FUZZY COMPLETE LATTICES AND DISTANCE SPACES

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Abstract. In this paper, we introduce the notions of fuzzy join (resp. meet) complete lattices and distance spaces in complete co-residuated lattices. Moreover, we investigate the relations between Alexandrov pretopologies (resp. precotopologies) and fuzzy join (resp. meet) complete lattices, respectively. We give their examples.

1. Introduction

As an algebraic structure for many valued logic, a complete residuated lattice is an important mathematical tool [1-4, 6-11, 15, 16]. For an extension of classical rough sets introduced by Pawlak [12, 13], many researchers [1, 6-11] developed \(L\)-lower and \(L\)-upper approximation operators in complete residuated lattices. By using the concepts of lower and upper approximation operators, fuzzy concepts, information systems and decision rules are investigated in complete residuated lattices [1-4, 6-11, 15, 16].

Zhang et al. [17, 18] introduced the notion of fuzzy complete lattices using fuzzy partially order on a frame as generalizations of usual complete lattices. Based on residuated lattices as an extension of frame, Zhang [19] introduced the notions of partially orders, join, meet and fuzzy completeness.

Kim et al. [7-10] studied the properties of fuzzy join and meet completeness, \(L\)-fuzzy upper and lower approximation spaces and Alexandrov \(L\)-topologies with fuzzy partially ordered spaces in complete residuated lattices. Zheng and Wang [20] introduced complete co-residuated lattices. By using this concepts, lower and upper...
approximation operators, fuzzy rough sets and information systems are investigated [6].

In this paper, we introduce the concepts of fuzzy join and meet complete lattices using distance spaces instead of fuzzy partially ordered spaces [19] in complete co-residuated lattices. We show that fuzzy join (resp. meet) complete lattices and Alexandrov pretopologies (resp. precotopologies) are equivalent, respectively. Moreover, their properties and examples are investigated.

2. Preliminaries

**Definition 2.1** ([6, 20]). An algebra $(L, \land, \lor, \oplus, 0, 1)$ is called a complete co-residuated lattice if it satisfies the following conditions:

(Q1) $L = (L, \leq, \lor, \land, 0, 1)$ is a complete lattice where 0 is the bottom element and 1 is the top element.

(Q2) $a = a \oplus 0$, $a \ominus (b \oplus c) = (a \ominus b) \oplus c$ for all $a, b, c \in L$.

(Q3) $(\land_{i \in \Gamma} a_i) \oplus b = \land_{i \in \Gamma} (a_i \ominus b)$.

**Remark 2.2.** (1) An infinitely distributive lattice $(L, \leq, \lor, \land, \oplus = \lor, 0, 1)$ is a complete co-residuated lattice. In particular, the unit interval $([0, 1], \leq, \lor, \land, \oplus = \lor, 0, 1)$ is a complete co-residuated lattice [4,15].

(2) The unit interval with a right-continuous t-conorm $\oplus$, $([0, 1], \leq, \lor, \land, \oplus = +, \lor, 0, 1)$, is a complete co-residuated lattice [1,4,15].

(3) Let $(L, \leq, \ominus)$ be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \land \{z \in L \mid x \ominus z \geq y\}.$$ Then $(x \ominus y) \geq z$ if $x \geq (y \ominus z)$.

(4) $([0, \infty], \leq, \lor, \ominus = +, \land, \ominus = \land, \lor, 0, 1)$ is a commutative unital co-quantale where

$$x \ominus y = \land \{z \in [0, \infty] \mid x + z \geq y\}$$

$$= \land \{z \in [0, \infty] \mid z \geq -x + y\} = (y - x) \lor 0,$$

$$\infty + a = a = a = \infty, \forall a \in [0, \infty], \infty \rightarrow \infty = 0.$$

In this paper, we assume $(L, \land, \lor, \ominus, \ominus = \land, \lor, 0, 1)$ is a complete co-residuated lattice.

For $\alpha \in L, A \in L^X$, we denote $(\alpha \ominus A), (\alpha \ominus A), \alpha_X \in L^X$ as $(\alpha \ominus A)(x) = \alpha \ominus A(x), (\alpha \ominus A)(x) = \alpha \ominus A(x), \alpha_X(x) = \alpha$.

**Lemma 2.3.** Let $(L, \land, \lor, \ominus, \ominus = \land, \lor, 0, 1)$ be a complete co-residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.
(1) If \( y \leq z \), \((x \oplus y) \leq (x \oplus z)\), then \( x \ominus y \leq x \ominus z \) and \( z \ominus x \leq y \ominus x \).

(2) \( x \ominus (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \ominus y_i) \) and \( (\bigwedge_{i \in I} x_i) \ominus y = \bigwedge_{i \in I} (x_i \ominus y) \).

(3) \( x \ominus (\bigwedge_{i \in I} y_i) \leq \bigwedge_{i \in I} (x_i \ominus y_i) \).

(4) \( (\bigvee_{i \in I} x_i) \ominus y \leq \bigwedge_{i \in I} (x_i \ominus y) \).

(5) \( x \ominus (x \ominus y) \geq y \), \((x \ominus y) \ominus y \leq x \) and \((x \ominus y) \ominus (y \ominus z) \geq x \ominus z \).

(6) \( (x \ominus y) \ominus z = x \ominus (y \ominus z) = y \ominus (x \ominus z) \).

(7) \( x \ominus y \geq (x \ominus z) \ominus (y \ominus z) \), \(x \ominus y \geq (y \ominus z) \ominus (x \ominus z)\) and \((x \ominus y) \ominus (z \ominus w) \leq (x \ominus z) \ominus (y \ominus w) \).

(8) \( x \ominus x = 0 \), \( 0 \ominus x = x \). Moreover, \( x \ominus y = 0 \) iff \( x \geq y \).

**Proof.** (1) Since \( y = y \land z \), \( x \ominus y = x \ominus (y \land z) = (x \ominus y) \land (x \ominus z) \). Then \((x \ominus y) \leq (x \ominus z)\).

Since \( y \leq z \leq x \ominus (x \ominus z) \), \( x \ominus y \leq x \ominus z \). Since \( x \leq y \ominus (y \ominus x) \leq z \ominus (y \ominus x) \), \( z \ominus x \leq y \ominus x \).

