A WEIERSTRASS SEMIGROUP AT A GENERALIZED FLEX
ON A PLANE CURVE

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Abstract. We consider a Weierstrass semigroup at a generalized flex on a smooth plane curve. We find the candidates of a Weierstrass semigroup at a 2-flex of higher multiplicity on a smooth plane curve of degree $d \geq 5$, and give some examples to show the existence of them.

1. Introduction and Preliminaries

Let $C$ be a smooth complex projective plane curve of degree $d \geq 4$. Let $P$ be a point on $C$. We divide the lines on the plane into three types according to the intersection multiplicity at $P$:

(1) $I(C \cap \ell_0, P) = 0$;
(2) $I(C \cap \ell_1, P) = 1$;
(3) $I(C \cap \ell_2, P) \geq 2$;

where $I(C \cap \ell, P)$ means the intersection multiplicity of $C$ and $\ell$ at $P$. We call $\ell_2$ the tangent line to $C$ at $P$ and denote it by $T_P C$. If $I(C \cap T_P C, P) > 2$, then we call $P$ the inflection point or a flex on $C$. One can generalize the notion of this concept by replacing the lines by curves of some given degree $m$. At each point $P$, for each natural number $m < d$, there exists a curve $F_m$ of degree $m$ which have the highest order of contact with $C$. We call such a curve $F_m$ as an osculating curve of degree $m$ at $P$ to $C$. Note that an osculating curve $F_m$ need not be irreducible. We are interested in the case that $F_m$ is irreducible. The point $P$ on $C$ is called an $m$-flex if $F_m$ is irreducible and $I(C \cap F_m, P) > \frac{m(m+3)}{2}$ where $F_m$ is an osculating curve of

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degree $m$. Note that the number $\frac{m(m+3)}{2}$ is the dimension of the system of curves of degree $m$. Obviously, a 1-flex means a flex in our notation.

In this paper, we consider a Weierstrass semigroup at 2-flexes, which is also known as sextactic points.

The following are well known:

**Lemma 1.1.** On a smooth plane curve of degree $d \geq 4$, the canonical series is cut out by the system of all curves of degree $d - 3$.

**Lemma 1.2** ([5], Bertini’s theorem). The generic element of a linear system is smooth away from the base locus of the system.

**Lemma 1.3** ([2], Bezout’s theorem). Let $C_m$ and $C_n$ be plane curves of degree $m$ and $n$, respectively. If they have no common component, then we have

$$\sum_{P \in C_m \cap C_n} I(C_m \cap C_n) = mn.$$ 

**Lemma 1.4** ([4], Namba’s lemma). Let $C_1$, $C_2$ and $C$ be three plane curves, and let $P$ be a smooth point on $C$. If $I(C \cap C_1, P) \geq m$ and $I(C \cap C_2, P) \geq m$, then $I(C_1 \cap C_2, P) \geq m$.

**Corollary 1.5.** Let $C$ be a plane curve and $P$ a smooth point on $C$. Let $C_1$ and $C_2$ be plane curves defined by the polynomial $h_1$ and $h_2$, respectively. If $\min\{I(C \cap C_1, P), I(C \cap C_2, P)\} > (\deg h_1)(\deg h_2)$ and $h_2$ is irreducible, then $h_1$ is a multiple of $h_2$.

**Proof.** By Namba’s lemma, $I(C_1 \cap C_2, P) \geq \min\{I(C \cap C_1, P), I(C \cap C_2, P)\} > (\deg h_1)(\deg h_2)$. By Bezout’s theorem, $C_1$ and $C_2$ have a common component. Since $C_2$ is irreducible, $C_2$ is the common component of them.

For a point $P$ on a smooth curve $C$ of genus $g$, $P$ is a Weierstrass point if the gap sequence $G_P = \{n \in \mathbb{N}_0 \mid \text{there exists a canonical divisor } K \text{ with } I(C \cap K, P) = n - 1\}$ is different from $\{1, 2 \rightarrow g\}([1])$. We call the sequence $\{I(C \cap K, P) \mid K \text{ is a canonical divisor of } C\}$ as an order sequence of canonical divisors at $P$. Thus $P$ is a Weierstrass point if the order sequence of canonical divisors at $P$ is not $\{0, 1 \rightarrow g - 1\}$. Recall that there are only finite number of Weierstrass points on $C$, which means that the order sequence of canonical divisors at a point is exactly $\{0, 1 \rightarrow g - 1\}$ except for a finite number of points, i.e., Weierstrass points. For a
smooth plane curve $C$ of degree $d$, by Lemma 1.1, the order sequence of canonical divisors at $P$ is the set \( \{ I(C \cap f_{d-3}, P) \mid f_{d-3} \text{ is a polynomial of degree } d - 3 \} \).

