EVALUATIONS OF THE ROGERS-RAMANUJAN CONTINUED FRACTION BY THETA-FUNCTION IDENTITIES

DAE HYUN PAEK

ABSTRACT. In this paper, we use theta-function identities involving parameters $l_{5,n}$, $l'_{5,n}$, and $l'_{4,n}$ to evaluate the Rogers-Ramanujan continued fractions $R(e^{-2\pi\sqrt{5/20}})$ and $S(e^{-\pi\sqrt{5/3}})$ for some positive rational numbers $n$.

1. INTRODUCTION

The Rogers-Ramanujan continued fractions $R(q)$ and $S(q)$, for $|q| < 1$, are defined by

$$R(q) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}}$$

and

$$S(q) = -R(-q).$$

In his first two letters to G. H. Hardy, S. Ramanujan asserted that

$$R(e^{-2\pi}) = \sqrt{\frac{5 + \sqrt{5}}{2} - \frac{\sqrt{5} + 1}{2}},$$

$$S(e^{-\pi}) = \sqrt{\frac{5 - \sqrt{5}}{2} - \frac{\sqrt{5} - 1}{2}},$$

and

$$R(e^{-2\pi\sqrt{5/20}}) = \frac{\sqrt{5}}{1 + \left(\frac{5^{3/4}}{2} \left(\frac{\sqrt{5} - 1}{2}\right)^{5/2} - 1\right)^{1/5}} - \frac{1 + \sqrt{5}}{2}.$$
See [4, 5] for more details about the proofs and historical remarks of these evaluations. Further explicit evaluations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for some positive rational numbers $n$ were given in [4, 5, 12, 13]. Ramanathan [8, 9, 10] evaluated $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 2, 5, \frac{7}{5}, \frac{17}{5}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 1, 5, \frac{3}{5}, \frac{3}{5}, \frac{22}{5}$. Ramanathan [11] also evaluated $R(e^{-\pi\sqrt{n}})$ for $n = \frac{2}{5}, \frac{7}{5}$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{2}{5}, \frac{22}{5}$ by using Kronecker’s limit formula. Berndt and Chan [4] established values of $R(e^{-2\pi\sqrt{n}})$ for $n = 4, 9, 16, 64$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5}, \frac{7}{5}, \frac{1}{15}, \frac{1}{35}$ by employing an eta-function identity. Meanwhile, Chan [6] evaluated $S(e^{-\pi\sqrt{3}})$ by using a modular equation. Berndt, Chan, and Zhang [5] derived formulas for the explicit evaluations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for positive rational numbers $n$ in terms of Ramanujan-Weber class invariants. In particular, they determined the values of $R(e^{-6\pi})$ and $S(e^{-\pi\sqrt{n}})$ for $n = 9, \frac{5}{3}, \frac{9}{5}, \frac{11}{5}, \frac{15}{7}, \frac{36}{5}, \frac{39}{5}, \frac{53}{5}, \frac{60}{5}, \frac{101}{5}$.

Yi [13] established some formulas and found values of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ by using modular equations of degrees 5 or 25. In addition, Yi [13] employed modular equations of degree 1, 5, 25 and 5p for any positive integer p so that she was able to compute $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 2, 3, 4, 9, 16, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{2}{5}, \frac{3}{5}, \frac{3}{7}, \frac{2}{3}, \frac{7}{5}, \frac{5}{7}, \frac{9}{5}, \frac{5}{9}, \frac{2}{5}, \frac{9}{2}, \frac{1}{15}, \frac{1}{25}, \frac{1}{17}, \frac{1}{16}, \frac{1}{35}, \frac{1}{25}, \frac{1}{35}, \frac{1}{45}, \frac{1}{35}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 1, 3, 9, 27, \frac{1}{3}, \frac{1}{15}, \frac{1}{7}, \frac{2}{9}, \frac{9}{1}, \frac{1}{15}, \frac{1}{25}, \frac{1}{15}, \frac{1}{35}, \frac{1}{35}$. Recently, Paek and Yi [7] evaluated $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{4}{5}, \frac{16}{5}$ by using modular equations of degree 5. Saikia [12] evaluated explicit values of $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 4, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 1, \frac{1}{3}, \frac{2}{5}, \frac{9}{5}$ by using parametrization of Ramanujan’s theta-functions.