(2) By (1), \( x \ominus (\bigvee_{i \in I} y_i) \geq \bigvee_{i \in I} (x \ominus y_i) \). Since \( x \ominus \bigvee_{i \in I} (x \ominus y_i) \geq \bigvee_{i \in I} (x \ominus y_i) \), \( x \ominus (\bigvee_{i \in I} y_i) \leq \bigvee_{i \in I} (x \ominus y_i) \).

By (1), \( (\bigwedge_{i \in I} x_i) \ominus y \geq \bigwedge_{i \in I} (x_i \ominus y) \). Since \( (\bigwedge_{i \in I} x_i) \ominus (\bigwedge_{i \in I} x_i) \ominus y \geq \bigwedge_{i \in I} (x_i \ominus (x_i \ominus y)) \geq y \), \( (\bigwedge_{i \in I} x_i) \ominus y \leq \bigwedge_{i \in I} (x_i \ominus y) \).

(3) and (4) are easily proved from (1).

(5) Since \( x \ominus y \geq x \ominus y \), \( x \ominus (x \ominus y) \geq y \). Moreover, \( x \geq (x \ominus y) \ominus y \). Since \( x \ominus (x \ominus y) \ominus (y \ominus z) \geq y \ominus (y \ominus z) \geq z \), \( x \ominus y \ominus (y \ominus z) \leq x \ominus z \).

(6) We have \( x \ominus y \ominus ((x \ominus y) \ominus z) \geq z \) if and only if \( x \ominus ((x \ominus y) \ominus z) \geq y \ominus z \).

Thus \((x \ominus y) \ominus z \geq x \ominus (y \ominus z) \).

Since \( x \ominus y \ominus ((x \ominus (y \ominus z)) \geq y \ominus (y \ominus z) \geq z \), \( x \ominus (y \ominus z) \geq (x \ominus y) \ominus z \).

Similarly, \((x \ominus y) \ominus z = y \ominus (x \ominus z) \).

(7) Since \((x \ominus z) \ominus (x \ominus y) \geq y \ominus z \), \( x \ominus y \geq (x \ominus z) \ominus (y \ominus z) \). Since \((x \ominus (x \ominus y) \ominus (y \ominus z) \geq z \), \( x \ominus y \geq (y \ominus z) \ominus (x \ominus z) \).

Since \( z \ominus w \leq x \ominus (x \ominus z) \ominus y \ominus (y \ominus w), (x \ominus y) \ominus (z \ominus w) \leq (x \ominus z) \ominus (y \ominus w) \).

(8) For \( x \in L \), \( x \ominus x = \bigwedge \{z \in L \mid x \ominus z \geq x\} = 0 \) and \( 0 \ominus x = \bigwedge \{z \in L \mid 0 \ominus z \geq x\} = x \).

**Definition 2.4.** Let \((L, \land, \lor, \ominus, \odot, 0, 1)\) be a complete co-residuated lattice. Let \( X \) be a set. A function \( d_X : X \times X \to L \) is called a **distance function** if it satisfies the following conditions:

(M1) \( d_X(x, x) = 0 \) for all \( x \in X \),

(M2) \( d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z) \), for all \( x, y, z \in X \),

(M3) if \( d_X(x, y) = d_X(y, x) = 0 \), then \( x = y \).\( \square \)
The pair \((X, d_X)\) is called a distance space.

**Remark 2.5.** (1) We define a distance function \(d_X : X \times X \to [0, \infty)\). Then \((X, d_X)\) is called a non-symmetric pseudo-metric space.

(2) Let \((L, \land, \lor, \ominus, \oslash, 0, 1)\) be a complete co-residuated lattice. Define a function \(d_L : L \times L \to L\) as \(d_L(x, y) = x \ominus y\). By Lemma 2.3 (5) and (8), \((L, d_L)\) is a distance space.

Moreover, we define a function \(d_{L_X} : L^X \times L^X \to L\) as \(d_{L_X}(A, B) = \bigvee_{x \in X} (A(x) \ominus B(x))\). Then \((L^X, d_{L_X})\) is a distance space.

(3) We define a function \(d_{[0, \infty]^X} : [0, \infty]^X \times [0, \infty]^X \to [0, \infty]\) as \(d_{[0, \infty]^X}(A, B) = \bigvee_{x \in X} (A(x) \ominus B(x)) = \bigvee_{x \in X} ((B(x) - A(x)) \lor 0)\). Then \([0, \infty]^X, d_{[0, \infty]^X}\) is a non-symmetric pseudo-metric space.

(4) If \((X, d_X)\) is a distance space and we define a function \(d_X^{-1}(x, y) = d_X(y, x)\), then \((X, d_X^{-1})\) is a distance space.

(5) Let \((L, \land, \lor, \ominus, \oslash, 0, 1)\) be a complete co-residuated lattice. Let \((X, d_X)\) be a distance space and define \((d_X \oplus d_X)(x, z) = \bigwedge_{y \in X} (d_X(x, y) \oplus d_X(y, z))\) for each \(x, z \in X\). By (M2), \((d_X \oplus d_X)(x, z) \geq d_X(x, z)\) and \((d_X \oplus d_X)(x, z) \leq d_X(x, x) \oplus d_X(x, z) = d(x, z)\). Hence \((d_X \oplus d_X) = d_X\).

(6) If \(d_X\) is a distance function and \(d_X^{-1}(x, y) = d_X(y, x)\) for each \(x, y \in X\), then \(d_X^{-1}\) is a distance function.

3. **Fuzzy Complete Lattices and Distance Spaces**

**Definition 3.1.** Let \((X, d_X)\) be a distance space and \(A \in L^X\).

(1) A point \(x_0\) is called a fuzzy join of \(A\), denoted by \(x_0 = \sqcup_X A\), if it satisfies

\((J1)\) \(A(x) \geq d_X(x_0, x)\),
\n\((J2)\) \(\bigvee_{x \in X} (A(x) \ominus d_X(x_0, x)) \geq d_X(y, x_0)\).

The pair \((X, d_X)\) is called fuzzy join complete if \(\sqcup_X A\) exists for each \(A \in L^X\).

