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APPLICATIONS OF CLASS NUMBERS AND BERNOULLI NUMBERS TO HARMONIC TYPE SUMS

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Abstract. Divisibility properties of harmonic numbers by a prime number p have been a recurrent topic. However, finding the exact p-adic orders of them is not easy. Using class numbers of number fields and Bernoulli numbers, we compute the exact p-adic orders of harmonic type sums. Moreover, we obtain an asymptotic formula for generalized harmonic numbers whose p-adic orders are exactly one.

1. Introduction

Here we study the exact p-adic order of harmonic numbers via class numbers of some number fields and Bernoulli numbers. There are many results concerning the p-adic orders of harmonic numbers. However, most of them provide a p-adic lower bound for such numbers, as it is indeed hard to find the exact p-adic orders of these numbers. In this note, we find the exact p-adic orders of some generalized harmonic type sums using the properties of class numbers of certain number fields and Bernoulli numbers.

Let K be a field and $\mathcal{O}_K$ be its ring of integers. There is a positive integer $h(K)$ which is called the class number of K, and it is related to how $\mathcal{O}_K$ is far from being a principal ideal domain. In fact $h(K) = 1$ if and only if $\mathcal{O}_K$ is a principal ideal domain. The number $h(K)$ is related to the Dedekind zeta function of K and also some L-functions. For some particular number fields, there are some exact formulas for $h(K)$. For instance, if $K = \mathbb{Q}(\sqrt{-p})$ where $p > 3$ is a prime number which is congruent to 3 modulo 4, then by [9, Chapters 1 & 6] we know that

$$h(K) = \frac{1}{(2 - (\frac{2}{p}))} \sum_{m < \frac{p}{2}} \left(\frac{m}{p}\right),$$

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where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Harmonic numbers are defined by the terms of the sequence of partial sums of the harmonic series, namely

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

for $n \geq 1$. The order of growth of harmonic numbers is well-known, precisely we have

$$\begin{aligned} h_n &\sim \log n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} \\ &= \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots \end{aligned} \tag{1.1}$$

as n tends to infinity, where γ is Euler's constant and B_m is the mth Bernoulli number, and these numbers are defined via the coefficients of the generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m.$$

Here are some Bernoulli numbers: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$ and $B_4 = -\frac{1}{30}$. Moreover, one has that $B_{2n+1} = 0$ for $n \geq 1$. The von Staudt-Clausen theorem states that for an even integer $m \geq 2$, we have

$$B_m = A_m - \sum_{p-1|m} \frac{1}{p}$$

for some $A_m \in \mathbb{Z}$ (see for instance [18, Theorem 3, Chapter 15]). Therefore for an even integer m, the denominator of B_m is the product

$$\prod_{p-1|m} p.$$

For more on Bernoulli numbers, polynomials and some explicit formulas, we refer the reader to [20]. Given an odd prime p, Babbage's result [2] states that $h_{p-1} \equiv 0 \pmod p$. Besides, for $p \geq 5$ Wolstenholme [30] obtained that

$$h_{p-1} \equiv 0 \left(\bmod p^2\right). \tag{1.2}$$

Wolstenholme also showed that

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{(p-1)^2} \equiv 0 \pmod p, \tag{1.3}$$

where $p \geq 5$ is a prime. For a comprehensive survey on Wolstenholme's theorem, the reader may consult [21].

The nth generalized harmonic number of order s is defined as

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}, \tag{1.4}$$

which is a partial sum of the Riemann Zeta function $\zeta(s)$ defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s > 1$. There are also Wolstenholme's type congruences for generalized harmonic numbers, for instance, it was proved in [11] that if p is a prime with $p-1 \nmid s$, then one has

$$H_{p-1}^{(s)} \equiv 0 \pmod{p}. \tag{1.5}$$

Observe that congruence (1.5) generalizes congruence (1.3).

The other generalization of harmonic numbers is hyperharmonic numbers. They first occurred in the book of Conway and Guy [8]: the nth hyperharmonic number of order r is defined recursively by

$$h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)}, \quad r \geq 2,$$

where $h_n^{(1)} = h_n$ as the initial case. By [8], $h_n^{(r)}$ can be computed in terms of binomial coefficients and harmonic numbers

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}).$$

There is no harmonic number which is an integer except 1 [28]. Extending [28], the authors [14] proved that almost all hyperharmonic numbers are not integers. This gives an almost positive answer to Mező's conjecture [22]. Moreover, this idea was further generalized in [1]. However, Mező's conjecture is false and it was refuted by the second author [24]. Numerous p-adic order calculations for hyperharmonic numbers can be found in [15].

Generalized hyperharmonic numbers [10] are defined by

$$H_n^{(s,r)} := \sum_{k=1}^{n} H_k^{(s,r-1)},$$

where $H_n^{(s,1)} = H_n^{(s)}$ and $r \geq 2$. Note that generalized hyperharmonic numbers extend both $h_n^{(r)}$ and $H_n^{(s)}$ at the same time. In [10], a combinatorial identity was given for generalized hyperharmonic numbers

$$H_n^{(s,r)} = \sum_{j=1}^{n} \binom{n-j+r-1}{r-1} \frac{1}{j^s}, \tag{1.6}$$

and this was generalized to all hypersequences in [16]. We define the p-adic order of an integer as follows: for $a \in \mathbb{Z}$ we denote

$$\nu_p(a) := \begin{cases} m & \text{if } p^m \parallel a \text{ and } a \neq 0, \\ \infty & \text{if } a = 0, \end{cases}$$

as the *p-adic order* of a. Here $p^m \parallel a$ means $p^m \mid a$ but $p^{m+1} \nmid a$. We extend the p-adic order to a rational number $q = a/b \in \mathbb{Q}$ by $\nu_p(q) = \nu_p(a) - \nu_p(b)$ where $a, b \in \mathbb{Z}$. Now, we focus on the divisibility properties of harmonic type sums via Bernoulli numbers. Note that congruence (1.5) means that the p-adic order of $H_{p-1}^{(s)}$ is at least 1, and congruence (1.2) says that the p-adic order of h_{p-1} is at least 2 for $p \geq 5$. In general, it is very hard to compute the exact p-adic order of harmonic numbers. To illustrate, the only known primes p where the p-adic order of h_{p-1} is equal to 3 are 16843 and 2124679, which are in fact Wolstenholme primes. In this note, we find the exact p-adic order for a large class of harmonic type sums. Following the literature, given a prime p and a positive even integer m, we say that the tuple (p, m) is a *regular pair* if p does not divide B_m. A prime p is called *regular* if the tuple (p, m) is a regular pair for every even $m \leq p - 3$. Equivalently, a prime p is called regular if p does not divide the class number of the cyclotomic field $\mathbb{Q}(e^{2\pi i/p})$. This equivalence was proved by Kummer. He also proved Fermat's Last Theorem for a prime exponent which is regular. Unfortunately, the infinitude of regular primes is not known and there are infinitely many irregular primes where the smallest one is 37. Next, we state our results. In our first theorem, we obtain an asymptotic formula for the number of positive integers s up to a given real number x, for which the corresponding p-adic valuation of $H_{p-1}^{(s)}$ is exactly 1. We also infer that the density of such numbers is positive. Note that the regularity of p affects the density.