In this paper, we use theta-function identities involving parameters $l_{5, n}, l'_{5, n}$, and $l'_{5, 4m}$ to show how to evaluate $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{3}{5}, \frac{9}{5}, \frac{15}{4}, \frac{1}{4}, \frac{45}{4}$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5}, \frac{9}{5}, \frac{15}{4}, \frac{1}{4}, \frac{45}{4}$, where $m$ is any positive integer. Furthermore, we establish explicit evaluations of these for $m = 1, 2, 3$.

Ramanujan’s theta-function $\psi$ is defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

where $|q| < 1$.

Recall the parameters $l_{k, n}$ and $l'_{k, n}$ for the theta-function $\psi$ introduced in [14]. For any positive real numbers $k$ and $n$, define $l_{k, n}$ and $l'_{k, n}$ by

$$l_{k, n} = \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)}$$

and

$$l'_{k, n} = \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)},$$

where $q = e^{-\pi\sqrt{n}/k}$. 
We employ the following formulas in [12, Theorem 3.1] to evaluate $R(e^{-2\pi \sqrt{n/20}})$ and $S(e^{-\pi \sqrt{n/5}})$ in terms of $l'_5,n$ and $l_{5,n}$, respectively.

(1.3)  \[ \frac{1}{R^5(e^{-2\pi \sqrt{n/20}})} - 11 - R^5(e^{-2\pi \sqrt{n/20}}) = \sqrt{5} l'^2_{5,n} \left( \frac{\sqrt{5} l'^2_{5,n} - 5}{\sqrt{5} l'^2_{5,n} - 1} \right)^2 \]

and

(1.4)  \[ \frac{1}{S^5(e^{-\pi \sqrt{n/5}})} + 11 - S^5(e^{-\pi \sqrt{n/5}}) = \sqrt{5} l^2_{5,n} \left( \frac{\sqrt{5} l^2_{5,n} + 5}{\sqrt{5} l^2_{5,n} + 1} \right)^2. \]

From (1.3) and (1.4), we have

(1.5)  \[ R^5(e^{-2\pi \sqrt{n/20}}) = \sqrt{a^2 + 1} - a, \]

where

\[ 2a = 11 + \sqrt{5} l'^2_{5,n} \left( \frac{\sqrt{5} l'^2_{5,n} - 5}{\sqrt{5} l'^2_{5,n} - 1} \right)^2 \]

and

(1.6)  \[ S^5(e^{-\pi \sqrt{n/5}}) = \sqrt{b^2 + 1} - b, \]

where

\[ 2b = -11 + \sqrt{5} l^2_{5,n} \left( \frac{\sqrt{5} l^2_{5,n} + 5}{\sqrt{5} l^2_{5,n} + 1} \right)^2. \]

Consequently, in order to compute $R(e^{-2\pi \sqrt{n/20}})$ and $S(e^{-\pi \sqrt{n/5}})$, it suffices to evaluate $l'^2_{5,n}$ and $l^2_{5,n}$, respectively. Thus, in this paper, we employ theta-function identities involving $l_{5,n}$, $l'_{5,n}$, and $l'_{5,4n}$ to find some new explicit values of the Rogers-Ramanujan continued fraction.