2. A 2-flex which is a Weierstrass Point

Let $C$ be a smooth plane curve of degree $d \geq 4$. For each natural number $1 \leq k \leq d - 1$, the number $i_k := i_k(P)$ means the number $I(C \cap F_k, P)$ where $F_k$ is the osculating curve of degree $k$ at $P$ to $C$.

**Lemma 2.1.** Let $C$ be a smooth plane curve of degree $d \geq 4$ and $P$ a 2-flex on $C$. Then $i_1 = 2$ and $i_2 > 5$.

**Proof.** Since $P$ is a 2-flex, the osculating conic $F_2$ at $P$ to $C$ is irreducible and $i_2 = I(C \cap F_2, P) > 5$. Let $\ell_2$ be the tangent line to $C$ at $P$. Then $I(C \cap \ell_2, P) \geq 2$. If $I(C \cap \ell_2, P) \geq 3$, then $I(F_2 \cap \ell_2, P) \geq 3 > (\deg F_2)(\deg \ell_2)$, $F_2$ contains $\ell_2$ as a component, which is a contradiction since $F_2$ is an irreducible conic. Thus $i_1 = I(C \cap \ell_2, P) = 2$. \qed

We are interested in a 2-flex point $P$ which is a Weierstrass point.

If $d = 4$, then the genus of $C$ is 3 and the lines cut out the canonical series whose order sequence at $P$ is $\{0, 1, 2\}$. Thus $P$ is not a Weierstrass point. Thus we consider only $d \geq 5$.

**Remark 2.2.** If $d \geq 5$ and $m = 2$ then $F_2$ is unique. If $F_2$ and $G_2$ are two different osculating conics to $C$ at $P$ then $\min\{I(C \cap F_2, P), I(C \cap G_2, P)\} > 5$ so $I(F_2 \cap G_2, P) > 5$ by Nambs’s lemma. But $I(F_2 \cap G_2, P) \leq 4$ by Bertini’s theorem, which is a contradiction.

**Theorem 2.3.** Let $C$ be a smooth plane curve of degree $d \geq 5$ and $P$ a 2-flex on $C$. If $i_2(P) = I(C \cap F_2, P) \geq 2\lfloor \frac{d}{2} \rfloor + 2$ for an irreducible conic $F_2$, then $P$ is a Weierstrass point of $C$.

**Proof.** It suffices to show that there exists a polynomial $f_{d-3}$ such that $I(C \cap f_{d-3}, P) \geq g = \frac{(d-1)(d-2)}{2}$.

When $d$ is odd, we let $d = 2k + 1$, $k \geq 2$. Since $d - 3 = 2(k - 1)$, the degree of $F_2^{k-1}$ is a polynomial of degree $d - 3$. We have

\[
I(C \cap F_2^{k-1}, P) \geq (k - 1)(2\lfloor \frac{d}{2} \rfloor + 2) = \frac{d - 3}{2}(d + 1) \geq g.
\]
When \( d \) is even, we let \( d = 2k \), \( k \geq 3 \). Since \( d - 3 = 2(k - 2) + 1 \), the degree of \( F_2^{k-2} \ell_2 \) is a polynomial of degree \( d - 3 \), where \( \ell_2 \) is the tangent line to \( C \) at \( P \). We have
\[
I(C \cap F_2^{k-2} \ell_2, P) \geq (k - 2)(2 \lfloor \frac{d}{2} \rfloor + 2) + 2 = (d - 2)(d + 2) + 2 \geq g,
\]
since \( d \geq 6 \) for even \( d \).

Thus \( P \) is a Weierstrass point in both cases.

In next theorem, we give the order sequence at \( P \) when \( i_2(P) \) is a high value, i.e., \( i_2(P) \geq 2(d - 3) + 1 \).