Theorem 1.1. *Let $p \geq 5$ be a prime number and $x \in \mathbb{R}_{>0}$. Define*

$$V_p(x) := \left| \left\{ s \leq x \,\middle|\, \nu_p\left(H_{p-1}^{(s)}\right) = 1 \right\} \right|. \tag{1.7}$$

Then, there exists an effectively computable constant $c_p \in \left(0, \frac{1}{2}\right)$, depending only on p, such that

$$V_p(x) = c_p x + \mathcal{O}_p(1).$$

Moreover, if p is a regular prime, then

$$c_p = \frac{p-1}{2p}.$$

Furthermore, if p is an irregular prime, then more than the half of the positive integers s satisfy the inequality

$$\nu_p\left(H_{p-1}^{(s)}\right) > 1.$$

For instance, if the prime number p satisfies $5 \leq p < 37$, then we know that p is regular. Therefore,

$$c_p = \frac{p-1}{2p}.$$

However, when $p = 37$, we compute that

$$c_{37} = \frac{17}{37} < \frac{p-1}{2p}.$$

Our second theorem generalizes the corresponding result of [27], and this theorem will be used several times in the proof of our other results. Note that if we take $s \leq p-3$, then we obtain the corresponding results given by Glaisher [13] and Sun [27]. Here, we extend the range of s.

Theorem 1.2. *Let $p > 3$ be a prime number and s a positive integer. Assume that s is even, $s' \equiv s \pmod{p(p-1)}$ where $0 \leq s' < p(p-1)$ and put $a = \left\lceil \frac{s'}{p-1} \right\rceil$. If $p-1$ does not divide s, then*

$$H_{p-1}^{(s)} \equiv \frac{s}{s+a} pB_{ap-(s'+a)} \pmod{p^2}. \tag{1.8}$$

Moreover, if $s > 1$ is odd, $p-1 \nmid s+1$, $s'' \equiv s \pmod{p^2(p-1)}$ where $0 \leq s'' < p^2(p-1)$ and $b = \left\lceil \frac{s''+1}{p-1} \right\rceil$, then

$$H_{p-1}^{(s)} \equiv -\frac{s(s+1)}{2(b+s+1)} p^2 B_{bp-(b+s''+1)} \pmod{p^3}. \tag{1.9}$$

Using the class number of certain imaginary quadratic number fields, we can compute the exact p-adic order of some generalized harmonic numbers which is given in the following theorem.

Theorem 1.3. *Let $p \geq 5$ be a prime number with $p \equiv 3 \pmod 4$ and $s \geq 1$ a positive integer.*

(1) *If s is of the form $(kp + a - \frac{1}{2})(p-1) - 1$ where $k \geq 0$ and $a \geq 1$ are integers such that $p \nmid 2a+1$, then $\nu_p\left(H_{p-1}^{(s)}\right) = 1$.*

(2) *If $s = (\ell p^2 + b - \frac{1}{2})(p-1) - 2$ where $\ell \geq 0$ and $b \geq 1$ are integers such that $p \nmid 4b^2 + 8b + 3$, then $\nu_p\left(H_{p-1}^{(s)}\right) = 2$.*

To illustrate Theorem 1.3, we provide some examples by SageMath [23] which can be found in Table 1.

According to Table 1, note that the form of s given in Theorem 1.3 holds for the cases $(p, s) \in \{(7, 1), (7, 2), (7, 8), (11, 3), (11, 4)\}$. Also to get $\nu_p\left(H_{p-1}^{(s)}\right) \in \{1, 2\}$, it is not necessary to take s of the form given in Theorem 1.3, as we see for $(p, s) \in \{(7, 3), (7, 4)\}$. Moreover, the conditions $p \nmid 2a+1$ and $p \nmid 4b^2+8b+3$ are necessary for Theorem 1.3, which can be observed in the cases $(p, s) = (7, 14)$ and $(p, s) = (7, 7)$, respectively.

Short outline of the paper: In Section 2, we will provide the proofs of Theorems 1.1-1.3 together with two corollaries related to generalized harmonic numbers. In Section 3, we will deal with generalized hyperharmonic numbers and we extend Theorem 1.3 to these numbers. In the same section, we will also prove some other corollaries related to generalized hyperharmonic numbers.

TABLE 1. The values and the corresponding p-adic valuations of $H_{p-1}^{(s)}$ for the given values of p and s.

p	s	$H_{p-1}^{(s)}$	$\nu_p\left(H_{p-1}^{(s)}\right)$
7	1	49/20	2
7	2	5369/3600	1
7	3	28567/24000	2
7	4	14011361/12960000	1
7	6	47464376609/46656000000	0
7	7	940908897061/933120000000	3
7	8	168646392872321/167961600000000	1
7	14	78368963754768444899394 89/7836416409600000000000000	2
11	3	19164113947/16003008000	2
11	4	43635917056897/40327580160000	1

2. Generalized harmonic numbers

Suppose that $p \geq 5$ is a fixed prime number. In order to compute the constant c_p given in Theorem 1.1 effectively, we will need the number of elements in the set

$$R(p) := \{2 \leq \ell < p-1 \mid \nu_p(B_\ell) = 0\}. \tag{2.1}$$

Note that the cardinality of this set is

$$|R(p)| \leq \frac{p-3}{2} \tag{2.2}$$

as $B_\ell = 0$ for any odd $\ell > 1$. Also, this cardinality is closely related to *the index of irregularity of* p which is defined as

$$i(p) = |\{2 \leq i < p-1 \mid i \text{ is even and } \nu_p(B_i) \geq 1\}|.$$

There are several theoretical and computational results on the bounds and the exact values of the index of irregularity $i(p)$ for a given prime $p > 3$ and we refer the reader to [3–6, 17, 19, 25, 26, 29] for these results. To give a closed formula for the mentioned constant c_p, we will use the following lemma.

