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# GORENSTEIN FLAT-COTORSION MODULES OVER FORMAL TRIANGULAR MATRIX RINGS

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Abstract. Let $A$ and $B$ be rings and $U$ be a $(B,A)$-bimodule. If ${}_BU$ has finite flat dimension, $U_A$ has finite flat dimension and $U\otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$, then the Gorenstein flat-cotorsion modules over the formal triangular matrix ring $T=\left(\begin{smallmatrix}A & 0\\ U & B\end{smallmatrix}\right)$ are explicitly described. As an application, it is proven that each Gorenstein flat-cotorsion left $T$-module is flat-cotorsion if and only if every Gorenstein flat-cotorsion left $A$-module and $B$-module is flat-cotorsion. In addition, Gorenstein flat-cotorsion dimensions over the formal triangular matrix ring $T$ are studied.

## Introduction

Throughout, all rings are associative with a unit. Let $T=\left(\begin{smallmatrix}A & 0\\ U & B\end{smallmatrix}\right)$ be a formal triangular matrix ring, where $A$ and $B$ are any associative rings and $U$ is a $(B,A)$-bimodule. It is very interesting and important to describe various categories of $T$-modules; see for instance Gorenstein projective and injective modules [4], pure projective and pure injective modules [8], Gorenstein flat modules [10], Gorenstein projective modules [14] and $\Omega$-Gorenstein modules [13].

In recent years, it has been evident that one should pay more and more attention to the category of Gorenstein flat modules that are also cotorsion. Christensen, Estrada and Thompson [3] introduced Gorenstein flat-cotorsion modules. Christensen, Estrada, Liang, Thompson, Wu and Yang [2] further studied Gorenstein flat-cotorsion modules. The main purpose of this paper is to describe the category of Gorenstein flat-cotorsion $T$-modules. More accurately, let $M=\left(\begin{smallmatrix}M_1\\ M_2\end{smallmatrix}\right)_{\varphi^M}$ be a left $T$-module. If ${}_BU$ has finite flat dimension, $U_A$ has finite flat dimension and $U\otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$, then it is proven that $M$ is a Gorenstein flat-cotorsion left $T$-module if and only if $M_1$ is a Gorenstein flat-cotorsion left $A$-module,

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$\operatorname{Coker}(\varphi^M)$ is a Gorenstein flat-cotorsion left $B$-module and $\varphi^M$ is a monomorphism.

## 1. Notation and terminology

Let $X = \cdots \longrightarrow X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \longrightarrow \cdots$ be a complex of left $R$-modules. The following standard notations are used: $\mathrm{Z}_i(X) = \operatorname{Ker}(\partial_i^X)$ and $\mathrm{C}_i(X) = \operatorname{Coker}(\partial_{i+1}^X)$. For the definitions of *Gorenstein projective module*, *Gorenstein injective module* and *Gorenstein flat module*, see Enochs and Jenda [5].

An $R$-module $M$ is said to be *flat-cotorsion* if it is flat and cotorsion. Recall from Christensen, Estrada and Thompson [10, Definition 4.3 and Proposition 1.3] that a *totally acyclic complex of flat-cotorsion modules* is an acyclic complex $T$ of flat-cotorsion modules such that the complexes $\operatorname{Hom}_R(T, W)$ and $\operatorname{Hom}_R(W, T)$ are acyclic for every flat-cotorsion $R$-module $W$. An $R$-module $G$ is called *Gorenstein flat-cotorsion* if there exists a totally acyclic complex of flat-cotorsion $R$-modules with $\mathrm{Z}_0(T) = G$.

The flat and injective dimensions of a left $R$-module $M$ are written $\operatorname{fd}_R M$ and $\operatorname{id}_R M$; $\operatorname{id}_{R^\circ} M$ denotes the injective dimension of a right $R$-module $M$. For other notation we refer to [13].

Recall the following facts for later use.

**Lemma 1.1** ([2, Lemma 3.2])**.** *An $R$-module $G$ is Gorenstein flat-cotorsion if and only if the following conditions are satisfied.*

(1) *$G$ is cotorsion.*
(2) $\operatorname{Ext}_R^{i\geqslant 1}(G, W) = 0$ *holds for every flat-cotorsion $R$-module $W$.*
(3) *There exist a complex $U = U_0 \to U_{-1} \to \cdots$ of flat-cotorsion $R$-modules and an injective quasi-isomorphism $\varepsilon\colon G \to U$ such that the morphism* $\operatorname{Hom}_R(\varepsilon, W)$ *is a quasi-isomorphism for every flat-cotorsion $R$-module $W$.*

**Lemma 1.2** ([8, Theorem 4.2])**.** *If $T$ is a right coherent ring, then $A$ and $B$ are right coherent rings.*

**Lemma 1.3** ([2, Theorem 5.3])**.** *Let $M$ be an $R$-module. If $M$ is Gorenstein flat and cotorsion, then it is Gorenstein flat-cotorsion. The converse holds if $R$ is right coherent.*

**Lemma 1.4** ([2, Proposition 3.4])**.** *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of cotorsion $R$-modules. If $G''$ is Gorenstein flat-cotorsion, then $G$ is Gorenstein flat-cotorsion if and only if $G'$ is Gorenstein flat-cotorsion.*