**Theorem 2.4.** Let \( C \) be a smooth plane curve of degree \( d \geq 5 \) and \( P \) a 2-flex point on \( C \). If \( i_2(P) \geq 2(d - 3) + 1 \), then the order sequences at \( P \) is
\[
\bigcup_{0 \leq \alpha \leq \frac{d-3}{2}} \{ \alpha i_2 \rightarrow \alpha i_2 + 2(d - 2\alpha - 3) \}.
\]
Moreover, such a curve \( C \) and a point \( P \in C \) exist, indeed the following curve and the point \( P \) satisfy the conditions.

\[
C_d : \lambda_1(y-x^2) + \lambda_2(y-x^2)(x^{d-2} + y^{d-2}) + \lambda_3 y^d + \lambda_4 x^{i_2(0.5)} y^{i_2(0.5)} \text{ and } P = (0, 0).
\]

**Proof.** Note that the canonical series is cut out by the curves of degree \( d - 3 \). First, we obtain the orders at \( P \) using polynomials of the form \( F_2^a \ell_0^a \ell_1^\beta_1 \ell_2^\beta_2 \) with \( 2\alpha + \beta_0 + \beta_1 + \beta_2 = d - 3 \). Here \( \ell_2 \) is the tangent line at \( P \) to \( C \), \( \ell_0 \) is any line not passing through \( P \), and \( \ell_1 \) is any line, distinct from \( \ell_2 \), passing through \( P \). We have
\[
I(F_2^a \ell_0^a \ell_1^\beta_1 \ell_2^\beta_2 \cap C, P) = \alpha i_2 + \beta_1 + 2\beta_2.
\]
For fixed \( \alpha \), \( 0 \leq \alpha \leq \frac{d-3}{2} \), we obtain
\[
\{ I(F_2^a \ell_0^a \ell_1^\beta_1 \ell_2^\beta_2 \cap C, P) \mid \beta_0 + \beta_1 + \beta_2 = d - 3 - 2\alpha \} = \{ \alpha i_2 \rightarrow \alpha i_2 + 2(d - 3 - 2\alpha) \}.
\]

Since \( i_2(P) \geq 2(d - 3) + 1 \), we can check that
\[
\bigcup_{0 \leq \alpha \leq \frac{d-3}{2}} \{ \alpha i_2 \rightarrow \alpha i_2 + 2(d - 3 - 2\alpha) \}
\]
is the disjoint union. Hence the cardinality of it is
\[
\sum_{0 \leq \alpha \leq \frac{d-3}{2}} (2(d - 3 - 2\alpha) + 1) = \frac{(d - 1)(d - 2)}{2} = g
\]
where \( g \) is the genus of \( C \). Thus it is exactly the order sequence of the point \( P \).

Since the order sequence is completely determined by the values \( i_1 = 2 \) and \( i_2 \geq 2(d - 3) + 1 \), it suffices to find a smooth curve of degree \( d \) admitting such values. In fact, by the Bertini’s theorem, \( C_d \) is smooth for general nonzero \( \lambda_i \)’s. If we let
$F_2 = y - x^2$, then $F_2$ is the osculating conic and $I(F_2 \cap C_d, P) = i_2$. Thus $C_d$ is a desired curve. \hfill \square

3. At a 2-flex of Order of Contact $2(d - 3)$

In the Theorem 2.4, if $i_2(P) \leq 2(d - 3)$, then we check that $\bigcup_{0 \leq \alpha \leq \frac{d-3}{2}} \{ \alpha i_2 \rightarrow \alpha i_2 + 2(d - 2\alpha - 3) \}$ is not a disjoint union, by counting the elements of sets. So the number of orders at $P$ appeared in the union is less than the genus $g$ of $C$. Then we must find more orders at $P$ not appeared in the union.

In this section, we consider the case $i_2(P) = 2(d - 3)$. In this case, the cardinality of the union is exactly $g - 1$ because the last element $2(d - 3)$ of the first set and the first element $i_2$ in the second set are coincide. Thus we must find one more order at $P$.

For $d = 5$, $2(d - 3) = 4$ cannot be $i_2$ since $i_2 > 5$.

For $d = 6$, we have $i_2 = 2(d - 3) = 6$. From [3], we obtain the order sequences $\{0 \rightarrow 8\} \cup \{r\}$ with $r \in \{10 \rightarrow 16\} \cup \{18\}$. Here $r = i_3$. Moreover, they proved that $i_3$ cannot be 17.

So we deal with the cases for $d = 7, 8$ and 9 in this section.

3.1. On a curve of degree 7 Let $d = 7$ and $i_2 = 2(d - 3) = 8$. In this case the orders determined by lines and the power of osculating conic are

$$\{0 \rightarrow 8\} \cup \{8 \rightarrow 12\} \cup \{16\} = \{0 \rightarrow 12\} \cup \{16\}$$

and its cardinality is $g - 1 = 14$. Hence we need to find one more order.

