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Teaching Moves for Students' Mathematical Proficiencies in Multiplication Lessons

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In this paper, we report the types of teaching moves a mathematics teacher educator attempted in his teaching of third-grade students at an urban elementary school in South Korea over two months. We analyze the lesson videos to find the patterns of teaching moves and speculate the link between the teaching and students' mathematical proficiencies recommended in the Common Core State Standards for Mathematical Practices. Closely related teaching moves to the students' development of a certain mathematical proficiency would imply the exemplary practices that teachers—both inservice and preservice teachers—can implement in their classrooms.

Keywords: SMP, Standard for Mathematical Practices, teaching moves, teaching, practices, mathematical proficiencies, standards-based instruction

ZDM Classification: D75
MSC2010 Classification: 97C70

I. INTRODUCTION

It is widely agreed that how we teach is as important as what we teach. For several decades, researchers and mathematics educators have sought for effective teaching methods and strategies to help students improve their conceptual and procedural understandings. In result, numerous research has uncovered and documented important phenomena in mathematics education. Further, the field of mathematics education has collaboratively developed and provided policies and standards for more effective learning

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and teaching of mathematics (e.g., NCTM standards). We acknowledge the contributions of the research and the standards in raising students' performance as shown in the results of the recent Trends in International Mathematics and Science Study (TIMSS; Mullis, Martin, Foy, 2016), for example. We also appreciate the efforts of the community to develop and disseminate practical ways of teaching suggested in the standards. However, we admit that the caveat between the theory and the practice still exists. Teachers demand more tangible, applicable, and exemplary practices for them to enact in their classrooms.

In mathematics teacher education, we often hear our students (preservice teachers) whining about the difficulties of changing the classroom instructions from the way they experienced as grade school students to the new and different ways that their method course instructors demand. Because we tend to teach as we were taught, adopting a new pedagogy can be a challenge for those who did not experience learning through standards-based instructions. Researchers have shown that the employment of pedagogical recommendations is not always easy for the teachers (Cuban, 1993).

In this study, we are interested in finding practical ways of teaching to develop students' mathematical proficiencies that are recommended in the Common Core State Standards for Mathematical Practices (SMP: National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO, 2010]). There is little research that studies teacher's practices in relation with students' development of mathematical proficiencies. To move forward in providing mathematics teachers instructional practices that are both doable and effective in improving student's learning and doing mathematics, we seek for the pragmatic, mathematical teaching practices that support elementary school students' development of mathematical proficiencies. Therefore, the research questions that guided this study are the following:

1. What mathematical proficiencies did the participating students show throughout the lessons?
2. What teaching moves did the teacher make in relation to the students' mathematical proficiencies?

We illustrate actual teaching moments that are closely related to the "varieties of expertise" of students (NGA & CCSSO, 2010, p. 6). First, we examine the students' mathematical proficiencies (MPs) that are illustrated in the SMP in an authentic classroom setting. Then, we examine what type of teaching moves are made by the teacher to impact the students' MPs. We pay heed to the acts and words that the teacher makes both immediately before the students' exhibited MPs and throughout the lessons.

One of the authors of this article taught multiplication lessons to a third-grade class for six weeks during the school session in 2018. We analyze the recorded videos of his sequential multiplication lessons, focusing on students' MPs and teaching moves.

II. LITERATURE REVIEW

1. TEACHING MOVES

The term, ‘teaching moves’, is used interchangeably with or distinctively from ‘instructional moves’ and ‘teacher moves’ (McEwan, 2004; Stuart, 2018). Broadly, it has been used to describe teachers’ strategies, actions, or statements to achieve an aimed learning outcome (Nilson, 2016). Focusing on mathematics education, some researchers explored teaching moves to draw out specific outcomes in mathematics classrooms. Examples are teaching moves to promote equitable participation in mathematics classrooms (Wood, et al., 2019), teaching moves to support preschoolers' arithmetical accuracy (Banse, et al., 2020), and teaching moves to respond to students' mathematical thinking (Jacobs & Empson, 2016). This line of teaching moves studies in mathematics education stems from the studies of mathematical knowledge for teaching (MKT) that Ball et al. (2008) have advocated. The MKT is an intertwined mathematical knowledge with knowledge of pedagogy, students, and curriculum. Of two domains of MKT—Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK)—studies of teaching moves focus on teachers’ PCK and their practices.