Lemma 2.1. *Let* $p \geq 5$ *be a prime number and* $R(p)$ *be as in* (2.1). *Define*

$$R'(p) := \{2 \leq \ell' < p(p-1) \mid \nu_p(B_{\ell'}) = 0\}. \tag{2.3}$$

Then, we have $|R'(p)| = (p-1) \cdot |R(p)|$.

Proof. Observe that $R'(p)$ does not contain any odd integer, as $B_{\ell'} = 0$ for any odd $\ell' > 1$ (see [18, Proposition 15.1.1]). So, take any even positive integer $\ell' \in [2, p(p-1))$ and assume that $\ell \equiv \ell' \pmod{p-1}$ for some $0 \leq \ell < p-1$. By the von Staudt-Clausen theorem in [18, Theorem 3, Chapter 15], we know that $\nu_p(B_{\ell'}) = -1$ if and only if $p-1 \mid \ell'$. Therefore, none of the multiples of

$p-1$ in $[2, p(p-1))$ is an element of $R'(p)$. Also, since ℓ' is even, by Kummer's congruence (see for instance [18, Theorem 5, Chapter 15]) we have that

$$B_{\ell'} \equiv \frac{\ell'}{\ell} B_\ell \pmod{p}$$

whenever $p-1 \nmid \ell'$. Note that $p \nmid \ell$, as $\ell < p-1$. Hence, the condition $\nu_p(B_{\ell'}) = 0$ is equivalent to $\nu_p(\ell') = \nu_p(B_\ell) = 0$. This indicates that $\ell' \in R'(p)$ holds if and only if $p \nmid \ell'$ and $\ell \in R(p)$. In other words, any element ℓ' of $R'(p)$ satisfies the congruences

$$\ell' \equiv 1, 2, \ldots, p-1 \pmod{p}, \tag{2.4}$$

$$\ell' \equiv \ell \pmod{p-1} \tag{2.5}$$

for some $\ell \in R(p)$. Combining these solutions with the Chinese remainder theorem yields that there are $(p-1) \cdot |R(p)|$ possible common solutions modulo $p(p-1)$ for congruences (2.4) and (2.5). If we take the corresponding representatives in $[2, p(p-1))$, we conclude that $|R'(p)| = (p-1)|R(p)|$. □

Using the previous lemma, now we can give the corresponding estimation on $V_p(x)$.

2.1. Proof of Theorem 1.1

Let $p \geq 5$ be a prime number and $s \in \mathbb{Z}_{\geq 0}$. First of all, we emphasize that for any $t \in \mathbb{Z}_{>0}$, the condition $\nu_p\left(H_{p-1}^{(s+t\cdot\varphi(p^2))}\right) = 1$ is equivalent to $\nu_p\left(H_{p-1}^{(s)}\right) = 1$. To see this fact, observe that

$$H_{p-1}^{(s+t\cdot\varphi(p^2))} = \sum_{i=1}^{p-1} \frac{1}{i^{s+t\cdot\varphi(p^2)}} \equiv \sum_{i=1}^{p-1} \frac{1}{i^s} = H_{p-1}^{(s)} \pmod{p^2} \tag{2.6}$$

by Euler's Theorem. Now, define

$$\overline{V_p} := \left\{0 \leq s < \varphi(p^2) \mid \nu_p\left(H_{p-1}^{(s)}\right) = 1\right\} \tag{2.7}$$

and let x be any positive real number. To estimate $V_p(x)$ which is defined in (1.7), we divide the interval $[0, x]$ as

$$[0, x] = \bigcup_{j=0}^{k-1} [j \cdot \varphi(p^2), (j+1) \cdot \varphi(p^2)) \cup [k \cdot \varphi(p^2), x]$$

for some $k \in \mathbb{Z}_{\geq 0}$ where $x < (k+1) \cdot \varphi(p^2)$. By congruence (2.6), note that each interval of the form $[j \cdot \varphi(p^2), (j+1) \cdot \varphi(p^2))$ contains $|\overline{V_p}|$ many s values for which $\nu_p\left(H_{p-1}^{(s)}\right) = 1$. Also, notice that the interval $[k \cdot \varphi(p^2), x]$ contains at most $\varphi(p^2)$ many integers. Thus, we have

$$V_p(x) = \left|\overline{V_p}\right| \cdot k + \mathcal{O}\left(\varphi(p^2)\right)$$

$$= \left|\overline{V_p}\right| \cdot \left\lfloor \frac{x}{\varphi(p^2)} \right\rfloor + \mathcal{O}_p(1)$$
$$= \left|\overline{V_p}\right| \cdot \frac{x}{p(p-1)} + \mathcal{O}_p(1)$$
$$= c_p x + \mathcal{O}_p(1),$$

where

$$c_p = \frac{\left|\overline{V_p}\right|}{p(p-1)}.$$

From now on, our aim is to find the number of elements in $\overline{V_p}$ in terms of $|R(p)|$ where $R(p)$ is defined in (2.1).

Take any $s \in \overline{V_p}$. Observe that

$$H_{p-1}^{(s)} = \sum_{i=1}^{p-1} i^{-s} \equiv \sum_{i=1}^{p-1} i^{\varphi(p^2)-s} \pmod{p^2}, \tag{2.8}$$

where $0 < \varphi(p^2) - s \leq \varphi(p^2)$. By Faulhaber's formula (see [18, Theorem 1, Chapter 15]), we know that for any integer $m > 0$, equations

$$\sum_{i=1}^{p-1} i^m = \frac{1}{m+1} \sum_{j=1}^{m+1} \binom{m+1}{j} B_{m+1-j} p^j$$
$$= pB_m + \frac{m}{2} p^2 B_{m-1} + \frac{m(m-1)}{6} p^3 B_{m-2} + \sum_{j=4}^{m+1} \binom{m+1}{j} B_{m+1-j} p^j$$