**Lemma 3.1.** We have $i_1 = 2$, $i_3 = 10$, and $i_4 \geq 16$.

**Proof.** By Lemma 2.1, $i_1 = 2$.

Let $\ell_2$ be the tangent line to $C$ at $P$. Since $I(C \cap F_2 \ell_2, P) = 10$, we have $i_3 \geq 10$. Suppose that $i_3 > 10$ and let $f_3$ be a cubic such that $I(C \cap f_3, P) = i_3 > 10$. By Corollary 1.5, $f_3 = F_2 \ell$ for some line $\ell$. Then $I(C \cap f_3, P) \leq 10$ which is a contradiction. Thus $i_3 = 10$.

Since $I(C \cap F_2^2, P) = 16$, we have $i_4 \geq 16$. \hfill \square

**Remark 3.2.** Let $f_4$ be a quartic such that $i_4 = I(C \cap f_4, P)$. By Namba’s lemma, we have $I(f_4 \cap F_2, P) = 8 = (\deg f_4)(\deg F_2)$. Here we can not apply Corollary 1.5, i.e., we can not say that $F_2$ is a component of $f_4$. 

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Lemma 3.3. The order sequence at \( P \) is \( \{0 \to 12\} \cup \{16\} \cup \{r\} \) for some \( r \in \{13, 14, 15\} \cup \{17 \to 28\} \). Moreover, such \( r \) is attained by an irreducible polynomial of degree 4.

Proof. Since the degree of canonical divisor is \( 2g - 2 = 28 \), any order of canonical divisor at \( P \) is in the set \( \{0 \to 28\} \). Thus one more order \( r \) is an element in the set \( \{13, 14, 15\} \cup \{17 \to 28\} \). Let the divisor \( rP \) be cut out by a curve \( f \). If \( \deg f \leq 3 \), then \( I(f \cap C, P) = r \geq 10 \). Namba’s lemma implies that \( I(f \cap F_2, P) \geq 8 \), which is bigger than \((\deg f)(\deg F_2) \leq 6 \). By Bezout’s theorem, and that \( f \) is a multiple of \( F_2 \), say \( f = F_2h \) with \( \deg h \leq 1 \). Then \( I(h \cap C, P) = r = 8 \geq 5 \), which is a contradiction. Thus the degree of \( f \) is 4 since any canonical divisor is cut out by a curve of degree 4. We can also prove that \( f \) is irreducible. Indeed, if \( f \) is reducible, then \( f \) is factored into polynomials of lower degree than 4. If \( f \) is factored into four lines, then \( 0 \leq I(C \cap f, P) \leq 8 \). If \( f \) is factored into one line and a cubic, then \( I(C \cap f, P) \leq 12 \). If \( f \) is factored into two lines and a conic, then \( I(C \cap f, P) \leq 12 \). If \( f \) is factored into two conics, then \( I(C \cap f, P) = 16 \) for the case \( f = F_2^2 \), and \( I(C \cap f, P) \leq 12 \) for the case \( f = F_2f_2 \), where \( f_2 \) is a conic different from \( F_2 \). Note that \( I(C \cap f_2, P) \leq 4 \) because \( I(F_2 \cap f_2, P) \leq 4 = (\deg F_2)(\deg f_2) \) by Bezout’s theorem.

Lemma 3.4. There is no smooth plane curve of degree 7 with a point at which the order sequence is \( \{0 \to 12\} \cup \{16\} \cup \{r\} \) with \( 27 \leq r \leq 28 \).

Proof. Suppose that such a curve exists. Then the Weierstrass gap sequence \( G(P) = \{1 \to 13\} \cup \{17\} \cup \{r + 1\} \) and \( H(P) = \{0\} \cup \{14, 15, 16\} \cup \{18 \to \} \setminus \{r + 1\} \). Since 14 and 15 are elements in \( H(P) \) and \( H(P) \) is a semigroup, 28 and 29 are elements of \( H(P) \). Thus \( r + 1 \neq 28, 29 \), which contradicts the assumption.

Lemma 3.5. There exists a smooth plane curve of degree 7 with a point at which the order sequence is \( \{0 \to 12\} \cup \{16\} \cup \{r\} \) with \( 24 \leq r \leq 26 \).

Proof. Let \( P = (0, 0) \) and
\[
\begin{align*}
  f_2 &= y - x^2 \\
  f_4 &= \lambda_1 f_2 + \lambda_2 x^2 f_2 + \lambda_3 y^4 \\
  f_7 &= \mu_1 f_4 + \mu_2 f_4 (x^3 + y^3) + \mu_3 (f_2)^3 x^{r-24-2\lfloor \frac{r-24}{2} \rfloor} y^{\lfloor \frac{r-24}{2} \rfloor},
\end{align*}
\]
and let \( C_7 \) be the curve with the equation \( f_7 \).