A point \(x_1\) is called a fuzzy meet of \(A\), denoted by \(x_1 = \sqcap_X A\), if it satisfies

\((M1)\) \(A(x) \geq d_X(x, x_1)\),
\n\((M2)\) \(\bigvee_{x \in X} (A(x) \ominus d_X(x_1, x)) \geq d_X(x_1, y)\).

The pair \((X, d_X)\) is called fuzzy meet complete if \(\sqcap_X A\) exists for each \(A \in L^X\).

The pair \((X, d_X)\) is called fuzzy complete if \(\sqcap_X A\) and \(\sqcup_X A\) exists for each \(A \in L^X\).

**Theorem 3.2.** Let \((X, d_X)\) be a distance space and \(\Phi \in L^X\).

(1) A point \(x_0\) is a fuzzy join of \(\Phi\) iff \(\bigvee_{x \in X} (\Phi(x) \ominus d_X(x, y)) = d_X(y, x_0)\).

(2) A point \(x_1\) is a fuzzy meet of \(\Phi\) iff \(\bigvee_{x \in X} (\Phi(x) \ominus d_X(x, y)) = d_X(x_1, y)\).
(3) If $\sqcup_X \Phi$ is a fuzzy join of $\Phi \in L^X$, then it is unique. Moreover, if $\sqcap_X \Phi$ is a fuzzy meet of $\Phi \in L^X$, then it is unique.

Proof. (1) Let $\sqcup_X \Phi$ be a fuzzy join of $\Phi \in L^X$. By (J1), since $\Phi(x) \geq d_X(\sqcup_X \Phi, x)$, we have $\Phi(x) \oplus d_X(y, \sqcup_X \Phi) \geq d_X(\sqcup_X \Phi, y) \oplus d_X(y, \sqcup_X \Phi) \geq d_X(y, x)$. Hence $d_X(y, \sqcup_X \Phi) \geq \bigvee_{x \in X}(\Phi(x) \oplus d_X(y, x))$. By (J2), $d_X(y, \sqcup_X \Phi) = \bigvee_{x \in X}(\Phi(x) \oplus d_X(y, x))$.

Conversely, $d_X(y, \sqcup_X \Phi) \geq (\Phi(x) \oplus d_X(y, x))$ if and only if $\Phi(x) \geq d_X(y, \sqcup_X \Phi) \oplus d_X(y, x)$. Put $y = \sqcup_X \Phi$. Then $\Phi(x) \geq d_X(\sqcup_X \Phi, x)$.

(2) It is similarly proved as (1).

(3) Let $x_1, x_2$ be fuzzy joins of $\Phi \in L^X$. For all $y \in X$, we have

$$\bigvee_{x \in X}(\Phi(x) \oplus d_X(y, x)) = d_X(y, x_1) = d_X(y, x_2).$$

Put $y = x_1$. Then $0 = d_X(x_1, x_1) = d_X(x_1, x_2)$. Put $y = x_2$. Then $0 = d_X(x_1, x_2)$.

Theorem 3.3. Let $(X, d_X)$ be a distance space and $A, B \in L^X$.

1. If $\sqcup_X A, \sqcap_X B$ exist, $d_LX(A, B) \geq d_X(\sqcup_X B, \sqcap_X A)$.

2. If $\sqcap_X A, \sqcap_X B$ exist, $d_LX(A, B) \geq d_X(\sqcap_X A, \sqcap_X B)$.

Proof. (1) For each $A, B \in L^X$, $d_LX(A, B) = \bigvee_{x \in X}(A(x) \oplus B(x)) \geq \bigvee_{x \in X}(A(x) \oplus d_X(\sqcap_X B, x)) \geq d_X(\sqcap_X B, \sqcap_X A)$.

(2) For each $A, B \in L^X$, $d_LX(A, B) = \bigvee_{x \in X}(A(x) \oplus B(x)) \geq \bigvee_{x \in X}(A(x) \oplus d_X(x, \sqcap_X B)) \geq d_X(\sqcap_X A, \sqcap_X B)$.

Lemma 3.4. Let $(X, d_X)$ be a distance space. Then the followings hold.

1. For each $z \in X$, $\sqcup_X d_X(z, -) = z$ and $\sqcap_X d_X(-, z) = z$.

2. For $\Phi \in L^X$, $\sqcup_X \Phi = \sqcap_X \bigwedge (\Phi(z) \oplus d_X(z, -))$ and $\sqcap_X \Phi = \sqcup_X \bigwedge (\Phi(z) \oplus d_X(-, z))$.

Proof. (1) Since $d_X(x, z) \oplus d_X(y, z) \geq d_X(y, x)$,

$$d_X(y, z) \geq \bigvee_{x \in X}(d_X(z, x) \oplus d_X(y, x)).$$

From the definition of $\sqcup_X d_X(z, -) = z$,

$$d_X(x, \sqcup_X d_X(z, -)) = \bigvee_{x \in X}(d_X(z, x) \oplus d_X(y, x)) \geq d_X(z, z) \oplus d_X(y, z) = d_X(y, z).$$

Hence $d_X(x, \sqcup_X d_X(z, -)) = \bigvee_{x \in X}(d_X(z, x) \oplus d_X(y, x)) = d_X(y, z)$. Thus $\sqcup_X d_X(z, -) = z$. Similarly, $d_X(\sqcap_X d_X(-, z), y) = \bigvee_{x \in X}(d_X(z, x) \oplus d_X(x, y)) = d_X(z, y)$. Thus $\sqcap_X d_X(-, z) = z$. 
(2) From the definitions of \( \sqcup X \wedge (F(z) \oplus d_X(z, -)) \) and \( \sqcap X \wedge (F(z) \oplus d_X(-, z)) \),
\[
d_X(y, \sqcup X (F(z) \oplus d_X(z, -))) = \bigvee_{x \in X} (F(z) \oplus d_X(z, x)) \oplus d_X(y, x) \\
= \bigvee_{x \in X} (F(z) \oplus d_X(z, x) \oplus d_X(y, x)) = \bigvee_{x \in X} (F(z) \oplus d_X(z, x) \oplus d_X(x, y)) \\
= \bigvee_{x \in X} (F(z) \oplus d_X(x, y)) = d_X(y, \sqcup F), \\
d_X(\sqcap X (F(z) \oplus d_X(-, z)), y) = \bigvee_{x \in X} (F(z) \oplus d_X(-, z)) \oplus d_X(x, y) \\
= \bigvee_{x \in X} (F(z) \oplus d_X(x, z) \oplus d_X(x, y)) = \bigvee_{x \in X} (F(z) \oplus d_X(x, z) \oplus d_X(x, y)) \\
= \bigvee_{x \in X} (F(z) \oplus d_X(x, y)) = d_X(\sqcap F, y).
\]
\[
\square
\]

\textbf{Theorem 3.5.} Let \((X, d_X)\) be a distance space. Then the following are equivalent:

1. \( \sqcup X \Phi \) exists for every \( \Phi \in L_X \).
2. \( \sqcap X \Phi \) exists for every \( \Phi \in L_X \).