hold. So, applying Faulhaber's formula to congruence (2.8), we get that

$$H_{p-1}^{(s)} \equiv pB_{\varphi(p^2)-s} + \frac{\varphi(p^2)-s}{2} p^2 B_{\varphi(p^2)-s-1} \pmod{p^2} \tag{2.9}$$

as $p > 3$ and $\nu_p(B_n) \geq -1$ for any integer $n \geq 0$, by the von Staudt-Clausen theorem. Now, we will separate our cases with respect to the parity of s. First, take an odd s. Note that $\varphi(p^2) - s$ is also odd, as $p - 1 \mid \varphi(p^2)$. If $\varphi(p^2) - s = 1$, then congruence (2.9) becomes

$$H_{p-1}^{(s)} \equiv pB_1 + \frac{p^2}{2} B_0 = \frac{p(p-1)}{2} \pmod{p^2}$$

which indicates that $\nu_p\left(H_{p-1}^{(s)}\right) = 1$, as $B_0 = 1$ and $B_1 = -\frac{1}{2}$. So, suppose $\varphi(p^2) - s > 1$. In that case $B_{\varphi(p^2)-s} = 0$, and thus

$$H_{p-1}^{(s)} \equiv \frac{\varphi(p^2)-s}{2} p^2 B_{\varphi(p^2)-s-1} \pmod{p^2} \tag{2.10}$$

by congruence (2.9). This yields that $\nu_p\left(H_{p-1}^{(s)}\right) = 1$ if and only if

$$\nu_p\left(\varphi(p^2) - s\right) = 0 \text{ and } \nu_p\left(B_{\varphi(p^2)-s-1}\right) = -1$$

as $p > 2$. By the von Staudt-Clausen theorem, we know that the latter condition is equivalent to $p-1 \mid \varphi(p^2)-s-1$, that is $s+1 \equiv 0 \pmod{p-1}$. Besides, we need to have $p \nmid s$ in order to get $\nu_p\left(\varphi(p^2)-s\right)=0$. Combining these two facts gives us the following necessary and sufficient conditions for s being in $\overline{V_p}$:

$$s+1 \equiv 0 \pmod{p-1}, \tag{2.11}$$

$$s+1 \equiv 0, 2, 3, \ldots, p-1 \pmod{p}. \tag{2.12}$$

This implies that there are $(p-1)$ solutions modulo $p(p-1)$ by the Chinese remainder theorem. Note that the case $\varphi(p^2)-s=1$ is covered by these solutions, as this is satisfied when $p(p-1) \mid s+1$. Thus, we conclude that there are $(p-1)$ many possible odd values of $s \in \overline{V_p}$.

Next, suppose that $s \in \overline{V_p}$ is even. Note that the case $s=0$ is impossible, as this yields that $H_{p-1}^{(s)} = p-1$. This indicates that $0 < \varphi(p^2)-s < \varphi(p^2)$ where $\varphi(p^2)-s$ is also even, as $p-1 \mid \varphi(p^2)$. Moreover, since congruence (2.9) holds for s, we see that

$$H_{p-1}^{(s)} \equiv pB_{\varphi(p^2)-s} \pmod{p^2} \tag{2.13}$$

as $B_1 = -\frac{1}{2}$ and $p > 2$. Hence, $\nu_p\left(H_{p-1}^{(s)}\right) = 1$ is satisfied if and only if $\nu_p\left(B_{\varphi(p^2)-s}\right)=0$. By the definition of $R'(p)$ which is given in (2.3), we deduce that the latter fact is also equivalent to $\varphi(p^2)-s \in R'(p)$, as $0 < \varphi(p^2)-s < p(p-1)$ and $\varphi(p^2)-s$ is even. Thus, there are $|R'(p)| = (p-1)\cdot|R(p)|$ many possible even values of s in $\overline{V_p}$. Combining this with the possible odd values in $\overline{V_p}$, we obtain that

$$\left|\overline{V_p}\right| = (p-1)+|R'(p)| = (p-1)(|R(p)|+1) > 0. \tag{2.14}$$

Recall that $c_p = \frac{\left|\overline{V_p}\right|}{p(p-1)}$. By (2.2) and (2.14), we see that

$$0 < c_p = \frac{\left|\overline{V_p}\right|}{p(p-1)} = \frac{(p-1)(|R(p)|+1)}{p(p-1)} = \frac{|R(p)|+1}{p} \leq \frac{p-1}{2p} < \frac{1}{2}. \tag{2.15}$$

If we take a regular prime p, then we know that all even numbers less than $p-1$ are in the set $R(p)$. In other words, $|R(p)|$ reaches its maximum value and it is equal to $\frac{p-3}{2}$. Thus, we conclude by (2.15) that $c_p = \frac{p-1}{2p}$ for a regular prime $p \geq 5$.

Finally, recall by (1.5) that $\nu_p\left(H_{p-1}^{(s)}\right) \geq 1$ whenever $p-1 \nmid s$. Also, if $p-1 \mid s$, then we have

$$H_{p-1}^{(s)} = \sum_{i=1}^{p-1} \frac{1}{i^s} \equiv \sum_{i=1}^{p-1} 1 \pmod{p}.$$

This indicates that $\nu_p\left(H_{p-1}^{(s)}\right) = 0$ if and only if $p-1 \mid s$. Hence, the density is

$$\frac{\left|\left\{s \leq x \,\middle|\, \nu_p\left(H_{p-1}^{(s)}\right) = 0\right\}\right|}{x} \sim \frac{1}{p-1}$$

as x tends to infinity. Thus, the density of the positive integers s for which $\nu_p\left(H_{p-1}^{(s)}\right) \geq 2$ is

$$1 - \left(c_p + \frac{1}{p-1}\right) = 1 - \frac{|R(p)|+1}{p} - \frac{1}{p-1} \tag{2.16}$$

by (2.15). For any irregular prime p, we know that

$$|R(p)| \leq \frac{p-3}{2} - 1 = \frac{p-5}{2}$$

by inequality (2.2). Assembling this and (2.16), we deduce that

$$1 - \left(c_p + \frac{1}{p-1}\right) \geq 1 - \left(\frac{p-3}{2p} + \frac{1}{p-1}\right) > \frac{1}{2}$$

as $p > 3$. In other words, more than the half of the positive integers satisfy the inequality $\nu_p\left(H_{p-1}^{(s)}\right) > 1$, when p is irregular. □

Example 2.2. As we mentioned in the introduction, take the first irregular prime $p = 37$. In that case, the set

$$R(p) = R(37) = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 34\}.$$

Therefore, we have

$$|R(37)| = 16 = \frac{p-5}{2} < \frac{p-3}{2},$$

and thus $c_{37} = \frac{17}{37} \approx 0.459\ldots$, as $32 \notin R(37)$.