For our study, we adopt Jacobs and Empson’s (2016) notion of teaching move which is defined as “a unit of teaching activity that has coherence with respect to a purpose” (p. 186). We use the term to identify teacher’s actions or words that may elicit students’ MPs. We also take conceptual breaks in the learning situation as the unit of teaching moves instead of breaking down every word or act that teacher and students make (Jacobs & Empson, 2016).

2. STUDIES ON MATHEMATICAL TEACHING MOVES

When Stigler and Hiebert (2004) probed into the differences and commonalities of the high achieving countries’ classroom teaching, they concluded that the ways of implementing lessons matter significantly. They found that the common aspect of higher-achieving countries did “not lie in the organization of classrooms, the kinds of technologies used, or the types of problems presented to students, but in the way in which teachers and students work on problems as the lesson unfolds (p. 15).” Researchers also argued the importance of teachers’ capacity of facilitating reform-based practices such as mathematical discourse (Stein, 2007; Webel & Yeo, 2021), representation (Goldin, 2000), and technology (Schuetz, 2018).

These practices were consistent with the curriculum guidelines and standards such as the *NCTM Standards* (NCTM, 1989, 1991, 1995, 2000), the National Research Council (NRC, 2001) report, *Adding it Up*, and the *Common Core State Standards of Mathematics* (CCSSM; National Governors Association Center for Best Practices and the Council of Chief State School Officer [NGA & CCSSO], 2010). The deep-rooted concern about 'processes and proficiencies' in mathematics education is well reflected in these standards and the report. The Principle and Standards for School Mathematics (PSSM; NCTM, 2000) as well as the strands of mathematical proficiency by NRC report addressed the areas of curriculum, teaching, and assessment relating to classroom practices.

A number of studies presented considerable results about how the non-traditional curricular programs characterized effective mathematical instruction (e.g., Ball, 1993; Boaler, 1998; Carpenter et al., 1989, 2000; Cobb et al., 1991; Fennema et al., 1996; Fuson & Briars, 1990; Hiebert & Wearne, 1993; Spillane, 1999; Wood & Sellers, 1996). However, the correlation between teaching practices and student learning was not always consistent across the studies. Upon reviewing such studies, Hiebert (2003) concluded that alternative or standards-based programs were more effective in teaching and learning mathematics appropriately. Despite our conformity to this stance, we find a lack of reporting of practical teaching moves that teachers can enact in their authentic classroom settings to comply with the standards and the philosophy for the reform-based instructions.

3. STUDENTS' MATHEMATICAL PROFICIENCIES

Drawn on such research and the NCTM Standards (NCTM, 1989, 1991, 1995, 2000), the Standards for Mathematical Practices (SMP) of the CCSSM (NGA & CCSSO, 2010) endorse the teaching that improves students' conceptual understanding of key mathematical ideas and skills. The SMP is the guidelines of how teachers should seek to develop the learning in their students. The SMP explicitly delineates what mathematics teachers should nurture in all K-12 students. The eight standards of the SMP include (emphasis added):

SMP 1. *Make sense of problems and persevere in solving them (MP 1)*

SMP 2. *Reason abstractly and quantitatively (MP 2)*

SMP 3. *Construct viable arguments and critique the reasoning of others (MP 3)*

SMP 4. *Model with mathematics (MP 4)*

SMP 5. Use appropriate tools strategically.

SMP 6. Attend to precision.

SMP 7. *Look for and make use of structure (MP 5)*

SMP 8. Look for and express regularity in repeated reasoning.

These SMPs can be interpreted in two ways by focusing on “who” does “what”—1) the “expertise that *mathematics educators* at all levels should seek to develop in their students (emphasis added; NGA & CCSSO, 2010, p. 6)” and 2) the mathematical proficiencies that *students* should develop in their learning of mathematics. Here, we are interested in finding the teaching moves that are linked with students’ mathematical proficiencies (MPs). In this paper, we consider five MPs of these eight SMPs--make sense of problems and persevere in solving them (MP 1), reason abstractly and quantitatively (MP 2), construct viable arguments and critique the reasoning of others (MP 3), model with mathematics (MP 4), look for and make use of structure (MP 5). Note the assigned number for each MP is not always consistent with the SMPs. For example, ‘attend to precision’ is SMP 6, but in our MP framework, it is MP 5. These five MPs are not only the most prevalent proficiencies that emerged throughout the lessons, but they also deliver the vision of mathematics education in Korea.