2. Evaluations of $l_{5,n}$ and $l'_{5,n}$

In this section, we compute $l^2_{5,n}$ and $l'^2_{5,n}$ for some positive rational numbers $n$ to evaluate $R(e^{-2\pi \sqrt{n/20}})$ and $S(e^{-\pi \sqrt{n/5}})$. We begin by recalling the known values of $l_{5,3}$, $l_{5,1/3}$, $l_{5,9}$, and $l'_{5,1/9}$ in [14] which will play key roles in evaluating the Rogers-Ramanujan continued fraction later on.

**Lemma 2.1** ([14, Theorem 4.8]). We have

(i)  $l_{5,3} = \sqrt{2 + \sqrt{5}},$

(ii) $l_{5,1/3} = \sqrt{-2 + \sqrt{5}},$

(iii) $l_{5,9} = \frac{1 + \sqrt{3}}{\sqrt{5} - \sqrt{3}}.$
(iv) $l_{5,1/9} = \frac{\sqrt{5} - \sqrt{3}}{1 + \sqrt{3}}$.

Note that the values of (i) and (ii) in Lemma 2.1 were given in [14] as follow:

\[ l_{5,3} = (38 + 17\sqrt{5})^{1/6} \quad \text{and} \quad l_{5,1/3} = (-38 + 17\sqrt{5})^{1/6}. \]

We now need a couple of theta-function identities: one shows a relation between $l'_{5,n}$ and $l'_{5,4n}$ and the other shows a relation between $l'_{5,n}$ and $l_{5,n}$ for any positive rational number $n$.

Lemma 2.2 ([7, Corollary 3.8]). For every positive real number $n$, we have

\begin{equation}
\sqrt{5} \left( l_{5,n}^2 + \frac{1}{l_{5,n}^2} \right) = \left( \frac{l'_{5,n}}{l_{5,4n}} \right)^2 + \left( \frac{l'_{5,4n}}{l'_{5,n}} \right)^2 + 4
\end{equation}

Identity (2.1) follows from a modular equation of degree 5 in [7] such as $P^2 + 5 \frac{Q^2}{P^2} = \left( \frac{P}{Q} \right)^2 + \left( \frac{Q}{P} \right)^2 + 4$, where $P = \frac{\psi(q)}{q^{1/2}\psi(q^2)}$ and $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^4)}$.

Lemma 2.3 ([7, Corollary 3.12]). For every positive real number $n$, we have

\begin{equation}
\left( l_{5,n}^2 - l_{5,n}^2 - \sqrt{5} \right) \left( \frac{1}{l_{5,2n}^2} - \frac{1}{l_{5,n}^2} - \sqrt{5} \right) = 1
\end{equation}

Identity (2.2) follows from a modular equation of degree 5 in [7] such as

\[ (P^2 - Q^2 - 5) \left( \frac{1}{P^2} - \frac{1}{Q^2} - 1 \right) = 1, \]

where $P = \frac{\psi(q)}{q^{1/2}\psi(q^2)}$ and $Q = \frac{\psi(q)}{q^{1/2}\psi(q^2)}$.

We are in position to evaluate $l_{5,n}^2$ for $n = 3, \frac{3}{4}, \frac{3}{12}, \frac{3}{16}$, and $\frac{1}{48}$.

Theorem 2.4. We have

(i) $l_{5,3}^2 = \frac{1}{2} (2 + \sqrt{3}) (1 + \sqrt{5})$,
(ii) $l_{5,1/3}^2 = \frac{1}{2} (2 + \sqrt{3}) (-1 + \sqrt{5})$,
(iii) $l_{5,3/4}^2 = \frac{4 + \sqrt{2} + \sqrt{30}}{2 - \sqrt{3} + \sqrt{15}}$,
(iv) $l_{5,1/12}^2 = \frac{4 - \sqrt{2} + \sqrt{30}}{-2 + \sqrt{3} + \sqrt{15}}$,
(v) $l_{5,3/16}^2 = \frac{4(2 + \sqrt{10}) + 2\sqrt{12 + 33\sqrt{10} + \sqrt{270(13 - 4\sqrt{10})}}}{17 + 17\sqrt{2} + 9\sqrt{3} - 7\sqrt{5} + 9\sqrt{6} - 5\sqrt{10} - 3\sqrt{15} - 3\sqrt{30}}$. 