2.2. Proof of Theorem 1.2

We will follow the same approach that was given in [27, Theorem 5.1]. By Faulhaber's formula given in [18, Theorem 1, Chapter 15], we see that for any $m \in \mathbb{Z}_{>0}$,

$$\sum_{i=1}^{p-1} i^m = pB_m + \frac{m}{2}p^2 B_{m-1} + \frac{m(m-1)}{6}p^3 B_{m-2} + \sum_{j=4}^{m+1} \binom{m+1}{j} B_{m+1-j} p^j. \tag{2.17}$$

By the von Staudt-Clausen theorem (see also [18, Proposition 15.2.1]), we infer that $\nu_p(B_n) \geq -1$ for any $n \in \mathbb{N}$. Moreover, as mentioned in the introduction, we have $B_n = 0$, if $n > 1$ is odd. Combining these two facts and equation (2.17), we obtain that

$$\sum_{i=1}^{p-1} i^m \equiv \begin{cases} pB_m & \left(\bmod\ p^2\right) & \text{if } m \text{ is even,} \\ \frac{m}{2}p^2 B_{m-1} & \left(\bmod\ p^3\right) & \text{if } m > 1 \text{ is odd} \end{cases} \tag{2.18}$$

as $B_1 = -\frac{1}{2}$. Let $s = s' + k\varphi(p^2)$ for some $k \in \mathbb{Z}_{\geq 0}$. Observe that

$$\begin{aligned} H_{p-1}^{(s)} = \sum_{i=1}^{p-1} \frac{1}{i^s} = \sum_{i=1}^{p-1} \left(\frac{1}{i}\right)^{s'+k\varphi(p^2)} &\equiv \sum_{i=1}^{p-1} \frac{1}{i^{s'}} \ \left(\bmod\ p^2\right) \\ &\equiv \sum_{i=1}^{p-1} i^{\varphi(p^2)-s'} \ \left(\bmod\ p^2\right) \end{aligned} \tag{2.19}$$

by Euler's Theorem. If we take an even s, then we derive that

$$H_{p-1}^{(s)} \equiv pB_{\varphi(p^2)-s'} \ \left(\bmod\ p^2\right) \tag{2.20}$$

using congruence (2.18). Notice that $s \equiv s' \pmod{p-1}$. By the divisibility assumption on s, we get that $p-1 \nmid s'$. By Kummer's congruences on Bernoulli numbers, we deduce that

$$\begin{aligned} B_{\varphi(p^2)-s'} &= B_{(p-a)(p-1)+a(p-1)-s'} \\ &\equiv \frac{(p-a)(p-1)+a(p-1)-s'}{a(p-1)-s'} B_{a(p-1)-s'} \pmod{p} \\ &\equiv \frac{a+ap-a-s'}{-a-s'} B_{a(p-1)-s'} \equiv \frac{s'}{s'+a} B_{ap-a-s'} \pmod{p}, \end{aligned} \tag{2.21}$$

where $a = \left\lceil \frac{s'}{p-1} \right\rceil$. By the definition of a, we know that $a \geq \frac{s'}{p-1} > a-1$ which implies that $a(p-1)-s' \geq 0$. Also we have $(a-1)(p-1)-s' < 0$. This indicates that $a(p-1)-s' < p-1$. Recall that $p-1 \nmid s'$. This leads to the fact that $0 < a(p-1)-s' < p$. Also, note that the congruence $s' \equiv s \pmod{p(p-1)}$ yields that $s' \equiv s \pmod{p}$. Combining this with (2.20) and (2.21), we get that

$$H_{p-1}^{(s)} \equiv \frac{s}{s+a} pB_{ap-(s'+a)} \ \left(\bmod\ p^2\right),$$

which is congruence (1.8).

For the second part of the theorem, assume that $s > 1$ is odd. Similar to the first case, we also obtain that

$$H_{p-1}^{(s)} \equiv \sum_{i=1}^{p-1} i^{\varphi(p^3)-s''} \ \left(\bmod\ p^3\right). \tag{2.22}$$

Recall that $p-1 \nmid s+1$ and $s'' \equiv s \ \left(\bmod\ p^2(p-1)\right)$. Therefore, we have $p-1 \nmid s''+1$. Note that $\varphi(p^3) - s'' > 0$, as $0 \leq s'' < p^2(p-1)$. Also,

$\varphi(p^3) - s'' = 1$ contradicts $p-1 \nmid s''+1$. Combining this fact with congruences (2.18) and (2.22), we get that

$$H_{p-1}^{(s)} \equiv \frac{\varphi(p^3) - s''}{2} p^2 B_{\varphi(p^3)-s''-1} \pmod{p^3} \tag{2.23}$$

as s'' is odd and $\varphi(p^3) - s'' > 1$. Again by Kummer's congruence,

$$\begin{aligned} B_{\varphi(p^3)-s''-1} &= B_{(p^2-b)(p-1)+b(p-1)-s''-1} \\ &\equiv \frac{(p^2-b)(p-1)+b(p-1)-s''-1}{b(p-1)-s''-1} B_{b(p-1)-s''-1} \pmod{p} \\ &\equiv \frac{s''+1}{b+s''+1} B_{bp-b-s''-1} \pmod{p}, \end{aligned} \tag{2.24}$$

where $b = \left\lceil \frac{s''+1}{p-1} \right\rceil$. Similar to the first part, observe that

$$0 < b(p-1) - s'' - 1 < p-1,$$

since $p-1 \nmid s''+1$. As $s'' \equiv s \pmod{p^2(p-1)}$, we have $s'' \equiv s \pmod{p}$. Therefore, we derive that

$$\begin{aligned} \frac{\varphi(p^3) - s''}{2} B_{\varphi(p^3)-s''-1} &\equiv \frac{p^3 - p^2 - s''}{2} \cdot \frac{s''+1}{b+s''+1} B_{bp-b-s''-1} \pmod{p} \\ &\equiv -\frac{s(s+1)}{2(b+s+1)} B_{bp-b-s''-1} \pmod{p}. \end{aligned}$$

Hence, we obtain that

$$H_{p-1}^{(s)} \equiv \frac{\varphi(p^3) - s''}{2} p^2 B_{\varphi(p^3)-s''-1} \equiv -\frac{s(s+1)}{2(b+s+1)} p^2 B_{bp-(b+s''+1)} \pmod{p^3}$$

by congruence (2.23). □

Before proving Theorem 1.3, we need two corollaries which are immediate consequences of Theorem 1.2.