\textbf{Proof.} (1) \( \Rightarrow \) (2). For every \( \Phi \in L_X \) and \( \bigvee (F(y) \oplus d_X(y, -)) \in L_X \), there exists \( z = \sqcup X (\bigvee (F(y) \oplus d_X(y, -))) \). We will show that \( z = \sqcap X \Phi \).

(M2) By the definition of \( \sqcup X (\bigvee (F(y) \oplus d_X(y, -))) \), by (J1),
\[
\bigvee (F(y) \oplus d_X(y, x)) \geq d_X(\sqcup X (\bigvee (F(y) \oplus d_X(y, -))), x) = d_X(z, x).
\]

(M1) Since \( (F(y) \oplus d_X(y, x)) \oplus \Phi(y) \geq d_X(y, x) \) iff \( \Phi(y) \geq (F(y) \oplus d_X(y, x)) \) and \( d_X(y, x) \),
\[
\Phi(y) \geq \bigvee_{x \in X} ((F(y) \oplus d_X(y, x)) \oplus d_X(y, x)) \\
\geq \bigvee_{x \in X} (\bigvee_{y \in X} (F(y) \oplus d_X(y, x)) \oplus d_X(y, x)) \\
= d_X(y, \sqcup X (\bigvee (F(y) \oplus d_X(y, -)))) = d_X(y, z).
\]

(2) \( \Rightarrow \) (1). For every \( \Psi \in L_X \) and \( \bigvee (F(y) \rightarrow d_X(\rightarrow, y)) \in L_X \), there exists \( w = \sqcap X (\bigvee (F(y) \rightarrow d_X(\rightarrow, y))) \). We will show that \( z = \sqcup X \Psi \).

(J2) Since \( w = \sqcap X (\bigvee (F(y) \rightarrow d_X(\rightarrow, y))) \),
\[
\bigvee (\Psi(y) \rightarrow d_X(x, y)) \geq d_X(x, \sqcap X (\bigvee (F(y) \rightarrow d_X(\rightarrow, y)))) = d_X(x, w).
\]

(J1) Since \( (F(y) \rightarrow d_X(y, x)) \oplus \Phi(y) \geq d_X(y, x) \) iff \( \Phi(y) \geq (F(y) \rightarrow d_X(y, x)) \) and \( d_X(y, x) \),
\[
\Psi(y) \geq \bigvee_{x \in X} ((\Psi(y) \rightarrow d_X(x, y)) \rightarrow d_X(x, y)) \\
\geq \bigvee_{x \in X} (\bigvee_{y \in X} (\Psi(y) \rightarrow d_X(x, y)) \rightarrow d_X(x, y)) \\
= d_X(\sqcap X (\bigvee (\Psi(y) \rightarrow d_X(\rightarrow, y))), y).
\]

Hence \( \sqcup X \Psi = \sqcap X (\bigvee (F(y) \rightarrow d_X(\rightarrow, y))) = w. \)

\textbf{Definition 3.6.} (1) A subset \( \tau \subset L_X \) is called an Alexandrov pretopology on \( X \) iff it satisfies the following conditions:

(O1) if \( A_i \in \tau \) for all \( i \in I \), then \( \bigcup_{i \in I} A_i \in \tau. \)

(O2) if \( A \in \tau \) and \( \alpha \in L \), then \( \alpha \cup A \in \tau. \)
(2) A subset \( \eta \subset L^X \) is called an *Alexandrov precotopology* on \( X \) iff it satisfies the following conditions:

- (CO1) if \( A_i \in \eta \) for all \( i \in I \), then \( \bigwedge_{i \in I} A_i \in \eta \).
- (CO2) if \( A \in \eta \) and \( \alpha \in L \), then \( \alpha \oplus A \in \eta \).

A subset \( \tau \subset L^X \) is called an *Alexandrov topology* on \( X \) iff it is both Alexandrov pretopology and Alexandrov precotopology on \( X \).

**Theorem 3.7.** Let \( \tau \subset L^X \). Define \( d_\tau : \tau \times \tau \to L \) as \( d_\tau(A, B) = \bigvee_{x \in X} (A(x) \ominus B(x)) \). Then the following statements hold.

1. \((\tau, d_\tau)\) is a distance space.
2. \( \sqcup_\tau \Phi \) is a fuzzy join of \( \Phi \in L^\tau \) iff \( \bigvee_{A \in \tau} (\Phi(A) \ominus d_\tau(B, A)) = d_\tau(B, \sqcup_\tau \Phi) \).
3. \( \sqcap_\tau \Phi \) is a fuzzy meet of \( \Phi \in L^\tau \) iff \( \bigvee_{A \in \tau} (\Phi(A) \ominus d_\tau(A, B)) = d_\tau(\sqcap_\tau \Phi, B) \).
4. If \( \sqcup_\tau \Phi \) is a fuzzy join of \( \Phi \in L^\tau \), then it is unique. Moreover, if \( \sqcap_\tau \Phi \) is a fuzzy meet of \( \Phi \in L^\tau \), then it is unique.

**Proof.** (1) (M1) For each \( A \in \tau \), \( d_\tau(A, A) = \bigvee_{x \in X} (A(x) \ominus A(x)) = 0 \).

- (M2) By Lemma 2.3(5), \( d_\tau(A, B) \oplus d_\tau(B, C) = \bigvee_{x \in X} (A(x) \ominus B(x)) \oplus \bigvee_{x \in X} (B(x) \ominus C(x)) \geq \bigvee_{x \in X} ((A(x) \ominus B(x)) \oplus (B(x) \ominus C(x))) \geq d_\tau(A, C) \), for all \( A, B, C \in \tau \).