**Lemma 1.5** ([6, Proposition 1.14])**.** *A left $T$-module $F = \binom{F_1}{F_2}_{\varphi^F}$ is flat if and only if $F_1$ is a flat left $A$-module,* $\operatorname{Coker}(\varphi^F)$ *is a flat left $B$-module and $\varphi^F$ is a monomorphism.*

**Lemma 1.6** ([15, Proposition 3.4]). *Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left $T$-module. If ${}_BU$ has finite flat dimension, then $M$ has finite flat dimension if and only if $M_1$ and $M_2$ have finite flat dimensions.*

**Lemma 1.7** ([9, Corollary 4.3]). *A left $T$-module $C = \binom{C_1}{C_2}_{\varphi^C}$ is a cotorsion left $T$-module if and only if $C_1$ is a cotorsion left $A$-module and $C_2$ is a cotorsion left $B$-module.*

## 2. Gorenstein flat-cotorsion modules

In this section, we will describe Gorenstein flat-cotorsion modules over formal triangular matrix ring. Notice that the class of Gorenstein flat modules is closed under extensions; see Šaroch and Šťovíček [12, Corollary 3.12]. Hence the condition "$T$ is right coherent" is unnecessary of Mao [10, Theorem 2.3] by the proof in it.

**Lemma 2.1** ([10, Theorem 2.3]). *Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left $T$-module. If ${}_BU$ has finite flat dimension and $U_A$ has finite flat or injective dimension, then the following conditions are equivalent.*

(1) *$M$ is a Gorenstein flat left $T$-module.*
(2) *$M_1$ is a Gorenstein flat left $A$-module,* $\mathrm{Coker}(\varphi^M)$ *is a Gorenstein flat left $B$-module and $\varphi^M$ is a monomorphism.*

*In this case, $U \otimes_A M_1$ is a Gorenstein flat left $B$-module if and only if $M_2$ is a Gorenstein flat left $B$-module.*

**Corollary 2.2.** *If ${}_BU$ has finite flat dimension and $U_A$ has finite flat or injective dimension, then each Gorenstein flat left $T$-module is flat if and only if Gorenstein flat left $A$-module and $B$-module are flat.*

*Proof.* Let $M_1$ be a Gorenstein flat left $A$-module and $M_2$ be a Gorenstein flat left $B$-module. By Lemma 2.1, $\mathbf{p}(M_1, M_2)$ is a Gorenstein flat left $T$-module and so it is a flat left $T$-module. It follows from Lemma 1.5 that $M_1$ is a flat left $A$-module and $M_2$ is a flat left $B$-module.

Conversely, let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a Gorenstein flat left $T$-module. By Lemma 2.1, $M_1$ is a Gorenstein flat left $A$-module, $\mathrm{Coker}(\varphi^M)$ is a Gorenstein flat left $B$-module and $\varphi^M$ is a monomorphism. Hence $M_1$ is a flat left $A$-module and $\mathrm{Coker}(\varphi^M)$ is a flat left $B$-module. It follows from Lemma 1.5 that $M$ is a flat left $T$-module. □

**Proposition 2.3.** *Let $T$ be a right coherent ring and $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left $T$-module. If ${}_BU$ has finite flat dimension, $U_A$ has finite flat or injective dimension and $U \otimes_A M_1$ is a cotorsion left $B$-module, then the following conditions are equivalent.*

(1) *$M$ is a Gorenstein flat-cotorsion left $T$-module.*
(2) *$M_1$ is a Gorenstein flat-cotorsion left $A$-module,* $\mathrm{Coker}(\varphi^M)$ *is a Gorenstein flat-cotorsion left $B$-module and $\varphi^M$ is a monomorphism.*

*In this case, $U \otimes_A M_1$ is a Gorenstein flat-cotorsion left $B$-module if and only if $M_2$ is a Gorenstein flat-cotorsion left $B$-module.*

*Proof.* It follows from Lemmas 2.1, 1.2, 1.3, 1.4 and 1.7. □

**Theorem 2.4.** *Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left $T$-module. If ${}_BU$ has finite flat dimension, $U_A$ has finite flat dimension and $U \otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$, then the following conditions are equivalent.*

(1) *$M$ is a Gorenstein flat-cotorsion left $T$-module.*
(2) *$M_1$ is a Gorenstein flat-cotorsion left $A$-module,* $\mathrm{Coker}(\varphi^M)$ *is a Gorenstein flat-cotorsion left $B$-module and $\varphi^M$ is a monomorphism.*

*In this case, $U \otimes_A M_1$ is a Gorenstein flat-cotorsion left $B$-module if and only if $M_2$ is a Gorenstein flat-cotorsion left $B$-module.*

*Proof.* (1) ⇒ (2): Let $M$ be a Gorenstein flat-cotorsion left $T$-module. It follows from Lemma 1.1 that $M$ is a cotorsion left $T$-module, $\mathrm{Ext}_T^{i\geqslant 1}(M, W) = 0$ holds for every flat-cotorsion left $T$-module $W$ and there exists an exact sequence of left $T$-modules

$$\mathbf{X}:\ 0 \longrightarrow \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \xrightarrow{\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}} \begin{pmatrix} X_1^0 \\ X_2^0 \end{pmatrix}_{\varphi^0} \xrightarrow{\begin{pmatrix} d_1^0 \\ d_2^0 \end{pmatrix}} \begin{pmatrix} X_1^1 \\ X_2^1 \end{pmatrix}_{\varphi^1} \xrightarrow{\begin{pmatrix} d_1^1 \\ d_2^1 \end{pmatrix}} \begin{pmatrix} X_1^2 \\ X_2^2 \end{pmatrix}_{\varphi^2} \longrightarrow \cdots$$

with each $\binom{X_1^i}{X_2^i}_{\varphi^i}$ flat-cotorsion for $i \geq 0$, and $\mathrm{Hom}_T(\mathbf{X}, W)$ is exact for every flat-cotorsion left $T$-module $W$. Hence $M_1$ is a cotorsion left $A$-module and $M_2$ is a cotorsion left $B$-module by Lemma 1.7.