Corollary 2.3. *Let $p \geq 5$ be a prime. Assume that s is even,*

$$s' \equiv s \pmod{p(p-1)},$$

where $0 \leq s' < p(p-1)$ and define $a' = \left\lceil \frac{s'}{p-1} \right\rceil$. If $p-1$ and p do not divide s and $(p, a'(p-1) - s')$ is a regular pair then $\nu_p\left(H_{p-1}^{(s)}\right) = 1$. In particular, if p is a regular prime with $p \nmid s$ and $p-1 \nmid s$, then $\nu_p\left(H_{p-1}^{(s)}\right) = 1$.

Proof. Since $(p, a'(p-1) - s')$ is a regular pair, we know that $p \nmid B_{a'(p-1)-s'}$. As a consequence of a theorem related to Bernoulli numbers given in [12], we obtain that if p divides the integer $m \geq 0$, then p also divides the numerator of B_m. Hence, we get that $p \nmid a'(p-1) - s'$. This implies that $p \nmid s' + a'$. Moreover, the congruence $s' \equiv s \pmod{p(p-1)}$ yields that $s' \equiv s \pmod{p}$

and $s' \equiv s \pmod{p-1}$. Thus $p \nmid s + a'$. Also since $p \nmid s$ and $p-1 \nmid s$, one has that

$$\nu_p\left(\frac{s}{s+a'}B_{a'(p-1)-s'}\right) = 0$$

by the von Staudt-Clausen theorem, as $s' \equiv s \pmod{p-1}$. This in turn implies that

$$\nu_p\left(H_{p-1}^{(s)}\right) = 1$$

by Theorem 1.2. As $0 < a'(p-1) - s' < p-1$, one can also deduce the same result by the definition of a regular prime when $p-1 \nmid s$. □

Corollary 2.4. *Let $p \geq 5$ be a prime. Suppose that s is odd,*

$$s'' \equiv s \pmod{p^2(p-1)},$$

where $0 \leq s'' < p^2(p-1)$ and $b' = \left\lceil \frac{s''+1}{p-1} \right\rceil$. If $p \nmid s(s+1)$, $p-1 \nmid s+1$ and $(p, b'(p-1) - s'' - 1)$ is a regular pair, then $\nu_p\left(H_{p-1}^{(s)}\right) = 2$. In particular, if $p, p-1 \nmid s+1$ and p is a regular prime with $p \nmid s$, then the same result follows.

Proof. First of all, note that if $s = 1$, then $\nu_p\left(H_{p-1}^{(s)}\right) = 2$ by congruence (1.2). So, take $s > 1$. As the conditions of Theorem 1.2 are satisfied, we can use this result to obtain the p-adic orders of corresponding generalized harmonic numbers. Since $(p, b'(p-1) - s'' - 1)$ is a regular pair, we deduce that $p \nmid b'(p-1) - s'' - 1$ by the properties of Bernoulli numbers [12]. Note that $s'' \equiv s \pmod{p^2(p-1)}$ implies that $p \nmid b'(p-1) - s - 1$, which is also equivalent to the fact that $p \nmid b' + s + 1$. Recall that $p \nmid s(s+1)$. Combining all of these facts leads to $\nu_p\left(H_{p-1}^{(s)}\right) = 2$ by Theorem 1.2 and by a similar argument as in the previous corollary. Also if p is regular with $p \nmid s$ and $p, p-1 \nmid s+1$, then the same result follows as $0 < b'(p-1) - s'' - 1 < p-1$ and each such pair (p, m) is regular for $m \leq p-3$. □

Now, we are ready to prove our last theorem from the introduction.

2.3. Proof of Theorem 1.3(1)

Let $p \geq 5$ be a prime with $p \equiv 3 \pmod 4$. Let $h(p)$ be the class number of the quadratic imaginary field $\mathbb{Q}(\sqrt{-p})$. Then, it is known that (see [9, Chapter 1 & 6])

$$h(p) = \frac{1}{(2 - (\frac{2}{p}))} \sum_{m < \frac{p}{2}} \left(\frac{m}{p}\right), \tag{2.25}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol whose values can be -1, 0 and 1. Hence, we have

$$1 \leq h(p) < p/2 < p.$$

In particular, we obtain that p does not divide $h(p)$. Moreover, we know by [7] that

$$h(p) \equiv -2B_{\frac{p+1}{2}} \pmod{p}.$$

By the previous argument, one sees that the p-adic order of $B_{\frac{p+1}{2}}$ is 0. This indicates that $\left(p, \frac{p+1}{2}\right)$ is a regular pair for $p \equiv 3 \pmod{4}$. So if $s = (kp + a - \frac{1}{2})(p-1) - 1$ for some $k \geq 0$ and $a \geq 1$, then $s \equiv -\frac{2a+1}{2} \pmod{p}$. As $p \nmid 2a+1$, we get that $p \nmid s$. Let $s' = a(p-1) - \frac{p+1}{2}$. Thus, we have that $s \equiv s' \pmod{p(p-1)}$. Observe that s is even and

$$\frac{s'}{p-1} = \frac{\left(a - \frac{1}{2}\right)(p-1) - 1}{p-1} = a - \left(\frac{1}{2} + \frac{1}{p-1}\right).$$

Therefore $a = \left\lceil \frac{s'}{p-1} \right\rceil$, as $0 < \frac{1}{2} + \frac{1}{p-1} < 1$ for $p > 3$. Note that $s \equiv s' \equiv -\frac{p+1}{2} \pmod{p-1}$, and since $0 < \frac{p+1}{2} < p-1$ for $p > 3$, we derive that $p - 1 \nmid s$. As $\frac{p+1}{2} = a(p-1) - s'$ and $\left(p, \frac{p+1}{2}\right)$ is a regular pair, by Corollary 2.3 we conclude that $\nu_p\left(H_{p-1}^{(s)}\right) = 1$. □