For (i), let $l = \sqrt{2} + \sqrt{5}$ in Lemma 2.1(i), we find that
\[ 2x^2 - 4(1 + \sqrt{5})x + 3 + \sqrt{5} = 0. \]
Solving the equation for $x$ and using $x > 1$, we complete the proof.

For (iii), let $n = \frac{3}{4}$ in (2.1). Putting $l_{5,3}^2 = x$ and $l_{5,3}^2 = \frac{1}{2}(2 + \sqrt{3})(1 + \sqrt{5})$ obtained in (i), we deduce that
\[ (2 - \sqrt{3} + \sqrt{10})x^2 - 8x - 2 - \sqrt{3} - \sqrt{10} = 0. \]
Solving the last equation for $x$ and using $x > 1$, we complete the proof.

The proofs of (ii), (iv), (v), and (vi) are similar to those of (i) or (iii).

**Remark 1.** Letting $n = \frac{3}{64}$ and $\frac{1}{192}$ in (2.1) and using the value $l_{5,3/16}^2$ and $l_{5,1/48}^2$ in Theorem 2.4(v) and (vi), respectively, we can also evaluate $l_{5,3/64}^2$ and $l_{5,1/192}^2$. Hence, by the same argument, we can evaluate $l_{5,3/4m}^2$ and $l_{5,1/3.4m}^2$ for every nonnegative integer $m$.

We now evaluate $l_{5,3/4}^2$ and $l_{5,1/12}^2$.

**Theorem 2.5.** We have

(i) $l_{5,3/4}^2 = \frac{4 + \sqrt{2} - 5\sqrt{3} - 2\sqrt{5} + \sqrt{15} + \sqrt{30}}{4 - 2\sqrt{3} + 2\sqrt{15}}$
\[ + \frac{\sqrt{26 - 84\sqrt{2} + 18\sqrt{3} + 63\sqrt{5} - 47\sqrt{6} - \sqrt{10} + 38\sqrt{15}}}{(2 + \sqrt{3})\sqrt{2} + 6\sqrt{5} + 4\sqrt{15}}, \]

(ii) $l_{5,1/12}^2 = \frac{4 - \sqrt{2} - 5\sqrt{3} + 2\sqrt{5} - \sqrt{15} + \sqrt{30}}{-4 + 2\sqrt{3} + 2\sqrt{15}}$
\[ + \frac{\sqrt{-26 - 84\sqrt{2} - 18\sqrt{3} + 63\sqrt{5} - 47\sqrt{6} + \sqrt{10} + 38\sqrt{15}}}{(2 + \sqrt{3})\sqrt{-2} + 6\sqrt{5} + 4\sqrt{15}}. \]

Proof. The results follow directly from (2.2) and Theorem 2.4(ii) and (iv) with the help of Mathematica.

**Remark 2.** Letting $n = \frac{3}{16}$ and $\frac{1}{38}$ in (2.2) and using the value $l_{5,3/16}^2$ and $l_{5,1/48}^2$ in Theorem 2.4(v) and (vi), respectively, we can also evaluate $l_{5,3/16}^2$ and $l_{5,1/48}^2$. By the same argument as before, we are able to evaluate $l_{5,3/4m}^2$ and $l_{5,1/3.4m}^2$ for every positive integer $m$.

