Then $0 \neq a^n x \in N_i$ for all $i \in I$. Since N_i is a weakly semiprime module, then $ax \in N_i$ for all $i \in I$. Therefore, $a^{n-1}x = a^{n-2}(ax) \in N_i$ for all $i \in I$ and so $a^{n-1}x \in N$. Hence, $\bigcap_{i\in I} N_i$ is a weakly quasi n-absorbing submodule of M for all positive integer $n \geq 2$. □

Theorem 2.19. *Let M_1, M_2 be R-modules with $M = M_1 \oplus M_2$, n be a positive integer and N_1 (resp., N_2) be a proper submodule of M_1 (resp., M_2). Then the following statements are equivalent*:

(1) *$N_1 \oplus M_2$ (resp., $M_1 \oplus N_2$) is a weakly quasi n-absorbing submodule of M which is not a quasi n-absorbing submodule.*
(2) *If N_1 (resp., N_2) is a weakly quasi n-absorbing submodule of M_1 (resp., M_2) which is not a quasi n-absorbing submodule of M_1 (resp., M_2) and $a^n M_2 = 0$ (resp., $a^n M_1 = 0$) for every unbreakable element a of N_1 (resp., N_2).*

The proof of the previous theorem needs the following lemma.

Lemma 2.20. *Let M_1, M_2 be R-modules with $M = M_1 \oplus M_2$, n be a positive integer and N_1 (resp., N_2) be proper weakly quasi n-absorbing submodule of M_1 (resp., M_2). Let $a \in R$. Then the following statements are equivalent*:

(1) *a is an unbreakable element of N_1 (resp., N_2).*
(2) *a is an unbreakable element of $N_1 \oplus M_2$ (resp., $M_1 \oplus N_2$).*

Proof. Assume that a is an unbreakable element of N_1. Then there exists $x \in M_1$ with $a^n x = 0$ and neither $a^n \in (N_1 :_R M_1)$ nor $a^{n-1}x \in N_1$. Then $a^n(x, 0) = (0, 0)$ and neither $a^n \in (N_1 \oplus M_2 :_R M_1 \oplus M_2)$ nor $a^{n-1}(x, 0) \in N_1 \oplus M_2$. Hence, a is an unbreakable element of $N_1 \oplus M_2$. Conversely, assume that $a \in R$ is an unbreakable element of $N_1 \oplus M_2$. So there exists $(x, y) \in M_1 \oplus M_2$ with $a^n(x, y) = (0, 0)$ and neither $a^n \in (N_1 \oplus M_2 :_R M_1 \oplus M_2)$ nor $a^{n-1}(x, y) \in N_1 \oplus M_2$. Hence, $a^n x = 0$ for $x \in M_1$ and neither $a^n \in (N_1 :_R M_1)$ nor $a^{n-1}x \in N_1$. Thus, a is an unbreakable element of N_1.

With similar proof as above, one can easily show that a is an unbreakable element of N_2 if and only if a is an unbreakable element of $M_1 \oplus N_2$. □

Proof of Theorem 2.19. (1) $\Rightarrow$ (2) Assume that $N_1 \oplus M_2$ is a weakly quasi n-absorbing submodule of M which is not a quasi n-absorbing submodule. Then by Proposition 2.11, $N_1 \simeq \frac{N_1 \oplus M_2}{0 \oplus M_2}$ is a weakly n-absorbing submodule of M_1. Now, by Lemma 2.20, it follows that N_1 is not a quasi n-absorbing submodule of M_1 since N_1 admits an element which is unbreakable $a \in R$, as a is an unbreakable element of $N_1 \oplus M_2$. It remains to show that if a is an unbreakable element of N_1, then $a^n M_2 = 0$. Assume by the way of contradiction that a is an unbreakable element of N_1 and $a^n M_2 \neq 0$. Then $a^n y \neq 0$ for some $y \in M_2$. Since a is an unbreakable element of N_1, then there exists $x \in M_1$ with $a^n x = 0$ and neither $a^n \in (N_1 :_R M_1)$ nor $a^{n-1}x \in N_1$. Since $0 \neq a^n(x, y) \in N_1 \oplus M_2$, then the fact that $N_1 \oplus M_2$ is a weakly quasi n-absorbing submodule of $M_1 \oplus M_2$ and $a^n \notin (N_1 \oplus M_2 : M_1 \oplus M_2)$ give that $a^{n-1}x \in N_1$, which is a contradiction. Hence, $a^n M_2 = 0$.