2.4. Proof of Theorem 1.3(2)

First remind from the proof of Theorem 1.3(1) that $\left(p, \frac{p+1}{2}\right)$ is a regular pair. Take s of the form $(\ell p^2 + b - \frac{1}{2})(p-1) - 2$ for some $\ell \geq 0$ and $b \geq 1$. Recall that s is odd and if $s'' = (b - \frac{1}{2})(p-1) - 2$, then $s'' \equiv s \pmod{p^2(p-1)}$. Similar to the proof of Theorem 1.3(1), we see that $s \equiv -\frac{2b+3}{2} \pmod{p}$. Hence, we get $s + 1 \equiv -\frac{2b+1}{2} \pmod{p}$. As $p \nmid 4b^2 + 8b + 3 = (2b+1)(2b+3)$, we deduce that $p \nmid s(s+1)$. Thus, we obtain that

$$\frac{s''+1}{p-1} = \frac{\left(b - \frac{1}{2}\right)(p-1) - 1}{p-1} = b - \left(\frac{1}{2} + \frac{1}{p-1}\right).$$

Since $p > 3$, we derive that $b = \left\lceil \frac{s''+1}{p-1} \right\rceil$ as $0 < \frac{1}{2} + \frac{1}{p-1} < 1$. Moreover, we have $s'' + 1 = b(p-1) - \frac{p+1}{2}$. Therefore, $(p, b(p-1) - s'' - 1) = \left(p, \frac{p+1}{2}\right)$ is a regular pair. Furthermore, we observe that $s + 1 \equiv s'' + 1 \equiv -\frac{p+1}{2} \pmod{p-1}$. As $p > 3$, we get that $0 < \frac{p+1}{2} < p - 1$. Thus $p - 1 \nmid s + 1$. Now by Corollary 2.4, we conclude that $\nu_p\left(H_{p-1}^{(s)}\right) = 2$. □

3. Generalized hyperharmonic numbers

The next corollary extends Theorem 1.3 to generalized hyperharmonic numbers.

Corollary 3.1. *Let $p \geq 5$ be a prime number with $p \equiv 3 \pmod{4}$ and $s \geq 1$ a positive integer. If the conditions of Theorem 1.3(1) hold and r is of the* *form* mp^2+1 *for some* $m \geq 0$*, then the exact* p*-adic valuation of the generalized hyperharmonic number* $H_{p-1}^{(s,r)}$ *is* 1*. Moreover, if* $r \equiv 1 \pmod{p^3}$*, then* $\nu_p\left(H_{p-1}^{(s,r)}\right) = 2$ *under the conditions on* p *and* s *given in Theorem 1.3*(2)*.*

Proof. Let $r \equiv 1 \pmod{p^2}$. Then, observe by [16, Proposition 2.1] that

$$\begin{aligned} H_{p-1}^{(s,r)} &= \sum_{j=1}^{p-1} \binom{p-1-j+r-1}{r-1} \frac{1}{j^s} \\ &= \sum_{j=1}^{p-1} \frac{r(r+1)\cdots(r+(p-j-2))}{(p-j-1)!} \cdot \frac{1}{j^s} \\ &\equiv \sum_{j=1}^{p-1} \frac{1\cdot 2\cdots(p-j-1)}{(p-j-1)!} \cdot \frac{1}{j^s} \equiv \sum_{j=1}^{p-1} \frac{1}{j^s} \pmod{p^2}. \end{aligned}$$

Namely, $H_{p-1}^{(s,r)} \equiv H_{p-1}^{(s)} \pmod{p^2}$ when $r \equiv 1 \pmod{p^2}$. As the conditions on p and s given in Theorem 1.3(1) are satisfied, one has that $\nu_p\left(H_{p-1}^{(s,r)}\right) = 1$. Similarly, if $r \equiv 1 \pmod{p^3}$, then we deduce that

$$H_{p-1}^{(s,r)} = \sum_{j=1}^{p-1} \frac{r(r+1)\cdots(r+(p-j-2))}{(p-j-1)!} \cdot \frac{1}{j^s} \equiv \sum_{j=1}^{p-1} \frac{1}{j^s} \pmod{p^3}.$$

In that case $\nu_p\left(H_{p-1}^{(s,r)}\right) = 2$, if the conditions of Theorem 1.3(2) are satisfied. □

Recall by Table 1 that the conditions of Corollary 3.1 are satisfied when we have $(p,s) \in \{(7,8),(11,3)\}$. So, if we take $(p,s,r) = (7,8,99)$, then computations performed by SageMath [23] show that

$$H_{p-1}^{(s,r)} = H_6^{(8,99)} = \frac{7353180000659238390827}{83980800000000} \quad \text{and} \quad \nu_7\left(H_6^{(8,99)}\right) = 1,$$

which is also verified by the first part of Corollary 3.1. Similarly, choosing $(p,s,r) = (11,3,1332)$ leads to

$$H_{p-1}^{(s,r)} = H_{10}^{(3,1332)} = \frac{332492623235968636273146368852 27}{889056000}$$

and $\nu_{11}\left(H_{10}^{(3,1332)}\right) = 2$, as it is expected from the second part of Corollary 3.1.

Corollary 3.2. *Let* $p \geq 5$ *be a prime number,* $s > 2$ *be an even positive integer and* $s_0 \equiv s \pmod{p^2(p-1)}$ *with* $0 \leq s_0 < p(p-1)$*. Denote* $c = \left\lceil \frac{s_0}{p-1} \right\rceil$*. If* $p-1 \nmid s$*,* $p \nmid s(s+1)$*,* $(p, c(p-1)-s_0)$ *is a regular pair and* $r \equiv 2 \pmod{p^3}$*, then* $\nu_p\left(H_{p-1}^{(s,r)}\right) = 2$*.*

Proof. Let $r \equiv 2 \pmod{p^3}$. Observe that

$$\binom{p-1-j+r-1}{r-1} \equiv \frac{2\cdot 3\cdots(p-j)}{(p-j-1)!} \equiv p-j \pmod{p^3}.$$

So, we have

$$\begin{aligned} H_{p-1}^{(s,r)} &\equiv \sum_{j=1}^{p-1} \frac{p-j}{j^s} \pmod{p^3} \\ &= p\sum_{j=1}^{p-1}\frac{1}{j^s} - \sum_{j=1}^{p-1}\frac{1}{j^{s-1}} \equiv pH_{p-1}^{(s)} - H_{p-1}^{(s-1)} \pmod{p^3}. \end{aligned} \tag{3.1}$$