We now evaluate $l_{5,n}^2$ for $n = 9, \frac{9}{4}, \frac{1}{9}$, and $\frac{1}{36}$.
Theorem 2.6. We have

(i) \[ l_{5,9}^2 = (2 + \sqrt{3})(2 + \sqrt{5} + 2\sqrt{2 + \sqrt{5}}) \],

(ii) \[ l_{5,9/4}^2 = \frac{(2 + \sqrt{3})(5 + 2\sqrt{2} + \sqrt{5} + \sqrt{6 + 6\sqrt{5}})}{2\sqrt{2}(-2 + \sqrt{5} + \sqrt{15} + 2\sqrt{2 + \sqrt{5}})} \],

(iii) \[ l_{5,1/9}^2 = (2 - \sqrt{3})(2 + \sqrt{5} + 2\sqrt{2 + \sqrt{5}}) \],

(iv) \[ l_{5,1/36}^2 = \frac{(2 - \sqrt{3})(5 + 2\sqrt{2} + \sqrt{5} - \sqrt{6 + 6\sqrt{5}})}{\sqrt{2}(-2 + \sqrt{5} - \sqrt{15} + 2\sqrt{2 + \sqrt{5}})} \].

Proof. For (i), let \( n = 3 \) in (2.2). Putting \( l_{5;9}^2 \) and \( l_{5;9}^1 \) in Lemma 2.1(iii), we find that

\[ x^2 - 2(2 + \sqrt{5})x + 2 + \sqrt{3} = 0. \]

Solving the equation for \( x \) and using \( x > 1 \), we complete the proof.

For (ii), let \( n = \frac{9}{4} \) in (2.1). Putting \( l_{5;9/4}^2 = x \) and the value \( l_{5;9}^2 \) obtained in (i), we deduce that

\[ \frac{-2 + \sqrt{5} + \sqrt{15} + 2\sqrt{2 + \sqrt{5}}}{2 + \sqrt{3}} x^2 - 4x - \frac{2 - \sqrt{5} + \sqrt{15} + 2\sqrt{2 + \sqrt{5}}}{2 - \sqrt{3}} = 0. \]

Solve the last equation for \( x \) with the help of Mathematica and use \( x > 1 \) to complete the proof.

The proofs of (iii) and (iv) are similar to those of (i) and (ii), respectively. \( \square \)

Remark 3. Letting \( n = \frac{9}{16} \) and \( \frac{1}{144} \) in (2.1) and using the value of \( l_{5;9/4}^2 \) and \( l_{5;1/36}^2 \) in Theorem 2.6(ii) and (iv), respectively, we can also evaluate \( l_{5;9/16}^2 \) and \( l_{5;1/144}^2 \).

Hence we can evaluate \( l_{5;9/4}^2 \) and \( l_{5;1/36}^2 \) for every nonnegative integer \( m \).

We end this section by evaluating \( l_{5;9/4}^2 \) and \( l_{5;1/36}^2 \).

Theorem 2.7. We have

(i) \[ l_{5,9/4}^2 = \frac{(c - 4)\sqrt{c} + \sqrt{(c - 4)(c^2 + 4)}}{2\sqrt{5}c} \],

where \( c = -1 + \frac{\sqrt{5}(2 + \sqrt{3})(5 + 2\sqrt{2} + \sqrt{5} + 2\sqrt{6 + 6\sqrt{5}})}{\sqrt{2}(-2 + \sqrt{5} + \sqrt{15} + 2\sqrt{2 + \sqrt{5}})} \),

(ii) \[ l_{5,1/36}^2 = \frac{(c - 4)\sqrt{c} + \sqrt{(c - 4)(c^2 + 4)}}{2\sqrt{5}c} \],

where \( c = -1 + \frac{\sqrt{5}(2 - \sqrt{3})(5 + 2\sqrt{2} + \sqrt{5} - 2\sqrt{6 + 6\sqrt{5}})}{\sqrt{2}(-2 + \sqrt{5} - \sqrt{15} + 2\sqrt{2 + \sqrt{5}})} \).
Proof. The results follow directly from (2.2) and Theorem 2.6(ii) and (iv) with the help of Mathematica.