(2) $\Rightarrow$ (1) Assume that N_1 is a weakly quasi n-absorbing which is not quasi n-absorbing submodule of M_1 and $a^n M_2 = 0$ for every unbreakable a element of N_1. Let $b \in R$ and $(x, y) \in M_1 \oplus M_2$ with $0 \neq b^n(x, y) \in N_1 \oplus M_2$. If $0 \neq b^n x \in N_1$, then either $b^n \in (N_1 \oplus M_2 :_R M)$ or $b^{n-1}(x, y) \in N_1 \oplus M_2$. Now, suppose that $b^n = 0$ and neither $b^n \in (N_1 :_R M_1)$ nor $b^{n-1} \in N_1$, then b is an unbreakable element of N_1. From assumption, we have $b^n M_2 = 0$, and so $b^n(x, y) = 0$, which is a contradiction. Therefore, either $b^n \in (N_1 \oplus M_2 :_R M_1 \oplus M_2)$ or $b^{n-1}(x, y) \in N_1 \oplus M_2$. Finally, we conclude that $N_1 \oplus M_2$ is a weakly quasi n-absorbing submodule of M. Now the fact $N_1 \oplus M_2$ is not a quasi n-absorbing submodule of M follows from Lemma 2.20. The proof is complete. □

Now we establish some facts for $N_1 \bigoplus N_2$ to be a quasi n-absorbing submodule of $M_1 \oplus M_2$ for some positive integer $0 < n$.

Theorem 2.21. *Let M_1, M_2 be R-modules and N_1 (resp., N_2) be a submodule of M_1 (resp., M_2). If $N_1 \oplus N_2$ is a weakly quasi n-absorbing submodule of $M = M_1 \oplus M_2$ that is not quasi n-absorbing submodule for some positive integer $n > 0$, then one of the following two assertions hold*:

1. *N_1 and N_2 are weakly quasi n-absorbing submodules and if there exists an unbreakable element a of N_1, then $a^n N_2 = 0$.*
2. *N_1 and N_2 are weakly quasi n-absorbing submodules and if there exists an unbreakable element b of N_2, then $b^n N_1 = 0$.*

Proof. (1) Suppose that $N_1 \oplus N_2$ is a weakly quasi n-absorbing submodule that is not quasi n-absorbing submodule of M. Let $a \in R$ and $x \in M_1$ with $0 \neq a^n x \in N_1$. Then $0 \neq a^n(x, 0) \in N_1 \oplus N_2$ which is a weakly quasi n-absorbing submodule of M. It follows that $a^{n-1}x \in N_1$ or $a^n \in (N_1 :_R M_1)$. Hence, N_1 is a weakly quasi n-absorbing submodule of M_1. The same argument shows that N_2 is a weakly quasi n-absorbing submodule of M_2. Now, suppose that N_1 admits an unbreakable element $a \in R$. Then $a^n x = 0$ but neither $a^n \in (N_1 :_R M_1)$ nor $a^{n-1}x \in N_1$ for some $x \in M_1$. Assume that $a^n N_2 \neq 0$.

Then there exists $z \in N_2$ such that $0 \neq a^n z \in N_2$, so $0 \neq a^n(x,z) = (0, a^n z) \in N_1 \oplus N_2$ which is a weakly quasi n-absorbing submodule of M. So, either $a^{n-1}(x,z) \in N_1 \oplus N_2$ or $a^n \in (N_1 \oplus N_2 :_R M)$. Therefore, $a^{n-1}x \in N_1$ or $a^n \in (N_1 :_R M_1)$, which is a contradiction. Hence, $a^n N_2 = 0$.