Then, for general nonzero \( \lambda_i \)'s and \( \mu_i \)'s we can check the following:
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(1) $C_7$ is a smooth curve of degree 7, since the base locus of the system is only one point $\{(0, 0, 1)\}$ at which a generic member is smooth. By Bertini’s theorem, $C_7$ is smooth.

(2) $I(f_2 \cap C_7, P) = 8$, since $I(f_2 \cap C_7, P) = I(f_2 \cap f_4, P) = I(f_2 \cap y^4, P)$.

(3) $I(T_P C_7 \cap C_7, P) = 2$, since $I(T_P C \cap C_7, P) = I(y \cap C_7, P)$.

(4) $I(f_4 \cap C_7, P) = r$, since $I(f_4 \cap C_7, P) = I(f_4 \cap (f_2)^3 x^{r-24-2(16-2)} y^{r-24})$

$= 3I(f_4 \cap f_2, P) + (r - 24 - 2(16-2))I(f_4 \cap x, P) + (\lfloor \frac{r - 24}{2} \rfloor)I(f_4 \cap y, P)$

$= 12I(y \cap f_2, P) + (r - 24 - 2(16-2)) + 2(\lfloor \frac{r - 24}{2} \rfloor)$

Thus the $C_7$ is a desired curve.

Lemma 3.6. There exists a smooth plane curve of degree 7 with a point at which the order sequence is $\{0 \rightarrow 12\} \cup \{16\} \cup \{r\}$ with $17 \leq r \leq 22$.

Proof. Let $P = (0, 0)$ and

$f_2 = y - x^2$
$f_4 = \lambda_1 f_2 + \lambda_2 x^2 f_2 + \lambda_3 y^4$
$f_7 = \mu_1 f_4 + \mu_2 f_4 (x^3 + y^3) + \mu_3 (f_2)^2 x^{r-16-2(16-2-2)} y^{r-24}$

and let $C_7$ be the curve with the equation $f_7$.

Then, for general nonzero $\lambda_i$’s and $\mu_i$’s we can check the following:

(1) $C_7$ is a smooth curve of degree 7.

(2) $I(f_2 \cap C_7, P) = 8$.

(3) $I(T_P C_7 \cap C_7, P) = 2$.

(4) $I(f_4 \cap C_7, P) = r$.

Thus the $C_7$ is a desired curve.

Lemma 3.7. There exists a smooth plane curve of degree 7 with a point at which the order sequence is $\{0 \rightarrow 12\} \cup \{16\} \cup \{r\}$ with $13 \leq r \leq 15$.

Proof. Let $P = (0, 0)$ and

$f_2 = y - x^2$
$f_4 = \lambda_1 f_2 + \lambda_2 x^2 f_2 + \lambda_3 y^4$
$f_7 = \mu_1 f_4 + \mu_2 f_4 (x^3 + y^3) + \mu_3 (f_2)^2 x^{r-8-2(16-2-2)} y^{r-24}$

and let $C_7$ be the curve with the equation $f_7$.

Then, for general nonzero $\lambda_i$’s and $\mu_i$’s we can check the following:

(1) $C_7$ is a smooth curve of degree 7.

(2) $I(f_2 \cap C_7, P) = 8$.

(3) $I(T_P C_7 \cap C_7, P) = 2$.

(4) $I(f_4 \cap C_7, P) = r$.

Thus the $C_7$ is a desired curve.
and let $C_7$ be the curve with the equation $f_7$.

Then, for general nonzero $\lambda_i$’s and $\mu_i$’s we can check the following:

1. $C_7$ is a smooth curve of degree 7.
2. $I(f_2 \cap C_7, P) = 8$.
3. $I(T_P C_7 \cap C_7, P) = 2$.
4. $I(f_4 \cap C_7, P) = r$.

Thus the $C_7$ is a desired curve.

**Theorem 3.8.** Let $P$ be a 2-flex of order of contact 8 on a smooth plane curve of degree 7. Then the order sequence at $P$ is one of $\{0 \to 12\} \cup \{16\} \cup \{r\}$ for $r \in \{13 \to 15\} \cup \{17 \to 26\}$.

Also there exists a smooth plane curve of degree 7 with a 2-flex point $P$ at which the order sequence is $\{0 \to 12\} \cup \{16\} \cup \{r\}$ for $r \in \{13 \to 15\} \cup \{17 \to 22\} \cup \{24 \to 26\}$.

**Remark 3.9.** In the set $\{13, 14, 15\} \cup \{17 \to 28\}$ of all candidates for $r$, we proved or disproved the existence of a smooth curve of degree 7 corresponding to each integer except for the number 23.