Remark 4. As mentioned in Remark 3, if we evaluate \( t^2_{5,9/16} \) and \( t^2_{5,1/144} \), then we have \( t^2_{5,9/16} \) and \( t^2_{5,1/144} \). Thus we are able to compute \( t^2_{5,9/4} \) and \( t^2_{5,1/9.4} \) for every positive integer \( m \).

3. Evaluations of \( R(q) \) and \( S(q) \)

In view of Remark 1, we can evaluate \( R(e^{-2\pi \sqrt{n}}) \) for \( n = \frac{3}{3^m} \) and \( \frac{1}{15.3^m} \), where \( m \) is any positive integer. We evaluate the cases for \( m = 1, 2, \) and 3.

Theorem 3.1. We have

(i) \[ R^3(e^{-2\pi \sqrt{3/20}}) = \frac{57 - 50\sqrt{3} + 16\sqrt{5} - 36\sqrt{15}}{1 + 3\sqrt{5} + 2\sqrt{15}}, \]

(ii) \[ R^3(e^{-2\pi \sqrt{1/60}}) = \frac{57 - 50\sqrt{3} - 16\sqrt{5} + 36\sqrt{15}}{1 - 3\sqrt{5} - 2\sqrt{15}}, \]

(iii) \[ R^3(e^{-2\pi \sqrt{3/80}}) = \sqrt{a^2 + 1} - a, \text{ where} \]

\[ a = \frac{11}{2} + \frac{5\sqrt{5}}{2} \left( \frac{4 + \sqrt{2} + \sqrt{30}}{2 - \sqrt{3} + \sqrt{15}} \right) \left( \frac{4 + \sqrt{2} - 5\sqrt{3} - 2\sqrt{5} + \sqrt{15} + \sqrt{30}}{2 - \sqrt{3} - 4\sqrt{5} - 5\sqrt{6} - \sqrt{10} + \sqrt{15}} \right)^2, \]

(iv) \[ R^3(e^{-2\pi \sqrt{1/240}}) = \sqrt{a^2 + 1} - a, \text{ where} \]

\[ a = \frac{11}{2} - \frac{5\sqrt{5}}{2} \left( \frac{4 - \sqrt{2} + \sqrt{30}}{2 - \sqrt{3} - \sqrt{15}} \right) \left( \frac{4 - \sqrt{2} - 5\sqrt{3} + 2\sqrt{5} - \sqrt{15} + \sqrt{30}}{2 - \sqrt{3} + 4\sqrt{5} + 5\sqrt{6} - \sqrt{10} - \sqrt{15}} \right)^2, \]

(v) \[ R^3(e^{-2\pi \sqrt{3/320}}) = -2 + (a^2 + 2a + 2)(\sqrt{a^2 + 1} - a), \text{ where} \]

\[ a = -\frac{1}{2} + \frac{10\sqrt{2} + 4\sqrt{5} + \sqrt{5}}{17 + 17\sqrt{2} + 9\sqrt{3} - 7\sqrt{5} + 9\sqrt{6} - 5\sqrt{10} - 3\sqrt{15} - 3\sqrt{30}} \]

\[ 12 + 33\sqrt{10} + \sqrt{270(13 - 4\sqrt{10})}, \]

(vi) \[ R^3(e^{-2\pi \sqrt{1/960}}) = -2 + (a^2 + 2a + 2)(\sqrt{a^2 + 1} - a), \text{ where} \]

\[ a = -\frac{1}{2} + \frac{10\sqrt{2} + 4\sqrt{5} + \sqrt{5}}{-17 + 17\sqrt{2} - 9\sqrt{3} - 7\sqrt{5} + 9\sqrt{6} + 5\sqrt{10} - 3\sqrt{15} + 3\sqrt{30}} \]

\[ 12 + 33\sqrt{10} - \sqrt{270(13 - 4\sqrt{10})}. \]

Proof. The results follow directly from (1.5) and Theorem 2.4 with the help of Mathematica.
In view of Remark 2, we can evaluate \( S(e^{-\pi \sqrt{n}}) \) for \( n = \frac{3}{5^{m-1}} \) and \( \frac{1}{15^{m-1}} \), where 
\( m \) is any positive integer. We evaluate the cases for \( m = 1 \) and 2.