Next we show that $M_1$ is a Gorenstein flat-cotorsion left $A$-module. There is an exact sequence

$$\mathbf{X}_1:\ 0 \longrightarrow M_1 \xrightarrow{\tau_1} X_1^0 \xrightarrow{d_1^0} X_1^1 \xrightarrow{d_1^1} X_1^2 \xrightarrow{d_1^2} \cdots,$$

which is $U \otimes_A -$ exact by [4, Lemma 2.3] as each $X_1^i$ is a flat left $A$-module by Lemma 1.5. Let $\tau_1 : M_1 \to X_1^0$ and $\tau_2 : M_2 \to X_2^0$ be the inclusions. Consider the following commutative diagram of left $B$-modules.

$$\begin{array}{ccc} U \otimes_A M_1 & \xrightarrow{1\otimes_A \tau_1} & U \otimes_A X_1^0 \\ {\scriptstyle \varphi^M}\big\downarrow & & \big\downarrow{\scriptstyle \varphi^0} \\ M_2 & \xrightarrow{\tau_2} & X_2^0, \end{array}$$

where $\varphi^0$ is a monomorphism by Lemma 1.5. It follows that $\varphi^M$ is a monomorphism. Now there is an exact sequence

$$0 \longrightarrow U \otimes_A M_1 \xrightarrow{\varphi^M} M_2 \longrightarrow \mathrm{Coker}(\varphi^M) \longrightarrow 0,$$

where $U \otimes_A M_1$ and $M_2$ are cotorsion left $B$-modules. Hence $\mathrm{Coker}(\varphi^M)$ is a cotorsion left $B$-module. Let $D$ be a flat-cotorsion left $A$-module. It follows from Lemmas 1.5 and 1.7 and assumption that $\binom{D}{U\otimes_A D}$ is a flat-cotorsion left $T$-module. There is an exact sequence of cotorsion left $T$-modules

$$0 \to \begin{pmatrix} 0 \\ U \otimes_A D \end{pmatrix} \to \begin{pmatrix} D \\ U \otimes_A D \end{pmatrix} \to \begin{pmatrix} D \\ 0 \end{pmatrix} \to 0\,,$$

where $\binom{0}{U\otimes_A D}$ has finite flat dimension by Lemma 1.6. It follows from [2, Theorem 4.5 and Remark 4.6] that $\mathrm{Ext}_T^{i\geq 1}(\binom{M_1}{M_2}, \binom{0}{U\otimes_A D}) = 0$. Hence the sequence

$$0 \to \mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} 0 \\ U \otimes_A D \end{pmatrix}) \to \mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} D \\ U \otimes_A D \end{pmatrix}) \to \mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} D \\ 0 \end{pmatrix}) \to 0$$

is exact. Since $\mathrm{Hom}_T(\mathbf{X}, \binom{0}{U\otimes_A D})$ and $\mathrm{Hom}_T(\mathbf{X}, \binom{D}{U\otimes_A D})$ are acyclic, the complex $\mathrm{Hom}_T(\mathbf{X}, \binom{D}{0})$ is acyclic. As $(\mathbf{q}, \mathbf{h})$ is an adjoint pair, there is an isomorphism $\mathrm{Hom}_T(\mathbf{X}, \binom{D}{0}) \cong \mathrm{Hom}_A(\mathbf{X}_1, D)$ and so $\mathrm{Hom}_A(\mathbf{X}_1, D)$ is acyclic. Notice that

$$\mathrm{Ext}_T^{i\geq 1}(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \begin{pmatrix} D \\ U \otimes_A D \end{pmatrix}) = 0$$

and so it follows from [9, Lemma 3.2] that

$$\mathrm{Ext}_A^{i\geq 1}(M_1, D) \cong \mathrm{Ext}_T^{i\geq 1}(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \begin{pmatrix} D \\ 0 \end{pmatrix}) = 0.$$

Thus $M_1$ is a Gorenstein flat-cotorsion left $A$-module by Lemma 1.1.

Next we prove that $\mathrm{Coker}(\varphi^M)$ is a Gorenstein flat-cotorsion left $B$-module. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U \otimes_A M_1 & \xrightarrow{\varphi^M} & M_2 & \longrightarrow & \mathrm{Coker}(\varphi^M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes_A \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \overline{\tau_2}} & & \\
0 & \longrightarrow & U \otimes_A X_1^0 & \xrightarrow{\varphi^0} & X_2^0 & \longrightarrow & \mathrm{Coker}(\varphi^0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes_A d_1^0} & & \downarrow{\scriptstyle d_2^0} & & \downarrow{\scriptstyle \overline{d_2^0}} & & \\
0 & \longrightarrow & U \otimes_A X_1^1 & \xrightarrow{\varphi^1} & X_2^1 & \longrightarrow & \mathrm{Coker}(\varphi^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Since the first column and the second column are exact, the third column is exact with $\mathrm{Coker}(\varphi^i)$ flat-cotorsion for $i \geq 0$. Let $H$ be a flat-cotorsion left