**3.2. On a curve of degree 8** In this case the orders determined by lines and the power of osculating conic are

\[ \{0 \to 10\} \cup \{10 \to 16\} \cup \{20 \to 22\} = \{0 \to 16\} \cup \{20 \to 22\} \]

and its cardinality is $g - 1 = 20$. Hence we need to find one more order.

Using Bezout’s theorem and Namba’s Lemma, we have $i_1 = 2$, $i_2 = 10$, $i_3 = 12$, $i_4 = 20$, and $i_5 \geq 22$.

Since the degree of canonical divisor is $2g - 2 = 40$, any order of canonical divisor at $P$ is in the set $\{0 \to 40\}$. Thus one more order $r$ is an element in the set $\{17, 18, 19\} \cup \{23 \to 40\}$. Let the divisor $rP$ be cut out by a curve $f$. If $\deg f \leq 4$, then $I(f \cap C, P) = r \geq 17$. Namba’s lemma implies that $I(f \cap F_2, P) \geq 10$, which is bigger than $(\deg f)(\deg F_2) \leq 8$. By Bezout’s theorem, and that $f$ is a multiple of $F_2$, say $f = F_2h$ with $\deg h \leq 2$. Then $I(h \cap C, P) = r - 10 \geq 7$ and hence $h = F_2$ and $r = 20$ by Bezout’s theorem, which is a contradiction to the choice of $r$. Thus the degree of $f$ is 5 since any canonical divisor is cut out by a curve of degree 5. We can also prove that $f$ is irreducible in a similar way.
Lemma 3.10. There exists a smooth plane curve of degree 8 with a point at which the order sequence is \( \{0 \rightarrow 16\} \cup \{20 \rightarrow 22\} \cup \{40\} \), i.e., \( r = 40 \).

Proof. Let \( P = (0,0) \) and
\[
\begin{align*}
f_2 &= y - x^2 \\
f_5 &= \lambda_1(y - x^2) + \lambda_2 x^3(y - x^2) + \lambda_3 y^5 \\
f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3 + y^3) + \mu_3 (f_2)^4,
\end{align*}
\]
and let \( C_8 \) be the curve defined by the equation \( f_8 \).

Then, for general nonzero \( \lambda_i \)'s and \( \mu_i \)'s we can check the following:

1. \( C_8 \) is a smooth curve of degree 8.
2. \( I(f_2 \setminus C_8; P) = 10 \).
3. \( I(T_P C_8 \setminus C_8; P) = 2 \).
4. \( I(f_5 \setminus C_8; P) = 40 \).

Thus the \( C_8 \) is a desired curve.

Lemma 3.11. There is no smooth plane curve of degree 8 with a point at which the order sequence is \( \{0 \rightarrow 16\} \cup \{20 \rightarrow 22\} \cup \{r\} \) for \( 35 \leq r \leq 39 \).

Proof. Suppose that such a curve exists. Then the Weierstrass gap sequence \( G(P) = \{1 \rightarrow 17\} \cup \{21 \rightarrow 23\} \cup \{r+1\} \) and \( H(P) = \{0\} \cup \{18, 19, 20\} \cup \{24 \rightarrow\} - \{r+1\} \). Since 18, 19 and 20 are elements in \( H(P) \) and \( H(P) \) is a semigroup, every integers from 36 to 40 are elements of \( H(P) \). However, since the number \( r + 1 \) belongs to this set, it is a contradiction.

Lemma 3.12. There exists a smooth plane curve of degree 8 with a point at which the order sequence is \( \{0 \rightarrow 16\} \cup \{20 \rightarrow 22\} \cup \{r\} \) for \( 30 \leq r \leq 34 \).

Proof. Let \( P = (0,0) \) and
\[
\begin{align*}
f_2 &= y - x^2 \\
f_5 &= \lambda_1(y - x^2) + \lambda_2 x^3(y - x^2) + \lambda_3 y^5 \\
f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3 + y^3) + \mu_3 (f_2)^3 x^{r-30-2\frac{r-20}{2}} y^{\frac{r-20}{2}},
\end{align*}
\]
and let \( C_8 \) be the curve with the equation \( f_8 \).

Then, for general nonzero \( \lambda_i \)'s and \( \mu_i \)'s we can check the following:

1. \( C_8 \) is a smooth curve of degree 8.
2. \( I(f_2 \setminus C_8; P) = 10 \).
Thus the $C_8$ is a desired curve. \hfill \Box

**Lemma 3.13.** There exists a smooth plane curve of degree 8 with a point at which the order sequence is $\{0 \to 16\} \cup \{20 \to 22\} \cup \{r\}$ for $23 \leq r \leq 28$.