(2) Similar proof as assertion (1) above. □

Remark 2.22. Let N_1 (resp., N_2) be a submodule of M_1 (resp., M_2). If N_1 and N_2 are weakly quasi n-absorbing submodules, then $N_1 \oplus N_2$ need not be a weakly quasi n-absorbing submodule of $M_1 \oplus M_2$. For instance, take $M_1 = M_2 = \mathbb{Z}$ and $N_1 = 2^2\mathbb{Z}$, $N_2 = 3\mathbb{Z}$. It is clear that N_1 and N_2 are weakly quasi 2-absorbing submodules of $\mathbb{Z}$ since they are quasi 2-absorbing submodules. However, $N_1 \oplus N_2$ is not a weakly quasi 2-absorbing submodule of $M_1 \oplus M_2$ since $2^2.(3,3) \in 2^2\mathbb{Z} \oplus 3\mathbb{Z}$, but neither $2^2 = 4 \in (2^2\mathbb{Z} \oplus 3\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z}) = 12\mathbb{Z}$ nor $2(3,3) = (6,6) \in 2^2\mathbb{Z} \oplus 3\mathbb{Z}$.

The next theorem establishes about when the submodule $N_1 \bigoplus N_2$ is a weakly quasi $(n+1)$-absorbing submodule.

Theorem 2.23. *Let M_1, M_2 be R-modules and N_1 (resp., N_2) be a submodule of M_1 (resp., M_2). Consider the following assertions*:

(1) *N_1 is a weakly quasi n-absorbing submodule of M_1, N_2 is a quasi n-absorbing submodule of M_2 and $a^n y = 0$ whenever $a^n y \in N_2$, for some $a \in R$ and $y \in M_2$.*

(2) *N_2 is a weakly quasi n-absorbing submodule of M_2, N_1 is a quasi n-absorbing submodule of M_1 and $a^n x = 0$ whenever $a^n x \in N_1$, for some $a \in R$ and $x \in M_1$.*

If (1) or (2) holds, then $N_1 \bigoplus N_2$ is a weakly quasi $(n+1)$-absorbing submodule of M.

Proof. Suppose that (1) holds. Let $0 \neq a^{n+1}(x,y) \in N_1 \oplus N_2$ for some $a \in R$ and $(x,y) \in M$. From assumption $a^n(ay) = 0$ and $0 \neq a^n(ax) \in N_1$ and N_2 is a quasi n-absorbing submodule of M_2. Since N_1 is a weakly quasi n-absorbing submodule of M_1, it follows that $a^n x \in N_1$. On the other hand $a^n(ay) = 0 \in N_2$ and the fact that N_2 is a quasi n-absorbing submodule of M_2 gives $a^n y \in N_2$. Finally, $a^n(x,y) \in N_1 \oplus N_2$. Hence, $N_1 \bigoplus N_2$ is a weakly quasi $n+1$-absorbing submodule of M. The same argument if assertion (2) holds. □

The next proposition examines the weakly quasi n-absorbing submodules under localization.

Proposition 2.24. *Let N be a proper submodule of an R-module M and S be a multiplicative closed subset consisting entirely of nonzero divisor elements of R such that $(N :_R M) \cap S = \emptyset$. If N is a weakly quasi n-absorbing submodule of M, then $S^{-1}N$ is a weakly quasi n-absorbing submodule of $S^{-1}M$.*

Proof. Let $\frac{0}{1} \neq (\frac{a}{s_1})^n(\frac{m}{s_2}) \in S^{-1}N$. Then $0 \neq ua^n m \in N$ for some element u of S. So, $0 \neq (ua)^n m \in N$ which is a weakly quasi n-absorbing submodule of M. Therefore, $(ua)^{n-1}m \in N$ or $(ua)^n \in (N :_R M)$. Consequently, $\frac{u^{n-1}a^{n-1}m}{u^{n-1}s_1^{n-1}s_2} = (\frac{a}{s_1})^{n-1}(\frac{m}{s_2}) \in S^{-1}N$ or $\frac{u^n a^n}{u^n s_1^n} = (\frac{a}{s_1})^n \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$. Hence, $S^{-1}N$ is a weakly quasi n-absorbing submodule of $S^{-1}M$, as desired. □

The following proposition studies the weakly quasi n-absorbing property under homomorphism.