- (M3) If \( d_\tau(A, B) = d_\tau(B, A) = 0 \), by Lemma 2.3(8), \( A = B \). Hence \((\tau, d_\tau)\) is a distance space.

(2), (3) and (4) follow from Theorem 3.2.

**Theorem 3.8.** Let \((X, d_X)\) be a distance space. Then \((L^X, d_{L^X})\) is a complete lattice.

**Proof.** For every \( \Phi \in L^{L^X} \) and \( A \in L^X \), we obtain that \( \sqcap_{L^X} \Phi(x) = \bigwedge_{A \in L^X} (\Phi(A) \ominus A(x)) \) and \( \sqcup_{L^X} \Phi(x) = \bigvee_{A \in L^X} (\Phi(A) \ominus A(x)) \), since

\[
\begin{align*}
  d_{L^X}(\bigwedge_{A \in L^X} (\Phi(A) \ominus A(-)), B) &= \bigvee_{x \in X} (\bigwedge_{A \in L^X} (\Phi(A) \ominus A(x)) \ominus B(x)) \\
  &= \bigvee_{A \in L^X} (\Phi(A) \ominus \bigvee_{x \in X} (A(x) \ominus B(x))) \quad \text{(by Lemma 2.3(6))} \\
  &= d_{L^X}(\bigvee_{A \in L^X} (\Phi(A) \ominus d_{L^X}(A, B)), B), \\
  d_{L^X}(B, \bigvee_{A \in L^X} (\Phi(A) \ominus A(-))) &= \bigvee_{x \in X} (B(x) \ominus \bigvee_{A \in L^X} (\Phi(A) \ominus A(x))) \\
  &= \bigvee_{A \in L^X} (\bigvee_{x \in X} (B(x) \ominus A(x))) \quad \text{(by Lemma 2.3(6))} \\
  &= d_{L^X}(B, \sqcup_{L^X} \Phi).
\end{align*}
\]

**Theorem 3.9.** Let \( \tau \subset L^X \). Then the following statements are equivalent:

1. \((\tau, d_\tau)\) is fuzzy join complete.
2. \( \tau \) is an Alexandrov pretopology on \( X \).
Let $(\tau, d_\tau)$ be given. Define $\Phi : \tau \to L$ as $\Phi(A) = \alpha$ for $A \in \tau$ and $\Phi(B) = 1$, otherwise. Then
\[ \bigvee_{A \in \tau} (\Phi(A) \otimes A(x)) = \alpha \otimes A(x). \]
So, $\bigvee_{A \in \tau} = \alpha \oplus A \in \tau$.

(2) $\Rightarrow$ (1) For each $\Phi \in L^\tau$, by (O1) and (O2), $\bigvee_{C \in \tau} (\Phi(C) \otimes C) \in \tau$. Thus,
\[ d_\tau(B, \bigvee_{A \in \tau} (\Phi(A) \otimes A(x))) = d_\tau(B, \bigvee_{C \in \tau} (\Phi(C) \otimes C)) \]
(by Lemma 2.3(6)).

By Theorem 3.2(3), $\bigvee_{A \in \tau} = \bigwedge_{A \in \tau} = \alpha \oplus A \in \tau$.

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$\square$
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**Theorem 3.10.** Let $\tau \subseteq L^X$. Then the following statements are equivalent:

1. $(\tau, d_\tau)$ is fuzzy meet complete.
2. $\tau$ is an Alexandrov precotopology on $X$.

**Proof.** (1) $\Rightarrow$ (2) Since $(\tau, d_\tau)$ is fuzzy meet complete, for each $\Phi \in L^\tau$, we have
\[ d_\tau(B, \bigvee_{A \in \tau} \Phi(C) \otimes d_\tau(B, C)) = d_\tau(B, \bigvee_{C \in \tau} (\Phi(C) \otimes C)) \]
(by Lemma 2.3(6)).

By Theorem 3.2(3), $\bigwedge_{A \in \tau} = \bigvee_{A \in \tau} = \bigwedge_{A \in \tau} = \alpha \oplus A \in \tau$.

(CO1) Define $\Phi : \tau \to L$ as $\Phi(A) = \alpha$ for $A \in \tau$ and $\Phi(B) = 1$, otherwise. Then
\[ \bigwedge_{A \in \tau} (\Phi(A) \otimes A(x)) = \alpha \otimes A(x). \]
So, $\bigwedge_{A \in \tau} = \bigvee_{A \in \tau} = \alpha \oplus A \in \tau$.

(CO2) Let $\{A_i \in \tau \mid i \in \Gamma\}$ be given. Define $\Phi : \tau \to L$ as $\Phi(A_i) = 0$ for $i \in \Gamma$ and $\Phi(B) = 1$, otherwise. Then
\[ \bigwedge_{A \in \tau} (\Phi(A) \otimes A(x)) = \bigvee_{i \in \Gamma} (0 \oplus A_i(x)) = \bigvee_{i \in \Gamma} A_i(x). \]
So, $\bigwedge_{A \in \tau} = \bigvee_{A \in \tau} = \alpha \oplus A \in \tau$.

(2) $\Rightarrow$ (1) For each $\Phi \in L^\tau$, by (CO1) and (CO2), $\bigwedge_{A \in \tau} \Phi(C) \otimes C \in \tau$. Thus,
\[ d_\tau(\bigwedge_{A \in \tau} \Phi(C) \otimes C) = d_\tau(\bigwedge_{C \in \tau} (\Phi(C) \otimes C), B) \]
(by Theorem 3.2(6)).

By Theorem 3.2(3), $\bigwedge_{C \in \tau} = \bigvee_{A \in \tau} = \alpha \oplus A \in \tau$.  