**Theorem 3.2.** We have

(i) \( S^5(e^{-\pi \sqrt{3/5}}) = \frac{1}{4} \left( -3 - 5\sqrt{5} + \sqrt{30(5 + \sqrt{5})} \right) \),

(ii) \( S^5(e^{-\pi \sqrt{1/15}}) = \frac{1}{4} \left( -3 + 5\sqrt{5} + \sqrt{30(5 - \sqrt{5})} \right) \),

(iii) \( S^5(e^{-\pi \sqrt{3/30}}) = \frac{2 + (b^2 - 2b + 2)(\sqrt{b^2 + 1} - b)}{b^2} \), where

\[
b = \frac{1}{2} + \frac{\sqrt{5}(4 + \sqrt{2 - 5\sqrt{3} - 2\sqrt{5} + \sqrt{15} + \sqrt{30})}{4(2 - \sqrt{3} + \sqrt{15})} + \frac{\sqrt{5}(26 - 84\sqrt{2} + 18\sqrt{3} + 63\sqrt{5} - 47\sqrt{6} - \sqrt{10} + 38\sqrt{15})}{(4 + 2\sqrt{3})\sqrt{2 + 6\sqrt{5} + 4\sqrt{15}}}.
\]

(iv) \( S^5(e^{-\pi \sqrt{1/60}}) = \frac{2 + (b^2 - 2b + 2)(\sqrt{b^2 + 1} - b)}{b^2} \), where

\[
b = \frac{1}{2} + \frac{\sqrt{5}(4 - \sqrt{2 - 5\sqrt{3} + 2\sqrt{5} - \sqrt{15} + \sqrt{30})}{4(-2 + \sqrt{3} + \sqrt{15})} + \frac{\sqrt{5}(-26 - 84\sqrt{2} - 18\sqrt{3} + 63\sqrt{5} - 47\sqrt{6} + \sqrt{10} + 38\sqrt{15})}{(4 + 2\sqrt{3})\sqrt{-2 + 6\sqrt{5} + 4\sqrt{15}}}.
\]

**Proof.** The results are immediate consequences of (1.6), Lemma 2.1(i) and (ii), and Theorem 2.5 with the help of *Mathematica*. \( \square \)

See [13, Corollary 4.12(i) and (ii)] for alternative proofs of Theorem 3.2(i) and (ii), respectively.

In view of Remark 3, we are able to evaluate \( R(e^{-2\pi \sqrt{n}}) \) for \( n = \frac{9}{5^{m-1}} \) and \( \frac{1}{45^{m-1}} \), where \( m \) is any positive integer. We evaluate the cases for \( m = 1 \) and 2.

**Theorem 3.3.** We have

(i) \( R^5(e^{-2\pi \sqrt{9/20}}) = \sqrt{a^2 + 1} - a, \) where

\[
a = \frac{11}{2} + \frac{5}{2} \left( \frac{5 + 14\sqrt{5} - 10\sqrt{2 + 5\sqrt{5}}}{54 - 25\sqrt{3} + 16\sqrt{5} - 12\sqrt{15}} \right),
\]

(ii) \( R^5(e^{-2\pi \sqrt{1/180}}) = \sqrt{a^2 + 1} - a, \) where

\[
a = \frac{11}{2} + \frac{5}{2} \left( \frac{5 + 14\sqrt{5} - 10\sqrt{2 + 5\sqrt{5}}}{54 + 25\sqrt{3} + 16\sqrt{5} + 12\sqrt{15}} \right),
\]