**Proof.** Let $P = (0,0)$ and
\[
\begin{align*}
f_2 &= y - x^2 \\
f_5 &= \lambda_1(y - x^2) + \lambda_2 x^3(y - x^2) + \lambda_3 y^5 \\
f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3 + y^3) + \mu_3 f_2(x^{r-20-2}\lfloor \frac{r-10}{2} \rfloor y^{\lfloor \frac{r-10}{2} \rfloor}),
\end{align*}
\]
and let $C_8$ be the curve with the equation $f_8$.

Then, for general nonzero $\lambda_i$’s and $\mu_i$’s we can check the following:

1. $C_8$ is a smooth curve of degree 8.
2. $I(f_2 \cap C_8, P) = 10$.
3. $I(T_P C_8 \cap C_8, P) = 2$.
4. $I(f_5 \cap C_8, P) = r$.

Thus the $C_8$ is a desired curve. \hfill \Box

**Lemma 3.14.** There exists a smooth plane curve of degree 8 with a point at which the order sequence is $\{0 \to 16\} \cup \{20 \to 22\} \cup \{r\}$ for $17 \leq r \leq 19$.

**Proof.** Let $P = (0,0)$ and
\[
\begin{align*}
f_2 &= y - x^2 \\
f_5 &= \lambda_1(y - x^2) + \lambda_2 x^3(y - x^2) + \lambda_3 y^5 \\
f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3 + y^3) + \mu_3 f_2(x^{r-10-2}\lfloor \frac{r-10}{2} \rfloor y^{\lfloor \frac{r-10}{2} \rfloor}),
\end{align*}
\]
and let $C_8$ be the curve with the equation $f_8$.

Then, for general nonzero $\lambda_i$’s and $\mu_i$’s we can check the following:

1. $C_8$ is a smooth curve of degree 8.
2. $I(f_2 \cap C_8, P) = 10$.
3. $I(T_P C_8 \cap C_8, P) = 2$.
4. $I(f_5 \cap C_8, P) = r$.

Thus the $C_8$ is a desired curve. \hfill \Box
Theorem 3.15. Let $P$ be a 2-flex of order of contact 10 on a smooth plane curve of degree 8. Then the order sequence at $P$ is one of $\{0 \rightarrow 16\} \cup \{20 \rightarrow 22\} \cup \{r\}$ for $r \in \{17 \rightarrow 19\} \cup \{23 \rightarrow 34\} \cup \{40\}$.

Also there exists a smooth plane curve of degree 8 with a 2-flex point $P$ at which the order sequence is $\{0 \rightarrow 16\} \cup \{20 \rightarrow 22\} \cup \{r\}$ for $r \geq 17$ or $\{23 \rightarrow 28\} \cup \{30 \rightarrow 34\} \cup \{40\}$.

Remark 3.16. In the set $\{17, 18, 19\} \cup \{23 \rightarrow 40\}$ of all candidates for $r$, we proved or disproved the existence of a smooth curve of degree 8 corresponding to each integer except for the number 29.

3.3. On a curve of degree 9 In this case the orders determined by lines and the power of osculating conic are

$$\{0 \rightarrow 12\} \cup \{12 \rightarrow 20\} \cup \{24 \rightarrow 28\} \cup \{36\}$$

and its cardinality is $g - 1 = 27$. Hence we need to find one more order.

Using Bezout’s theorem and Namba’s Lemma, we have $i_1 = 2$, $i_2 = 12$, $i_3 = 14$, $i_4 = 24$, $i_5 = 26$, and $i_6 \geq 36$.

One more order $r$ is an element in the set

$$\{21, 22, 23\} \cup \{29 \rightarrow 35\} \cup \{37 \rightarrow 54\}.$$

Let the divisor $rP$ be cut out by a curve $f$. If $\deg f \leq 5$, then $I(f \cap C, P) = r \geq 21$. Namba’s lemma implies that $I(f \cap F_2, P) \geq 12$, and that $f$ is a multiple of $F_2$ by Bezout’s theorem. Let $f = hF_2$, $\deg h \leq 3$. Then we have $I(h \cap C, P) = r - 12 \geq 9$. By Namba’s theorem again, we conclude that $h$ is multiple of $F_2$. Thus $f = F_2^2$ or $f = F_2^2 \ell$ where $\ell$ is a line. Hence $I(f \cap C, P) = 24$, 25 or 26. Then this is a contradiction. Thus the degree of $f$ is 6. We can also prove that $f$ is irreducible in a similar way.