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$\square$
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Let \( \Phi \) be a fuzzy meet of \( \Phi \).

\[\text{Theorem 3.11.} \quad \text{Let } \Delta : L^X \to L^X \text{ be a map. The following statements are equivalent.} \]

1. \( d_{LX}(A, B) \geq d_{LX}(\Delta(A), \Delta(B)) \) for all \( A, B \in L^X \).
2. \( \alpha \oplus \Delta(A) \geq \Delta(\alpha \oplus A) \) for each \( \alpha \in L, A \in L^X \) and \( \Delta(A) \leq \Delta(B) \) for \( A \leq B \).
3. \( \Delta(\alpha \oplus A) \geq \alpha \oplus \Delta(A) \) for each \( \alpha \in L, A \in L^X \) and \( \Delta(A) \leq \Delta(B) \) for \( A \leq B \).

\[\text{Proof.} \quad (1) \implies (2). \text{ If } B \leq A, \text{ by Lemma 2.3(8), } d_{LX}(A, B) = 0 \text{ and } d_{LX}(\Delta(A), \Delta(B)) = 0. \text{ Thus } \Delta(B) \leq \Delta(A). \text{ Since } \alpha \geq d_{LX}(A, \alpha \oplus A) \geq d_{LX}(\Delta(A), \Delta(\alpha \oplus A)), \text{ we have } \alpha \oplus \Delta(A) \geq \Delta(\alpha \oplus A).
\]

(2) \implies (1). Put \( \alpha = d_{LX}(A, B) \). Then \( d_{LX}(A, B) \geq d_{LX}(\Delta(A), \Delta(B)), \text{ since } d_{LX}(A, B) \oplus \Delta(A) \geq d_{LX}(A, B \oplus A) \geq \Delta(B).
\]

(1) \implies (3). If \( A \leq B \), then \( \Delta(A) \leq \Delta(B) \). Since \( \alpha \geq d_{LX}(\alpha \oplus A, A) \geq d_{LX}(\Delta(\alpha \oplus A), \Delta(A)), \text{ we have } \Delta(\alpha \oplus A) \geq \alpha \oplus \Delta(A).
\]

(3) \implies (1). Put \( \alpha = d_{LX}(A, B) \). Then \( d_{LX}(A, B) \geq d_{LX}(\Delta(A), \Delta(B)) \), since \( \Delta(A) \geq d_{LX}(A, B \oplus B) \geq d_{LX}(A, B \oplus \Delta(B)). \]

\[\text{Theorem 3.12.} \quad \text{Let } \Delta : L^X \to L^X \text{ be a map. The following statements hold.} \]

1. \( \sqcup_{LX} \Delta^\rightarrow(\Phi) \) \( \leq \Delta(\sqcup_{LX} \Phi) \) for each \( \Phi \in L^{LX} \) where \( \Delta^\rightarrow(\Phi)(B) = \bigvee_{B = \Delta(A)} \Phi(A) \)
   iff \( \Delta(\alpha \oplus A) \geq \alpha \oplus \Delta(A) \) for each \( \alpha \in L, A \in L^X \) and \( \Delta(A) \leq \Delta(B) \) for \( A \leq B \).
2. \( \Delta(\sqcap_{LX} \Phi) \) \( \leq \sqcap_{LX} \Delta^\rightarrow(\Phi) \) for each \( \Phi \in L^{LX} \) iff \( \alpha \oplus \Delta(A) \geq \Delta(\alpha \oplus A) \) for each \( \alpha \in L, A \in L^X \) and \( \Delta(A) \leq \Delta(B) \) for \( A \leq B \).

\[\text{Proof.} \quad (1) \implies (2) \text{ For all } \Phi \in L^{LX},
\]

\[d_{LX}(B, \sqcup_{LX} \Phi) = \bigvee_{A \in L} (\Phi(A) \oplus d_{LX}(B, A)) = \bigvee_{A \in L} d_{LX}(B, \Phi(A) \oplus A) = d_{LX}(B, \bigvee_{A \notin L} \Phi(A) \oplus A), \]

\[d_{LX}(B, \sqcap_{LX} \Phi) = \bigvee_{C \in L} (\Phi(C) \oplus d_{LX}(B, C)) = \bigvee_{C \in L} d_{LX}(B, \Phi(C) \oplus C) = d_{LX}(B, \bigvee_{A \notin L} \Phi(A) \oplus C). \]

By Theorem 3.2(3), \( \sqcup_{LX} \Phi = \bigvee_{A \in L} (\Phi(A) \oplus A) \) and \( \sqcap_{LX} \Phi = \bigvee_{C \in L} (\Phi(C) \oplus C) \). Define \( \Phi_1 : L^X \to L \) as \( \Phi_1(A) = \alpha \) and \( \Phi_1(B) = 1 \), otherwise. Then

\[(\sqcup_{LX} \Phi_1)(x) = \bigvee_{D \in L} (\Phi_1(D) \oplus D(x)) = \alpha \oplus A(x).\]
Hence, since 

\[ D \to (\Phi_1)(B) = \bigvee_{B = D(A)} \Phi_1(A) \text{ and } D(\sqcup_{LX} \Phi_1) \geq \sqcup_{LX} D \to (\Phi_1) \text{ for all } \Phi_1 \in L^{LX}, \]

we have 

\[ \sqcup_{LX} D \to (\Phi_1)(x) = \bigvee_{C = D(A) \in LX} (D \to (\Phi_1))(C) \subseteq C(x) \]

\[ = \Phi_1(A) \otimes D(A)(x) = \alpha \otimes D(A)(x) \leq D(\sqcup_{LX} \Phi_1)(x) = D(\alpha \otimes A)(x). \]

Hence \( D \otimes D(A) \leq D(\alpha \otimes A). \)

Let \( A \leq B \) be given. Define \( \Phi_2 : L^X \to L \) as \( \Phi_2(A) = \Phi_2(B) = 0 \) and \( \Phi_2(C) = 1 \), otherwise. Then

\[ (\sqcup_{LX} \Phi_2)(x) = \bigvee_{D \in LX} (\Phi_2(D) \otimes D(x)) = A(x) \lor B(x) = B(x). \]