Proposition 2.25. *Let $f : M \to M'$ be a homomorphism of R-modules.*

1. *Assume that f is a monomorphism. If N' is a weakly quasi n-absorbing submodule of M', then $f^{-1}(N')$ is a weakly quasi n-absorbing submodule of M.*
2. *Assume that f is an epimorphism and* $\ker(f) \subseteq N$. *If N is a weakly quasi n-absorbing submodule of M, then $f(N)$ is a weakly quasi n-absorbing submodule of M'.*

Proof. (1) Assume that f is a monomorphism of R-modules and N' is a weakly quasi n-absorbing submodule of M'. Let $0 \neq a^n x \in f^{-1}(N')$ for some $a \in R$ and $x \in M$. Then $0 \neq a^n f(x) \in N'$ which is a weakly quasi n-absorbing submodule of M'. So, $a^n \in (N' :_R M')$ or $a^{n-1}f(x) \in N'$. Therefore, $a^n M' \subseteq N'$ or $f(a^{n-1}x) \in N'$. Hence, it follows that $a^n M \subseteq f^{-1}(N')$ or $a^{n-1}x \in f^{-1}(N')$. Thus, $a^n \in (f^{-1}(N') :_R M)$ or $a^{n-1}x \in f^{-1}(N')$, making $f^{-1}(N')$, a weakly quasi n-absorbing submodule of M.

(2) Assume that f is an epimorphism, $\ker(f) \subseteq N$ and N is a weakly quasi n-absorbing submodule of M. Let $a \in R$, $x' \in M'$ such that $0 \neq a^n x' \in f(N)$. Then there exists $x \in M$ such $x' = f(x)$. Since $0 \neq a^n x' = a^n f(x) = f(a^n x) \in f(N)$ and $\ker(f) \subseteq N$, then $0 \neq a^n x \in N$ which is a weakly quasi n-absorbing submodule of M. Therefore, $a^n \in (N :_R M)$ or $a^{n-1}x \in N$. And so $a^n M \subseteq N$ or $a^{n-1}x \in N$. It follows that $a^n M' \subseteq f(N)$ or $a^{n-1}f(x) \in f(N)$. Hence, $a^n \in (f(N) :_R M')$ or $a^{n-1}x' \in f(N)$. Finally, $f(N)$ is a weakly quasi n-absorbing submodule of M', as desired. □

We close this paper by studying about when the intersection of family of $(N_\alpha)_{\alpha \in I}$ is a weakly quasi n-absorbing submodule.

Theorem 2.26. *Consider $(N_\alpha)_{\alpha \in I}$ a chain of weakly quasi n-absorbing submodules of an R-module M. Then $N = \bigcap_{\alpha \in I} N_\alpha$ is a weakly quasi n-absorbing submodule of M.*

Proof. Let $0 \neq a^n x \in N$ for some $a \in R$ and $x \in M$. Clearly $0 \neq a^n x \in N_\alpha$ for each $\alpha \in I$. Two cases are then possible:

Case 1 : If $a^n \in (N_\alpha :_R M)$ for all $\alpha \in I$, then $a^n \in \bigcap(N_\alpha :_R M) = (\bigcap N_\alpha :_R M) = (N :_R M)$.

Case 2 : Assume that $a^n \notin (N_{\alpha'} :_R M)$ for some $\alpha' \in I$. Then $a^n \notin (N_\alpha :_R M)$ for all $N_\alpha \subseteq N_{\alpha'}$. Using the fact that N_α is a weakly quasi n-absorbing submodule of M for each $\alpha \in I$, then $a^{n-1}x \in N_\alpha$ for all $N_\alpha \subseteq N_{\alpha'}$. Consequently, it follows that $a^{n-1}x \in N = \bigcap_{\alpha \in I} N_\alpha$.

Finally, $N = \bigcap_{\alpha \in I} N_\alpha$ is a weakly quasi n-absorbing submodule of M, as desired. □

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