$B$-module. Then $\binom{0}{H}_0$ is a flat-cotorsion left $T$-module by Lemmas 1.5 and 1.7. Applying the functor $\mathrm{Hom}_B(-,H)$ to the first row in the above diagram, one has the exact sequence

$$0 \to \mathrm{Hom}_B(\mathrm{Coker}(\varphi^M), H) \to \mathrm{Hom}_B(M_2, H) \to \mathrm{Hom}_B(U \otimes_A M_1, H).$$

So $\mathrm{Hom}_B(\mathrm{Coker}(\varphi^M), H) \cong \mathrm{Ker}(\mathrm{Hom}_B(M_2, H) \to \mathrm{Hom}_B(U \otimes_A M_1, H)) \cong \mathrm{Hom}_T(\binom{M_1}{M_2}_{\varphi^M}, \binom{0}{H}_0)$, where the last isomorphism follows by a direct calculation. Similarly, one has $\mathrm{Hom}_B(\mathrm{Coker}(\varphi^i), H) \cong \mathrm{Hom}_T(\binom{X_1^i}{X_2^i}_{\varphi^i}, \binom{0}{H}_0)$ for $i \geq 0$. Hence the sequence

$$0 \to \mathrm{Coker}(\varphi^M) \to \mathrm{Coker}(\varphi^0) \to \mathrm{Coker}(\varphi^1) \to \mathrm{Coker}(\varphi^2) \to \cdots$$

is $\mathrm{Hom}_B(-,H)$ exact. Moreover, since $\mathrm{Ext}_T^{i\geq 1}(\binom{M_1}{M_2}, \binom{0}{H}) = 0$, it follows from Gillespie [7, p. 3389] that $\mathrm{Ext}_B^{i\geq 1}(\mathrm{Coker}(\varphi^M), H) = 0$. Hence $\mathrm{Coker}(\varphi^M)$ is a Gorenstein flat-cotorsion left $B$-module by Lemma 1.1.

(2) $\Rightarrow$ (1): Let $M_1$ be a Gorenstein flat-cotorsion left $A$-module, $\mathrm{Coker}(\varphi^M)$ a Gorenstein flat-cotorsion left $B$-module and let $\varphi^M$ be a monomorphism. Then $M_1$ is a cotorsion left $A$-module and $\mathrm{Coker}(\varphi^M)$ is a cotorsion left $B$-module by Lemma 1.1. As $\varphi^M$ is a monomorphism, there is an exact sequence of left $T$-modules

$$0 \to \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ \mathrm{Coker}(\varphi^M) \end{pmatrix} \to 0\,,$$

where $\binom{M_1}{U\otimes_A M_1}$ and $\binom{0}{\mathrm{Coker}(\varphi^M)}$ are cotorsion left $T$-modules by Lemma 1.7 and so is $\binom{M_1}{M_2}$. We will show that $\binom{M_1}{U\otimes_A M_1}$ and $\binom{0}{\mathrm{Coker}(\varphi^M)}$ are Gorenstein flat-cotorsion left $T$-modules. Then it follows from Lemma 1.4 that $M$ is a Gorenstein flat-cotorsion left $T$-module.

Since $M_1$ is a Gorenstein flat-cotorsion left $A$-module, by Lemma 1.1 there exists an exact sequence

$$\mathbf{Y}_1: \ 0 \longrightarrow M_1 \xrightarrow{\tau} Y^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} Y^2 \xrightarrow{d^2} \cdots$$

with $Y^i$ flat-cotorsion for $i \geq 0$, and $\mathrm{Ext}_A^{i\geq 1}(M_1, K) = 0$ for any flat-cotorsion left $A$-module $K$ and the complex $\mathrm{Hom}_A(\mathbf{Y}_1, K)$ is acyclic. As $U_A$ has finite flat dimension, the complex $U \otimes_A \mathbf{Y}_1$ is acyclic by [4, Lemma 2.3] and so $\mathrm{Tor}^A_{i\geq 1}(U, M_1) = 0$. Now there is an exact sequence of left $T$-modules

$$\mathbf{Y}: \ 0 \to \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \tau \\ 1 \otimes_A \tau \end{pmatrix}} \begin{pmatrix} Y^0 \\ U \otimes_A Y^0 \end{pmatrix} \xrightarrow{\begin{pmatrix} d^0 \\ 1 \otimes_A d^0 \end{pmatrix}} \begin{pmatrix} Y^1 \\ U \otimes_A Y^1 \end{pmatrix} \to \cdots$$

with each $\binom{Y^i}{U\otimes_A Y^i}$ flat-cotorsion for $i \geq 0$. Let $W = \binom{W_1}{W_2}$ be a flat-cotorsion left $T$-module. Since $(\mathbf{p}, \mathbf{q})$ is an adjoint pair, there are isomorphisms

$$\operatorname{Hom}_T(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}, \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}) \cong \operatorname{Hom}_A(M_1, W_1)$$

and $\operatorname{Hom}_T(\begin{pmatrix} Y^i \\ U\otimes_A Y^i \end{pmatrix}, \binom{W_1}{W_2}) \cong \operatorname{Hom}_A(Y^i, W_1)$ for $i \geq 0$. It follows that $\operatorname{Hom}_T(\mathbf{Y}, W) \cong \operatorname{Hom}_A(\mathbf{Y}_1, W_1)$ is acyclic as $W_1$ is a flat-cotorsion left $A$-module. By [9, Lemma 3.2] one has

$$\operatorname{Ext}_T^{i\geq 1}(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}, \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}) \cong \operatorname{Ext}_A^{i\geq 1}(M_1, W_1) = 0.$$

Hence $\binom{M_1}{U\otimes_A M_1}$ is a Gorenstein flat-cotorsion left $T$-module by Lemma 1.1.