Lemma 3.17. There is no smooth plane curve of degree 9 with a point at which the order sequence is $\{0 \rightarrow 20\} \cup \{24 \rightarrow 28\} \cup \{36\} \cup \{r\}$ for $r \in \{43 \rightarrow 47\} \cup \{51 \rightarrow 54\}$.

Proof. Suppose that such a curve exists. Then the Weierstrass gap sequence $G(P) = \{1 \rightarrow 21\} \cup \{25 \rightarrow 29\} \cup \{37\} \cup \{r + 1\}$ and $H(P) = \{0\} \cup \{22, 23, 24\} \cup \{30 \rightarrow 36\} \cup \{38 \rightarrow \} \cup \{r + 1\}$. Since $\{22, 23, 24, 30 \rightarrow 36\}$ is a subset of $H(P)$ and $H(P)$
is a semigroup, \( \{44 \rightarrow 48, 52 \rightarrow 55\} \) is a subset of \( H(P) \). However, since the gap \( r + 1 \) belongs to this set, it is a contradiction.

**Lemma 3.18.** There exists a smooth plane curve of degree 9 with a point at which the order sequence is \( \{0 \rightarrow 20\} \cup \{24 \rightarrow 28\} \cup \{36\} \cup \{r\} \) for \( r \in \{21 \rightarrow 23\} \cup \{29 \rightarrow 34\} \cup \{37 \rightarrow 42\} \cup \{48 \rightarrow 50\} \).

**Proof.** Let \( P = (0, 0) \) and

\[
\begin{align*}
  f_2 &= y - x^2 \\
  f_6 &= \lambda_1(y - x^2) + \lambda_2x^4(y - x^2) + \lambda_3y^6 \\
  f_9 &= \mu_1f_6 + \mu_2f_6(x^3 + y^3) + \mu_3(f_2)^4 \\
  f_{9,r} &= \begin{cases} 
  \nu_1f_9 + \nu_2(f_2)^4x^{r-48-2[\frac{r}{2}]}y^{\frac{r-48}{2}}, & \text{if } 48 \leq r \leq 50 \\
  \nu_1f_9 + \nu_2(f_2)^3x^{r-36-2[\frac{r}{2}]}y^{\frac{r-36}{2}}, & \text{if } 37 \leq r \leq 42 \\
  \nu_1f_9 + \nu_2(f_2)^2x^{r-24-2[\frac{r}{2}]}y^{\frac{r-24}{2}}, & \text{if } 29 \leq r \leq 34 \\
  \nu_1f_9 + \nu_2(f_2)x^{r-12-2[\frac{r}{2}]}y^{\frac{r-12}{2}}, & \text{if } 21 \leq r \leq 23 
  \end{cases}
\end{align*}
\]

and let \( C_{9,r} \) be the curve with the equation \( f_{9,r} \).

Then, for general nonzero \( \lambda_i \)'s, \( \mu_i \)'s and \( \nu_i \)'s we can check the following:

1. \( C_{9,r} \) is a smooth curve of degree 9.
2. \( I(f_2 \cap C_{9,r}, P) = 12 \).
3. \( I(T_PC_{9,r} \cap C_{9,r}, P) = 2 \).
4. \( I(f_5 \cap C_{9,r}, P) = r \).

Thus the \( C_{9,r} \) is a desired curve.

**Theorem 3.19.** Let \( P \) be a 2-flex of order of contact 12 on a smooth plane curve of degree 10. Then the order sequence at \( P \) is one of \( \{0 \rightarrow 20\} \cup \{24 \rightarrow 28\} \cup \{36\} \cup \{r\} \) for \( r \in \{21 \rightarrow 23\} \cup \{29 \rightarrow 34\} \cup \{37 \rightarrow 42\} \cup \{48 \rightarrow 50\} \).

Also there exists a smooth plane curve of degree 10 with a 2-flex point \( P \) at which the order sequence is \( \{0 \rightarrow 20\} \cup \{24 \rightarrow 28\} \cup \{36\} \cup \{r\} \) for \( r \in \{21 \rightarrow 23\} \cup \{29 \rightarrow 34\} \cup \{37 \rightarrow 42\} \cup \{48 \rightarrow 50\} \).

**Remark 3.20.** In the set \( \{21, 22, 23\} \cup \{29 \rightarrow 35\} \cup \{37 \rightarrow 54\} \) of all candidates for \( r \), we proved or disproved the existence of a smooth curve of degree 9 corresponding to each integer except for the number 35.
A WEIERSTRASS SEMIGROUP AT A GENERALIZED FLEX ON A PLANE CURVE

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