4. TEACHING MOVES FOR STUDENTS’ MATHEMATICAL PROFICIENCIES

The effectiveness of student learning depends on the learning goals that teachers have—*skill efficiency* or *conceptual understanding* (Hiebert & Grouws, 2007). If a teacher aims to improve students’ computation skills, students’ high achievement on only procedural skills and fluency such as accuracy and speed of calculation would be accounted for effectiveness. In the meantime, with a learning goal of students’ conceptual understanding, a teacher strives to help students’ mental connections--knowledge, ideas, and sense making. In this paper, we are not arguing which is superior to the other. Rather, we ratify Siegler and Alibali’s (2001) claim about these two. In their iterative model, they argued that “conceptual and procedural knowledge may develop in a hand-over-hand process, rather than one type strictly preceding the other (p. 347).” Star (2005) also asserted that conceptual and procedural knowledge complement each other. Moreover, he claimed that not only the type of knowledge, but the quality of knowledge should also be accounted for. Thus, we do not limit the effectiveness of teaching solely to students’ skill efficiency or conceptual understanding.

As to student achievement, regardless of the type of student knowledge, the NRC report (2001) asserted that “opportunity to learn is widely considered the single most important predictor of student achievement (p. 334).” It defined the opportunity to learn as “circumstances that allow students to engage in and spend time on academic tasks such as working on problems, exploring situations and gathering data, listening to explanations, reading texts, or conjecturing and justifying (pp. 333-334).” We emphasize the importance of providing *opportunities to learn* to young students, because we believe that the opportunity to learn is the key “to both explain the effects of particular kinds of teaching on particular kinds of learning and improve the alignment of teaching methods with learning goals (p. 379)." Thus, we focus on capturing the types of opportunities that the teacher educator provides to the students as well as the opportunities that students take from the teacher's teaching moments.

Nonetheless, we could not find research that examined the relationships between a teacher's specific moves--provision of opportunity to learn--in a classroom and the students' performance, needless to mention the link between teaching practices and the improvement of students' mathematical proficiencies that the SMP upholds. It would be meaningful to find explicit teaching moves that are linked with students' mathematical expertise.

III. RESEARCH METHODLODY

1. CASE STUDY

This paper draws on the embedded case study (Yin, 2014) that contains two sub-unit analyses (teacher and students) during a series of mathematical lessons. Case study is an appropriate method for this study because our purpose of this study is to illuminate the teacher's teaching practices and student's mathematical proficiencies by observing lessons (via watching the video recordings) and collecting data in natural settings.

The mathematics teacher educator, Jay (pseudonym), who taught multiplication lessons and the students of the classroom are selected for the case in this study for two key reasons: 1) Jay is not an in-service teacher, but a mathematics teacher educator in a college of education. This case of teaching by the teacher educator is unique because a teacher educator rarely teaches elementary school students over a long period of time. 2) The tasks are created by Jay and the given multiplication tasks are distinct, compared with what most third-grade teachers facilitate—most teachers use the tasks from a textbook or worksheet.

The validity and reliability of case study are often in doubt. In order to establish credibility of the study, 1) we analyze a series of recorded videos with content specific lessons over an extended time period rather than a few observations. The teacher's teaching of one mathematical topic for two months allows us to observe the teaching practice patterns and the changes in students. 2) We rely on member checking techniques. Because the subject of this study is one of the authors, it is possible to continuously check with him to verify our analysis and report. Member checking is an essential process for qualitative research, because it empowers the researcher to ensure the participant voices to be portrayed accurately by letting participants confirm or deny the interpretations of data and increase credibility to the qualitative study (Creswell & Miller, 2000; Lincoln & Guba, 1986).

2. BACKGROUND OF THE STUDY

This paper is a part of a larger study that investigates students' learning, the characteristics of the tasks, and the teaching practices when an experienced mathematics teacher educator planned and taught lessons in a real elementary classroom. Typically, a teacher educator in a college teaches preservice teachers theories and/or methods of teaching. Often, there is a subject emphasis such as mathematics education. Many mathematics teacher educators have teaching experiences in the K-12 education system before they become faculty members. However, they barely go back to teach younger students and attempt to teach them with the new ideas or skills that they learn during/after post graduate education.