It remains to show that $\begin{pmatrix} 0 \\ \operatorname{Coker}(\varphi^M) \end{pmatrix}$ is a Gorenstein flat-cotorsion left $T$-module. As $\operatorname{Coker}(\varphi^M)$ is a Gorenstein flat-cotorsion left $B$-module, by Lemma 1.1 there is an exact sequence

$$\mathbf{Z}_2: \ 0 \longrightarrow \operatorname{Coker}(\varphi^M) \xrightarrow{\tau} Z^0 \xrightarrow{d^0} Z^1 \xrightarrow{d^1} Z^2 \xrightarrow{d^2} \cdots$$

with $Z^i$ flat-cotorsion for $i \geq 0$, and $\operatorname{Ext}_B^{i\geq 1}(\operatorname{Coker}(\varphi^M), L) = 0$ for any flat-cotorsion left $B$-module $L$ and the complex $\operatorname{Hom}_B(\mathbf{Z}_2, L)$ is acyclic. Now there is an exact sequence of left $T$-modules

$$\mathbf{Z}: \ 0 \to \begin{pmatrix} 0 \\ \operatorname{Coker}(\varphi^M) \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 \\ \tau \end{pmatrix}} \begin{pmatrix} 0 \\ Z^0 \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 \\ d^0 \end{pmatrix}} \begin{pmatrix} 0 \\ Z^1 \end{pmatrix} \to \cdots$$

with each $\binom{0}{Z^i}$ flat-cotorsion for $i \geq 0$. Let $W = \binom{W_1}{W_2}$ be a flat-cotorsion left $T$-module. Per Lemma 1.5 there is an exact sequence $0 \longrightarrow U\otimes_A W_1 \xrightarrow{\varphi^W} W_2 \longrightarrow \operatorname{Coker}(\varphi^W) \longrightarrow 0$, where $W_2$ has finite flat dimension by Lemma 1.6. Since $(\mathbf{p}, \mathbf{q})$ is an adjoint pair, there are isomorphisms

$$\operatorname{Hom}_T(\begin{pmatrix} 0 \\ \operatorname{Coker}(\varphi^M) \end{pmatrix}, \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}) \cong \operatorname{Hom}_B(\operatorname{Coker}(\varphi^M), W_2)$$

and $\operatorname{Hom}_T(\binom{0}{Z^i}, \binom{W_1}{W_2}) \cong \operatorname{Hom}_B(Z^i, W_2)$ for $i \geq 0$. As $W_2$ is a cotorsion left $B$-module with finite flat dimension, it follows from [2, Theorem 4.5 and Remark 4.6] that $\operatorname{Hom}_T(\mathbf{Z}, W) \cong \operatorname{Hom}_B(\mathbf{Z}_2, W_2)$ is acyclic. By [9, Lemma 3.2] and [2, Theorem 4.5 and Remark 4.6] one has

$$\operatorname{Ext}_T^{i\geq 1}(\begin{pmatrix} 0 \\ \operatorname{Coker}(\varphi^M) \end{pmatrix}, \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}) \cong \operatorname{Ext}_B^{i\geq 1}(\operatorname{Coker}(\varphi^M), W_2) = 0.$$

Therefore, $\begin{pmatrix} 0 \\ \operatorname{Coker}(\varphi^M) \end{pmatrix}$ is a Gorenstein flat-cotorsion left $T$-module per Lemma 1.1.

Finally, if the equivalent conditions hold, then the last assertion follows from Lemma 1.4. □

*Remark* 2.5. (1) Suppose that the class of cotorsion left $A$-modules is closed under direct sums and $A = B$. If $U_A$ is projective, then $U \otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$.

(2) If $B$ is left Noetherian, ${}_BU$ is injective, $U_A$ is flat and $C$ is a left $A$-module, then one has $\mathrm{id}_B(U \otimes_A C) \leq \mathrm{id}_B U$. Hence $U \otimes_A C$ is a cotorsion left $B$-module.

(3) If $B$ is left Noetherian, ${}_BU$ is injective and ${}_AC$ is flat, then one has $\mathrm{id}_B(U \otimes_A C) \leq \mathrm{id}_B U$. Hence $U \otimes_A C$ is a cotorsion left $B$-module.

**Corollary 2.6.** *Let $T = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left $T$-module. Then the following conditions are equivalent.*

(1) *$M$ is a Gorenstein flat-cotorsion left $T$-module.*
(2) *$M_1$ and $\mathrm{Coker}(\varphi^M)$ are Gorenstein flat-cotorsion left $R$-modules, and $\varphi^M$ is a monomorphism.*
(3) *$M_2$ and $\mathrm{Coker}(\varphi^M)$ are Gorenstein flat-cotorsion left $R$-modules, and $\varphi^M$ is a monomorphism.*

*Proof.* It is an immediate consequence of Theorem 2.4. □

**Corollary 2.7.** *If ${}_BU$ has finite flat dimension, $U_A$ has finite flat dimension and $U \otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$, then each Gorenstein flat-cotorsion left $T$-module is flat-cotorsion if and only if Gorenstein flat-cotorsion left $A$-module and $B$-module are flat-cotorsion.*

*Proof.* By analogy with the proof of Corollary 2.2. □

Šaroch and Šťovíček introduced the following definition of a projectively coresolved Gorenstein flat module (a PGF-module for short); see [12].