Since \( D \to (\Phi_2)(B) = \bigvee_{B = D(A)} \Phi_2(A) \) and \( D(\sqcup_{LX} \Phi_2) \geq \sqcup_{LX} D \to (\Phi_2) \) for \( \Phi_2 \in L^{LX}, \)

\[ \sqcup_{LX} D \to (\Phi_2)(x) = \bigvee_{C = D(A) \in LX} (D \to (\Phi_2))(C) \subseteq C(x) \]

\[ = (\Phi_2(A) \otimes D(A)(x)) \lor (\Phi_2(B) \otimes D(B)(x)) = D(A)(x) \lor D(B)(x) \]

\[ \leq D(\sqcup_{LX} \Phi_1)(x) = D(A \lor B)(x) = D(B)(x). \]

Hence \( D(A) \leq D(B). \)

(\( \Leftarrow \)\) \( \sqcup_{LX} D \to (\Phi)(y) = \bigvee_{A \in LX} \Phi(A) \otimes D(A)(y) \]

\[ \leq D(\bigvee_{A \in LX} (\Phi(A) \otimes A))(y) = D(\sqcup_{LX} \Phi)(y). \]

(2) (\( \Rightarrow \)) For all \( \Phi \in L^{LX}, \)

\[ d_{LX}(\sqcap_{LX} \Phi, B) = \bigvee_{A \in LX} (\Phi(A) \otimes d_{LX}(A, B)) \]

\[ = \bigvee_{A \in LX} d_{LX}(\Phi(A) \otimes A, B) = d_{LX}(\bigwedge_{A \in LX} (\Phi(A) \otimes A), B), \]

\[ d_{LX}(\sqcap_{LX} D \to (\Phi), B) = \bigvee_{C \in LX} (D \to (\Phi))(C) \otimes d_{LX}(C, B)) \]

\[ = \bigvee_{C \in LX} ((\bigwedge_{D(A) = C} (\Phi(A) \otimes d_{LX}(C, B)))) \]

\[ = \bigvee_{A \in LX} (\Phi(A) \otimes d_{LX}(D(A), B) = \bigvee_{A \in LX} d_{LX}(\Phi(A) \otimes D(A), B) \]

\[ = d_{LX}(\bigwedge_{A \in LX} \Phi(A) \otimes D(A), B). \]

By Theorem 3.2(3), \( \sqcap_{LX} \Phi = \bigwedge_{A \in LX} (\Phi(A) \otimes A) \) and \( \sqcap_{LX} D \to (\Phi) = \bigwedge_{A \in LX} (\Phi(A) \otimes D(A)) \in L^X. \) Define \( \Phi_1 : L^X \to L \) as \( \Phi_1(A) = \alpha \) and \( \Phi_1(B) = 1 \), otherwise. Then

\[ (\sqcap_{LX} \Phi_1) = \bigwedge_{A \in LX} (\Phi_1(A) \otimes A) = \alpha \otimes A. \]

Since \( D \to (\Phi_1)(B) = \bigvee_{B = D(A)} \Phi_1(A) \) and \( D(\sqcap_{LX} \Phi_1) \leq \sqcap_{LX} D \to (\Phi_1) \) for \( \Phi_1 \in L^{LX}, \)

\[ \sqcap_{LX} D \to (\Phi_1)(y) = \bigwedge_{B \in LX} (\Phi_1(A) \otimes D(A)(y)) \]

\[ = \alpha \otimes D(A)(y) \geq D(\sqcap_{LX} \Phi_1)(y) = D(\alpha \otimes A)(y). \]

Hence \( D(\alpha \otimes A) \leq \alpha \otimes D(A) \in L^X. \)
Let $A \leq B$ be given. Define $\Phi_2 : L^X \to L$ as $\Phi_2(A) = \Phi_2(B) = 0$ and $\Phi_2(C) = 1$, otherwise. Then $(\sqcap_{D \in L^X} \Phi_2)(x) = \wedge_{D \in L^X} (\Phi_2(D) \oplus D(x)) = A(x) \land B(x) = A(x)$. Since $D^{-}(\Phi_2)(B) = \bigvee_{B = \Phi_2(A)} \Phi_2(A)$ and $D(\sqcap_{D \in L^X} \Phi_2) \leq \sqcap_{D \in L^X} D^{-}(\Phi_2)$ for $\Phi_2 \in L^{L^X}$,

$$\sqcap_{D \in L^X} D^{-}(\Phi_2)(x) = \bigvee_{C = \Phi_2(A) \in L^X} (D^{-}(\Phi_2)(C) \oplus C(x))$$

$$= (\Phi_2(A) \oplus D(A)(x)) \land (\Phi_2(B) \oplus D(B)(x)) = D(A)(x) \land D(B)(x)$$

$$\geq D(\sqcap_{D \in L^X} \Phi_1)(x) = D(A \land B)(x) = D(A).$$

Hence $D(A) \leq D(B)$.

$(\Leftarrow)$ $D(\sqcap_{D \in L^X} \Phi) \leq \sqcap_{D \in L^X} D^{-}(\Phi)$, since

$$\sqcap_{D \in L^X} D^{-}(\Phi) = \bigwedge_{A \in L^X} (\Phi(A) \oplus D(A))$$

$$\geq \bigwedge_{A \in L^X} D(\Phi(A) \oplus A) \geq D(\bigwedge_{A \in L^X} (\Phi(A) \oplus A)) = D(\sqcap_{D \in L^X} \Phi).$$

$\square$

**Theorem 3.13.** Let $D : L^X \to L^X$ be a map with $d_{L^X}(A, B) \geq d_{L^X}(D(A), D(B))$ for all $A, B \in L^X$. Then followings hold.

1. $\tau_D = \{A \in L^X \mid A \leq D(A)\}$ is an Alexandrov fuzzy pretopology, that is, $\tau_D$ is a fuzzy join complete lattice.
2. $\eta_D = \{A \in L^X \mid D(A) \leq A\}$ is an Alexandrov fuzzy precotopology, that is, $\eta_D$ is a fuzzy meet complete lattice.

**Proof.** (1) (O1) For each $A \in \tau_D$, by Theorem 3.11, $D(\alpha \land A) \geq \alpha \land D(A) \geq \alpha \land A$. Hence $(\alpha \land A) \in \tau_D$.

(O2) For each $A_i \in \tau_D$ for $i \in \Gamma$, $D(\bigvee_{i \in \Gamma} A_i) \geq \bigvee_{i \in \Gamma} D(A_i) \geq \bigvee_{i \in \Gamma} A_i$. Hence $\bigvee_{i \in \Gamma} A_i \in \tau_D$.