There have been a few mathematics teacher educators who attempted teaching K-12 students in a real classroom setting while on the faculty of a university. For example, Lampert (2001) analyzed and identified teaching problems that had emerged during her teaching mathematics to fifth grade students for a year. In a sense, our study is similar to Lampert's, because we focus on the teaching moves of one teacher with one group of students studying one subject during a rather longer period of time--over the course of one academic year for Lampert and six weeks for the multiplication topic for us. It would not be easy to implement specific instructional practices that a teacher educator aims for as well to assess their learning, if he/she teaches only one or a few class periods. Furthermore, it would not be sufficient time to observe the changes, if any, of students' attitude towards learning and effect of the practice on their achievement. The difference is that while Lampert described the holistic features of complex classroom teaching, we aim to seek the link between teacher's teaching moves and students' mathematical proficiencies.

It is meaningful for a teacher educator to experience how a theory can be put into practice in a real classroom setting with real grade students. That is because his/her teaching would be limited, if he/she has never experienced teaching real students in an actual classroom. It would also be meaningful to examine his/her practices with regard to the standards such as the SMP. That is because the practice standards are what teacher educators aim for the pre-service teachers to learn to teach. In this paper, we examine which particular teaching moves are observed in relation to each of the five MPs. We analyze the link between teacher's practice and students' growth of mathematical competencies that the SMP recommends.

3. JAY'S PROFILE

Jay is a mathematics teacher educator in a 4-year college of education in South Korea. He has taught prospective elementary school teachers mathematics education courses for about 18 years. Jay describes himself as a radical constructivist. During the interview, he expressed his beliefs that every student comes into a classroom with different prior knowledge, has different abilities to understand what they are taught, and further constructs their own knowledge. While interacting with pre-service teachers, he received some feedback about the difficulties of carrying out the pedagogical theories that he had taught. This intrigued him to study the practicality of a constructivist approach in preparing and teaching lessons for elementary school students. In his teaching of elementary school students, he tried to practice the most out of his teaching of pre-service teachers.

4. SETTING OF JAY'S TEACHINGS

Jay taught the students in one third-grade classroom of an urban elementary school in South Korea. This elementary school is affiliated to Jay's university and has a long-term partnership with the university. The elementary school teachers and the teacher educators often collaborate to help both elementary school students and pre-service teachers learn more effectively.

The third-grade class comprised twelve boys, twelve girls, and one homeroom teacher. The socio-economic status of all students in this class were in the mid to high. This class was one of three third-grade classes in this elementary school. The homeroom teacher was present to assist Jay's teaching throughout the lessons. From time to time, she restated the students' responses for Jay and/or other students to understand better due to the dialect of the students.

5. MATHEMATICAL TASKS

Jay taught two-digit by one-digit multiplication lessons across the 14 lessons. These lessons were held for two months (6/8/2018-7/25/2018). During the lessons, Jay provided 3 scaffolding tasks to the students as follows:

- Task 1: Finding the number of students
    - 1.1. How many third-grade students are in our school?
        - Numbers of students in grade 3 (3 classes): 24, 24, 24)
    - 1.2. How many total students are in grades 1-3 from our school?
        - Numbers of students in grade 1 (3 classes): 25, 24, 24
        - Numbers of students in grade 2 (3 classes): 25, 24, 25
        - Numbers of students in grade 3 (3 classes): 24, 24, 24
    - 1.3. Create your own school where there are several classes. You must use only two numbers for the number of students in each class.

- Task 2: Learn Addition and Multiplication through a children's book, *The 512 Ants on Sullivan Street*, by Carol A. Losi (Scholastic, 1997).
    - 2.1. How many ants in total were on Sullivan Street?
    - 2.2. Make your own pattern such as x2, x3, x4, ...
- Task 3: Today's Number

    Make a mathematical expression that makes 175. This expression should contain multiplication.

These tasks can be classified as rich mathematical tasks because they engage students in sense-making through discussion and scaffold their content knowledge of multiplication. These tasks showed gradual increase in the depth and breadth of knowledge, the levels of openness, and the interactions between teacher and students and among students (See Yeo, et al., 2020 for analysis of the tasks).

In this study, we analyze 4 consecutive lessons on Task 1. There are two reasons for choosing this task: 1) Task 1 is more appropriate to examine Jay's teaching practices than the other two tasks, because both Task 2 and Task 3 required the students' individual work without his instruction and 2) due to the intensity of this study, it required ample time to watch the video lessons, jot down Jay's actions as well as students' reactions for all eight SMPs.

Among many factors (including students' background, environment, tasks, etc.) that affect teacher's teaching and students' learning, we are solely interested in teacher's behavior or practice in the classroom. In the following section, we will explain how we analyzed Jay's classroom teaching practices.