**Definition.** A left $R$-module $M$ is called a *PGF-module* if there exists an acyclic complex $\mathbf{P}$ of projective left $R$-modules such that the complex $I \otimes_R \mathbf{P}$ is acyclic for every injective right $R$-module $I$ and $\mathrm{C}_0(P) \cong M$. Denote the class of PGF-modules by $\mathcal{PGF}$.

**Theorem 2.8.** *Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left $T$-module. If ${}_BU$ has finite flat dimension and $U_A$ has finite flat or injective dimension, then the following conditions are equivalent.*

(1) *$M$ is a PGF left $T$-module.*
(2) *$M_1$ is a PGF left $A$-module, $\mathrm{Coker}(\varphi^M)$ is a PGF left $B$-module and $\varphi^M$ is a monomorphism.*

*In this case, $U \otimes_A M_1$ is a PGF left $B$-module if and only if $M_2$ is a PGF left $B$-module.*

*Proof.* By analogy with the proof of [10, Theorem 2.3] and this time using [12, Lemma 3.8], one could obtain the result. □

## 3. Gorenstein flat-cotorsion dimensions

Recall that an $R$-complex $F$ is said to be *semi-flat* if it consists of flat modules and the functor $-\otimes_R F$ preserves acyclicity; see Avramov and Foxby [1]. An $R$-complex $C$ is said to be *semi-cotorsion* if it consists of cotorsion modules and $\mathrm{Hom}_R(F,C)$ is acyclic for every acyclic semi-flat $R$-complex $F$. Following Nakamura and Thompson [11] that an $R$-complex $W$ is called *semi-flat-cotorsion* if it is semi-flat and semi-cotorsion. Let $M$ be an $R$-complex. A semi-flat-cotorsion complex that is isomorphic to $M$ in the derived category $\mathcal{D}(R)$ is called a *semi-flat-cotorsion replacement* of $M$; see Christensen, Estrada, Liang, Thompson, Wu and Yang [2].

**Definition** ([2, Definition 4.1])**.** Let $M$ be an $R$-complex. The *Gorenstein flat-cotorsion dimension* of $M$, written $\mathrm{Gfcd}_R\, M$, is defined as

$$\mathrm{Gfcd}_R\, M \;=\; \inf\left\{ n\in\mathbb{Z} \;\middle|\; \begin{array}{l}\text{There is a semi-flat-cotorsion replacement}\\ W \text{ of } M \text{ with } \mathrm{H}_i(W)=0 \text{ for all } i>n \text{ and}\\ \text{with } \mathrm{C}_n(W) \text{ Gorenstein flat-cotorsion}\end{array}\right\}$$

with $\inf\varnothing=\infty$ by convention. We say that $\mathrm{Gfcd}_R\, M$ is *finite* if $\mathrm{Gfcd}_R\, M<\infty$.

Let $C$ be a cotorsion $R$-module. It follows from [2, Remark 4.6] that $\mathrm{Gfcd}_R\, C\leqslant n$ if and only if there exists an exact sequence

$$0\longrightarrow G\longrightarrow W_{n-1}\longrightarrow\cdots\longrightarrow W_0\longrightarrow C\longrightarrow 0$$

such that each $W_i$ is flat-cotorsion for $0\le i\le n-1$ and $G$ is Gorenstein flat-cotorsion. In particular, a cotorsion $R$-module $C$ is Gorenstein flat-cotorsion if and only if $\mathrm{Gfcd}_R\, C=0$ holds.

Set $\mathrm{LGFCD}(R)=\sup\{\mathrm{Gfcd}_R\, M\mid M \text{ is any left } R\text{-module}\}$.

**Lemma 3.1.** *Let $A$ be right Noetherian,* $\mathrm{LGFCD}(B)<\infty$*, ${}_BU$ be projective, $U_A$ have finite injective dimension and $M$ be a Gorenstein flat-cotorsion left $A$-module. If $U\otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$, then $U\otimes_A M$ is a Gorenstein flat-cotorsion left $B$-module.*

*Proof.* Let $M$ be a Gorenstein flat-cotorsion left $A$-module. Per Lemma 1.1 $M$ is a cotorsion left $A$-module and by assumption $U\otimes_A M$ is a cotorsion left $B$-module. Let $D$ be a flat-cotorsion left $B$-module. It follows from [2, Theorem 4.5] that $\mathrm{id}_B\, D<\infty$ as $\mathrm{LGFCD}(B)<\infty$. Since $A$ is right Noetherian, ${}_BU$ is projective and $\mathrm{id}_B\, D<\infty$, one has $\mathrm{fd}_A\,\mathrm{Hom}_B(U,D)\le\mathrm{id}_{A^o}\, U<\infty$. Let $F$ be a flat left $A$-module. The isomorphism $\mathrm{Ext}^1_A(F,\mathrm{Hom}_B(U,D))\cong\mathrm{Ext}^1_B(U\otimes_A F,D)=0$ yields that $\mathrm{Hom}_B(U,D)$ is a cotorsion left $A$-module. Now it follows from [2, Theorem 4.5] that