(2) (O1) For each $A \in \eta_D$, by Theorem 3.11, $D(\alpha \land A) \leq \alpha \land D(A) \leq \alpha \land A$. Hence $(\alpha \land A) \in \eta_D$.

(O2) For each $A_i \in \eta_D$ for $i \in \Gamma$, $D(\bigwedge_{i \in \Gamma} A_i) \leq \bigwedge_{i \in \Gamma} D(A_i) \leq \bigwedge_{i \in \Gamma} A_i$. Hence $\bigwedge_{i \in \Gamma} A_i \in \eta_D$. $\square$

**Example 3.14.** Let $X$ be a set and $R \in L^{X \times X}$. For each $A \in L^X$, define $D_1, D_2 : L^X \to L^X$ as follows:

$$D_1(A)(y) = \bigwedge_{x \in X} (R(x, y) \oplus A(x)), \quad D_2(A)(y) = \bigvee_{x \in X} (R(x, y) \ominus A(x)).$$

For each $A, B \in L^X$, the followings hold.

$$d_{L^X}(D_1(A), D_1(B)) = \bigvee_{y \in X} ((\bigwedge_{x \in X} (R(x, y) \oplus A(x))) \ominus (\bigwedge_{x \in X} (R(x, y) \ominus B(x))))$$
Let $\forall x \in X ((R(x,y) \ominus A(x)) \oplus (\bigwedge_{x \in X} (R(x,y) \oplus B(x))))$

$\leq \bigvee_{y \in X} \bigvee_{x \in X} ((R(x,y) \oplus A(x)) \ominus (R(x,y) \oplus B(x)))$

$\leq \bigvee_{x \in X} (A(x) \ominus B(x)) = d_{Lx}(A,B)$,

$d_{Lx}(D_2(A), D_2(B)) = \bigvee_{y \in X} (\bigvee_{x \in X} ((R(x,y) \ominus A(x)) \oplus (\bigwedge_{x \in X} (R(x,y) \oplus B(x))))$

$\leq \bigvee_{y \in X} ((R(x,y) \oplus A(x)) \ominus (R(x,y) \ominus B(x)))$

$\leq \bigvee_{x \in X} (A(x) \ominus B(x)) = d_{Lx}(A,B)$.

For each $i \in \{1,2\}$, by Theorems 3.12, 3.13 and 3.14, the followings hold.

(1) $\alpha \ominus D_i(A) \geq D_i(\alpha \ominus A)$ for each $\alpha \in L$, $A \in L^X$ and $D_i(A) \leq D_i(B)$ for $A \leq B$.

(2) $D_i(\alpha \ominus A) \geq \alpha \ominus D_i(A)$ for each $\alpha \in L$, $A \in L^X$ and $D_i(A) \leq D_i(B)$ for $A \leq B$.

(3) $\bigvee_{x \in X} D_i^{-}(\Phi) \leq D_i(\bigvee_{x \in X} \Phi)$ for each $\Phi \in L^X$ where $D_i^{-}(\Phi)(B) = \bigvee_{B \in D_i(A)} \Phi(A)$.

(4) $D_i(\bigwedge_{x \in X} \Phi) \leq \bigwedge_{x \in X} D_i^{-}(\Phi)$ for each $\Phi \in L^X$.

(5) $\tau_{D_i} = \{ A \in L^X \mid A \leq D_i(A) \}$ is an Alexandrov fuzzy pretopology, that is, $\tau_{D_i}$ is a fuzzy join complete lattice.

(6) $\eta_{D_i} = \{ A \in L^X \mid D_i(A) \leq A \}$ is an Alexandrov fuzzy precotopology, that is, $\eta_{D_i}$ is a fuzzy meet complete lattice.

**Example 3.15.** Let $X = \{x, y, z\}$, $A \in [0, \infty]^X$ with $A(x) = 8$, $A(y) = 3$, $A(z) = 9$.

(1) Define an Alexandrov pretopology as

$$\tau_X = \{ \alpha \ominus A \mid \alpha \in [0, \infty] \}.$$

By Theorem 3.7(1), $(\tau_X, d_{\tau_X})$ is a distance space. For each $\Phi : \tau_X \to [0, \infty]$, since $\bigvee_{C \in \tau_X} (\Phi(C) \ominus C) = \bigvee_{\alpha \in [0, \infty]} (\Phi(\alpha \ominus A) \ominus (\alpha \ominus A)) = \bigvee_{\alpha \in [0, \infty]} ((\Phi(\alpha \ominus A) \ominus \alpha) \ominus A)) \in \tau_X$, it follows that

$$d_{\tau_X}(B, \bigvee_{\alpha \in [0, \infty]} (\Phi(\alpha \ominus A) \ominus \alpha) \ominus A)) \in \tau_X.$$

By Theorem 3.2(2), $(\tau_X, d_{\tau_X})$ is a fuzzy join complete lattice.

(2) Define an Alexandrov pretopology as

$$\eta_X = \{ \alpha \ominus A \mid \alpha \in [0, \infty] \}.$$

By Theorem 3.7(1), $(\eta_X, d_{\eta_X})$ is a distance space. For each $\Phi : \eta_X \to [0, \infty]$, since $\bigwedge_{C \in \tau_X} (\Phi(C) \oplus C) \in \eta_X = \bigwedge_{\alpha \in [0, \infty]} ((\Phi(\alpha \ominus A) \ominus \alpha) \ominus A)) \in \eta_X$, we have
\[ d_{\eta_X}(\cap_{\eta_X} \Psi, B) = \bigvee_{C \in \eta_X} (\Psi(C) \ominus d_{\eta_X}(C, B)) \]
\[ = d_{\eta_X}(\bigwedge_{C \in \eta_X} (\Psi(C) \oplus C), B) \]
\[ = d_{\eta_X}(\bigwedge_{\alpha \in [0, \infty]} ((\Psi(\alpha \oplus A) \ominus \alpha) \oplus A), B). \]

By Theorem 3.2(3), \((\eta_X, d_{\eta_X})\) is a fuzzy meet complete lattice.

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