(iii) \( R^5(e^{-2\pi \sqrt{9/80}}) = \frac{-2 + (a^2 + 2a + 2)(\sqrt{a^2 + 1} - a)}{a^2}, \) where
EVALUATIONS OF $R(q)$ BY THETA-FUNCTION IDENTITIES

385

\[ a = \frac{1}{2} + \frac{\sqrt{5}(2 + \sqrt{3})(5 + 2\sqrt{2} + \sqrt{5} + \sqrt{6 + 6\sqrt{5}})}{2\sqrt{2}\left(-2 + \sqrt{5} + \sqrt{15} + 2\sqrt{2 + \sqrt{5}}\right)}, \]

(iv) $R_5(e^{-2\pi \sqrt{1/20}}) = \frac{-2 + (a^2 + 2a + 2)(\sqrt{a^2 + 1} - a)}{a^2}$, where

\[ a = \frac{1}{2} + \frac{\sqrt{5}(2 - \sqrt{3})(5 + 2\sqrt{2} + \sqrt{5} - \sqrt{6 + 6\sqrt{5}})}{2\sqrt{2}\left(-2 + \sqrt{5} - \sqrt{15} + 2\sqrt{2 + \sqrt{5}}\right)}. \]

Proof. The results follow directly from (1.5) and Theorem 2.4. We used Mathematica to verify (i)–(iv). \qed

In view of Remark 4, we can evaluate $S(e^{-\pi \sqrt{m}})$ for $n = \frac{9}{\sqrt{34}}$ and $\frac{1}{4\sqrt{34}m}$, where $m$ is any positive integer. We evaluate the cases for $m = 1$ and 2.

**Theorem 3.4.** We have

(i) $S_5(e^{-\pi \sqrt{9/5}})$

\[ = \frac{22 - 65\sqrt{5} - 32\sqrt{15} + 2\sqrt{5}(1850 + 1040\sqrt{3} - 143\sqrt{5} - 70\sqrt{15})}{4 + \sqrt{15}}, \]

(ii) $S_5(e^{-\pi \sqrt{1/45}})$

\[ = \frac{22 - 65\sqrt{5} + 32\sqrt{15} + 2\sqrt{5}(1850 - 1040\sqrt{3} - 143\sqrt{5} + 70\sqrt{15})}{4 - \sqrt{15}}, \]

(iii) $S_5(e^{-\pi \sqrt{3/20}}) = \frac{2 + (b^2 - 2b + 2)(\sqrt{b^2 + 1} - b)}{b^2}$, where

\[ b = \frac{-1 + \frac{c\sqrt{c} + \sqrt{(c - 4)(c^2 + 4)}}{\sqrt{2}\left(-2 + \sqrt{5} + \sqrt{15} + 2\sqrt{2 + \sqrt{5}}\right)}}{4\sqrt{c}}, \]

(iv) $S_5(e^{-\pi \sqrt{1/180}}) = \frac{2 + (b^2 - 2b + 2)(\sqrt{b^2 + 1} - b)}{b^2}$, where

\[ b = \frac{-1 + \frac{c\sqrt{c} + \sqrt{(c - 4)(c^2 + 4)}}{\sqrt{2}\left(-2 + \sqrt{5} - \sqrt{15} + 2\sqrt{2 + \sqrt{5}}\right)}}{4\sqrt{c}}, \]

\[ c = -1 + \frac{\sqrt{5}(2 - \sqrt{3})(5 + 2\sqrt{2} + \sqrt{5} - \sqrt{6 + 6\sqrt{5}})}{\sqrt{2}\left(-2 + \sqrt{5} - \sqrt{15} + 2\sqrt{2 + \sqrt{5}}\right)}. \]

Proof. The results follow from (1.6), Lemma 2.1(iii) and (iv), and Theorem 2.7. We used Mathematica to verify (i)–(iv). \qed
REFERENCES


Dae Hyun Paek, Department of Mathematics Education, Busan National University of Education, Busan 47503, Korea
Email address: paek@bnue.ac.kr