Notice that $s_0 \equiv s \pmod{p(p-1)}$. Since $s-1>1$ is odd, $c = \left\lceil \frac{s_0}{p-1} \right\rceil = \left\lceil \frac{(s_0-1)+1}{p-1} \right\rceil$ and $p-1$ does not divide $(s-1)+1 = s$, we obtain that

$$\begin{aligned} pH_{p-1}^{(s)} &\equiv \frac{s}{s+c}p^2 B_{cp-(c+s_0)} \pmod{p^3}, \\ H_{p-1}^{(s-1)} &\equiv -\frac{(s-1)s}{2(c+s)}p^2 B_{cp-(c+s_0)} \pmod{p^3} \end{aligned}$$

by Theorem 1.2. Combining these equations, we get that

$$H_{p-1}^{(s,r)} \equiv pH_{p-1}^{(s)} - H_{p-1}^{(s-1)} \equiv \frac{s(s+1)}{2(c+s)}p^2 B_{c(p-1)-s_0} \pmod{p^3}. \tag{3.2}$$

Since $(p, c(p-1)-s_0)$ is a regular pair, then $p \nmid B_{c(p-1)-s_0}$ which implies that $p \nmid c+s_0$ by [12]. Besides, using the von Staudt-Clausen theorem, one infers that

$$\nu_p\left(B_{c(p-1)-s_0}\right) = 0$$

as $s_0 \equiv s \pmod{p^2(p-1)}$ and $p-1 \nmid s$. Since $s \equiv s_0 \pmod p$ and $p \nmid s(s+1)$, we conclude that $\nu_p\left(H_{p-1}^{(s,r)}\right) = 2$. □

Corollary 3.3. *Let $p \geq 5$ be a prime number and $s>1$ be an odd natural number. Let $s_1 \equiv s \pmod{p(p-1)}$ where $0 \leq s_1 < p(p-1)$ and $a = \left\lceil \frac{s_1-1}{p-1} \right\rceil$. Assume that $p \nmid s-1$ and $p-1$ does not divide $s-1$ and s. Suppose also that $(p, a(p-1)-(s_1-1))$ is a regular pair and $r \equiv 2 \pmod{p^3}$. Then $\nu_p\left(H_{p-1}^{(s,r)}\right) = 1$.*

Proof. Suppose that $r \equiv 2 \pmod{p^3}$ and $s>1$ is odd. By congruence (3.1), we know that

$$H_{p-1}^{(s,r)} \equiv pH_{p-1}^{(s)} - H_{p-1}^{(s-1)} \pmod{p^3}. \tag{3.3}$$

As $p-1$ does not divide s, by congruence (1.5), one sees that

$$pH_{p-1}^{(s)} \equiv 0 \pmod{p^2}. \tag{3.4}$$

Also, since s is odd and $p-1 \nmid s-1$, by Theorem 1.2 again we get that

$$H_{p-1}^{(s-1)} \equiv \frac{s-1}{s-1+a} pB_{ap-(s_1-1+a)} \pmod{p^2},$$

where $a = \left\lceil \frac{s_1-1}{p-1} \right\rceil$. Since the pair $(p, ap-(s_1-1+a))$ is regular, $p \nmid B_{ap-(s_1-1+a)}$, and this in turn implies that $p \nmid ap-(s_1-1+a)$ by [12]. In addition, by the von Staudt-Clausen theorem, one obtains that

$$\nu_p\left(B_{ap-(s_1-1+a)}\right) = \nu_p\left(B_{a(p-1)-(s_1-1))}\right) = 0,$$

since $s_1 \equiv s \pmod{p(p-1)}$ and $p-1$ does not divide $s-1$. As $p \nmid s-1$, we deduce that $\nu_p\left(H_{p-1}^{(s-1)}\right) = 1$. Assembling this fact with congruences (3.3) and (3.4), we conclude that $\nu_p\left(H_{p-1}^{(s,r)}\right) = 1$. □

In order to give some examples for Corollary 3.2 and Corollary 3.3, we will take (p,s,r) from $\{(5,18,252),(11,4,1333)\}$ and $\{(7,11,688),(13,3,2199)\}$, respectively. Thanks to SageMath [23], we can give their values and corresponding p-adic valuations as follows:

- $H_4^{(18,252)} = \dfrac{79840540284431125787994 25}{2958148142320582656}$, $\nu_5\left(H_4^{(18,252)}\right) = 2$.
- $H_{10}^{(4,1333)} = \dfrac{151773816286600012246084601425511578 3}{40327580160000}$,
  $\nu_{11}\left(H_{10}^{(4,1333)}\right) = 2$.
- $H_6^{(11,688)} = \dfrac{5910645956568900471078065582293}{4534963200000000000}$,
  $\nu_7\left(H_6^{(11,688)}\right) = 1$.
- $H_{12}^{(3,2199)} = \dfrac{6365714215662050360007804164356023185517 1}{426000072960}$,
  $\nu_{13}\left(H_{12}^{(3,2199)}\right) = 1$.

Final Remark. One can also give other conditions on p, s and r to obtain

$$\nu_p\left(H_{p-1}^{(s,r)}\right) = 1 \quad \text{or} \quad \nu_p\left(H_{p-1}^{(s,r)}\right) = 2.$$

We do not deal with each case separately. Also in some cases, we may not need all of the conditions to prove such a result given in our corollaries (Corollary 3.2 and Corollary 3.3). For example, if we take $s \le p-3$ even and $r \equiv 2 \pmod{p^3}$ for $p \ge 5$, then automatically we have $p-1 \nmid s$ and $p \nmid s(s+1)$. In that case, we get that

$$H_{p-1}^{(s,r)} = \frac{p^2 s}{2} B_{p-1-s} \pmod{p^3}$$

by congruence (3.2). Thus, we conclude that $\nu_p\left(H_{p-1}^{(s,r)}\right) = 2$, if $(p, p-1-s)$ is a regular pair. Also if $1 < s < p-3$ is odd, $r \equiv 2 \pmod{p^3}$ and $(p, p-s)$ is a regular pair, then Corollary 3.3 follows, that is to say $\nu_p\left(H_{p-1}^{(s,r)}\right) = 1$.

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