$$\mathrm{Ext}^{i\ge1}_B(U\otimes_A M,D)\cong\mathrm{Ext}^{i\ge1}_A(M,\mathrm{Hom}_B(U,D))=0$$

and so $U\otimes_A M$ is a Gorenstein flat-cotorsion left $B$-module. □

**Theorem 3.2.** *Let $A$ be right Noetherian,* $\mathrm{LGFCD}(B) < \infty$*, ${}_BU$ be projective, $U_A$ have finite injective and flat dimensions and $M = \binom{M_1}{M_2}_{\varphi^M}$ be a cotorsion left $T$-module. If $U \otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$, then one has*

$$\max\{\mathrm{Gfcd}_A M_1, \mathrm{Gfcd}_B M_2\} \leq \mathrm{Gfcd}_T M \leq \max\{\mathrm{Gfcd}_A M_1 + 1, \mathrm{Gfcd}_B M_2\}.$$

*Proof.* First notice that $M_1$ is a cotorsion left $A$-module and $M_2$ is a cotorsion left $B$-module per Lemma 1.7.

Next we show that $\max\{\mathrm{Gfcd}_A M_1, \mathrm{Gfcd}_B M_2\} \leq \mathrm{Gfcd}_T M$. To this end, let $\mathrm{Gfcd}_T M = n < \infty$. There is an exact sequence of left $T$-modules

$$0 \to \begin{pmatrix} G_1^n \\ G_2^n \end{pmatrix}_{\varphi^n} \xrightarrow{\begin{pmatrix} d_1^n \\ d_2^n \end{pmatrix}} \begin{pmatrix} G_1^{n-1} \\ G_2^{n-1} \end{pmatrix}_{\varphi^{n-1}} \to \cdots \to \begin{pmatrix} G_1^0 \\ G_2^0 \end{pmatrix}_{\varphi^0} \xrightarrow{\begin{pmatrix} d_1^0 \\ d_2^0 \end{pmatrix}} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to 0$$

such that each $\binom{G_1^i}{G_2^i}_{\varphi^i}$ is Gorenstein flat-cotorsion for $0 \leq i \leq n$. By Theorem 2.4, all $G_1^i$ and $\mathrm{Coker}(\varphi^i)$ are Gorenstein flat-cotorsion. Hence each $U \otimes_A G_1^i$ is Gorenstein flat-cotorsion by Lemma 3.1 and so is each $G_2^i$ for $0 \leq i \leq n$. It follows that $\max\{\mathrm{Gfcd}_A M_1, \mathrm{Gfcd}_B M_2\} \leq \mathrm{Gfcd}_T M$.

Next we prove that $\mathrm{Gfcd}_T M \leq \max\{\mathrm{Gfcd}_A M_1 + 1, \mathrm{Gfcd}_B M_2\}$. Suppose that $\max\{\mathrm{Gfcd}_A M_1 + 1, \mathrm{Gfcd}_B M_2\} = l < \infty$. There is an exact sequence

$$0 \longrightarrow G_{l-1} \xrightarrow{\alpha_{l-1}} G_{l-2} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\alpha_1} G_0 \xrightarrow{\alpha_0} M_1 \longrightarrow 0$$

such that each $G_i$ is a Gorenstein flat-cotorsion left $A$-module. Take an exact sequence $W_0 \xrightarrow{\beta_0} M_2 \longrightarrow 0$ with $W_0$ flat-cotorsion. Hence there are short exact sequences $0 \longrightarrow K_1^{i+1} \longrightarrow G_i \xrightarrow{\pi_i} K_1^i \longrightarrow 0$, where $K_1^i = \mathrm{Ker}(\alpha_{i-1})$ is cotorsion for $1 \leq i \leq l-2$. Define $\gamma_0 : (U \otimes_A G_0) \oplus W_0 \to M_2$ by $\gamma_0(u \otimes_A g_0, w_0) = \varphi^M(u \otimes_A \alpha_0(g_0)) + \beta_0(w_0)$ for $u \in U, g_0 \in G_0$ and $w_0 \in W_0$. It is clear that $\gamma_0$ is an epimorphism. Now one has an exact sequence of left $T$-modules

$$0 \to \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^1} \to \begin{pmatrix} G_0 \\ (U \otimes_A G_0) \oplus W_0 \end{pmatrix} \xrightarrow{\begin{pmatrix} \alpha_0 \\ \gamma_0 \end{pmatrix}} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to 0\,.$$

Take an exact sequence $W_1 \xrightarrow{\beta_1} K_2^1 \longrightarrow 0$ with $W_1$ flat-cotorsion as $K_2^1$ is cotorsion. Define $\gamma_1 : (U \otimes_A G_1) \oplus W_1 \to K_2^1$ by $\gamma_1(u \otimes_A g_1, w_1) = \varphi^1(u \otimes_A \pi_1(g_1)) + \beta_1(w_1)$ for $u \in U, g_1 \in G_1$ and $w_1 \in W_1$. It is clear that $\gamma_1$ is an epimorphism. Now one has an exact sequence of left $T$-modules

$$0 \to \begin{pmatrix} K_1^2 \\ K_2^2 \end{pmatrix}_{\varphi^2} \to \begin{pmatrix} G_1 \\ (U \otimes_A G_1) \oplus W_1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \alpha_1 \\ \gamma_1 \end{pmatrix}} \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^1} \to 0.$$

Proceeding in this manner, one has an exact sequence of left $T$-modules

$$0 \to \begin{pmatrix} 0 \\ K_2^l \end{pmatrix} \to \begin{pmatrix} G_{l-1} \\ (U \otimes_A G_{l-1}) \oplus W_{l-1} \end{pmatrix} \to \cdots \to \begin{pmatrix} G_0 \\ (U \otimes_A G_0) \oplus W_0 \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to 0.$$

By Lemma 3.1 each $U \otimes_A G_i$ is Gorenstein flat-cotorsion. It follows from [2, Proposition 3.3] that each $(U \otimes_A G_i) \oplus W_i$ is Gorenstein flat-cotorsion. Now $K_2^l$ is Gorenstein flat-cotorsion since $\mathrm{Gfcd}_B\, M_2 \leq l$. By Theorem 2.4, $\begin{pmatrix} 0 \\ K_2^l \end{pmatrix}$ and each $\begin{pmatrix} G_i \\ (U\otimes_A G_i)\oplus W_i \end{pmatrix}$ are Gorenstein flat-cotorsion and so $\mathrm{Gfcd}_R\, M \leq l$. $\square$

**Corollary 3.3.** *Let $A$ be right Noetherian,* $\mathrm{LGFCD}(B) < \infty$, *${}_BU$ be projective, $U_A$ have finite injective and flat dimensions and $M = \binom{M_1}{M_2}_{\varphi^M}$ be a cotorsion left $T$-module. If $U \otimes_A C$ is a cotorsion left $B$-module for any cotorsion left $A$-module $C$, then $M$ has finite Gorenstein flat-cotorsion dimension if and only if $M_1$ and $M_2$ have finite Gorenstein flat-cotorsion dimensions.*

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## References

[1] L. L. Avramov and H.-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155. https://doi.org/10.1016/0022-4049(91)90144-Q

[2] L. W. Christensen, S. Estrada, L. Liang, P. Thompson, D. Wu, and G. Yang, *A refinement of Gorenstein flat dimension via the flat-cotorsion theory*, J. Algebra **567** (2021), 346–370. https://doi.org/10.1016/j.jalgebra.2020.09.024

[3] L. W. Christensen, S. Estrada, and P. Thompson, *Homotopy categories of totally acyclic complexes with applications to the flat-cotorsion theory*, in Categorical, homological and combinatorial methods in algebra, 99–118, Contemp. Math., 751, Amer. Math. Soc., RI, 2020. https://doi.org/10.1090/conm/751/15112

[4] E. E. Enochs, M. Cortés-Izurdiaga, and B. Torrecillas, *Gorenstein conditions over triangular matrix rings*, J. Pure Appl. Algebra **218** (2014), no. 8, 1544–1554. https://doi.org/10.1016/j.jpaa.2013.12.006

[5] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, De Gruyter Expositions in Mathematics, 30, Walter de Gruyter & Co., Berlin, 2000. https://doi.org/10.1515/9783110803662

[6] R. M. Fossum, P. A. Griffith, and I. Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Mathematics, Vol. 456, Springer-Verlag, Berlin, 1975.

[7] J. Gillespie, *The flat model structure on* Ch($R$), Trans. Amer. Math. Soc. **356** (2004), no. 8, 3369–3390. https://doi.org/10.1090/S0002-9947-04-03416-6

[8] A. Haghany, M. Mazrooei, and M. R. Vedadi, *Pure projectivity and pure injectivity over formal triangular matrix rings*, J. Algebra Appl. **11** (2012), no. 6, 1250107, 13 pp. https://doi.org/10.1142/S0219498812501071

[9] L. Mao, *Cotorsion pairs and approximation classes over formal triangular matrix rings*, J. Pure Appl. Algebra **224** (2020), no. 6, 106271, 21 pp. https://doi.org/10.1016/j.jpaa.2019.106271

[10] ———, *Gorenstein flat modules and dimensions over formal triangular matrix rings*, J. Pure Appl. Algebra **224** (2020), no. 4, 106207, 10 pp. https://doi.org/10.1016/j.jpaa.2019.106207

[11] T. Nakamura and P. Thompson, *Minimal semi-flat-cotorsion replacements and cosupport*, J. Algebra **562** (2020), 587–620. https://doi.org/10.1016/j.jalgebra.2020.07.001

[12] J. Šaroch and J. Šťovíček, *Singular compactness and definability for Σ-cotorsion and Gorenstein modules*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 23, 40 pp. https://doi.org/10.1007/s00029-020-0543-2

[13] D. Wu and C. Yi, *Ω-Gorenstein modules over formal triangular matrix rings*, Bull. Malays. Math. Sci. Soc. https://doi.org/10.1007/s40840-021-01169-w

[14] P. Zhang, *Gorenstein-projective modules and symmetric recollements*, J. Algebra **388** (2013), 65–80. https://doi.org/10.1016/j.jalgebra.2013.05.008

[15] R. Zhu, Z. Liu, and Z. Wang, *Gorenstein homological dimensions of modules over triangular matrix rings*, Turkish J. Math. **40** (2016), no. 1, 146–160. https://doi.org/10.3906/mat-1504-67

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