6. DATA COLLECTION AND ANALYSIS

In this study, the last author taught fourteen multiplication lessons to 3rd grade students, using the tasks that he had developed. Each lesson was held for about 40 minutes. Using two camcorders, we recorded every lesson for our primary data source. Two camcorders recorded the entire classroom; One camcorder captured front of the classroom with the blackboard to follow the teacher and the students and their work presented on the board. The other captured the rest of the students who worked on their own problem solving or listened to others. Each of these recordings was reviewed and transcribed for further analysis. In addition, we collected artifacts such as students' notebooks to support the students' comprehension of the content and their MPs.

In analyzing the video data, we watched both video recordings to compare the same moment by observing both the teacher and the students holistically and in-detail at the same time. With the purpose of investigating a phenomenon of the teacher educator's real classroom teaching, each of us watched individually the recordings several times. During our regular online meetings, we discussed what we had noticed from our observation of the mathematics classroom on the lesson videos. We spent a considerable time to come up with the agreement about the teacher's and the students' actions that are closely related with the SMP descriptions. We identified the patterns of teaching practices and development of mathematical proficiencies through the analysis using a framework on the basis of the SMP. In the following section, we further describe the analytic framework for the data analysis.

7. MATHEMATICAL PROFICIENCY (MP) FRAMEWORK

In this case study, the analytic approach involves a detailed description of the mathematics teacher educator's teaching moves and the setting within its environmental and cultural context. We examine both qualitative and quantitative components. We describe the practices in the multiplication lessons using the MP Framework (Figure 1). The framework for the analysis of Jay's and his students' mathematical behaviors is derived from the descriptions of the SMP (NGA & CCSSO, 2010). Our MP Framework guides us to analyze both Jay's teaching moves and his students' MPs by examining how they are closely related.

For this study, we keep the description of the Standards for Mathematical Practices (SMPs) for the selected mathematical proficiencies (MPs) but break it down to a few key components to examine the teaching moves more in-detail. These components guide us to study not only the frequency of each MP found in students' and Jay's moves, but also more detailed teaching moves. We look for their behaviors that are closely aligned with the descriptions of the SMPs as outlined in this framework. In other words, we focus on Jay's pedagogical actions and talks as well as those of the students from our video recordings of his multiplication lessons in relation with the 5 MPs.

After identifying the key components of each MP, we watched the lesson videos thoroughly to capture and categorize every action and talk that represented the components. We watched each video several times to make sure to focus on one MP at a time. Often, we happened to watch the same video clip again and again to check every important moment was obtained in our database for analysis. In the findings, we present patterns of teaching and learning in terms of teacher's and students' pedagogical behaviors rather than mathematical content knowledge.

Mathematical Proficiencies (MPs)	Components
MP 1 Make sense of problems and persevere in solving them.	● Make sense of problems ● Monitor the progress ● Make sense of the solution
MP 2 Reason abstractly and quantitatively.	● Ability to decontextualize ● Ability to contextualize
MP 3 Construct viable arguments and critique the reasoning of others.	● Construct viable arguments ● Critique the reasoning of others
MP 4 Model with mathematics.	● Apply the mathematics in everyday life ● Simplify a complicated situation using a mathematical model ● Analyze and interpret the model
MP 5 Look for and make use of structure.	● Discern a pattern or structure ● See complicated things as single objects

Figure 1. Mathematical Proficiencies Framework

VI. FINDINGS

Jay is not a traditional elementary school teacher, even if he had some experiences of teaching in elementary schools. Jay has emphasized student-centered learning when working with his preservice teachers. For this project, he continued his effort to implement reform-based teaching in an elementary school classroom.

In the following sections, we display anecdotal information and vignettes to illustrate the phenomenon of teaching practices and students' development of mathematical expertise in accordance with the MP Framework. We present the finding not only from the qualitative analysis, but also quantitatively to see if any pattern would be found between Jay's and the students' behaviors.

MP 1. MAKE SENSE OF PROBLEMS AND PERSEVERE IN SOLVING THEM

We focus on three important aspects that characterize MP 1–make sense of problems, monitor the progress, and make sense of the solution. As shown in Table 1 below, we noticed a number of moments that Jay helped students work on the given tasks in various ways throughout the lessons. The students also showed some proficiencies in making sense of the tasks and persisting to solve them. For all three tasks, Jay clarified the situation in the problem, explained it with simpler numbers, connected their backgrounds to the situation, and encouraged students to explain their reasoning of their solutions to others.

Table 1. Frequencies of MP 1 occurrences

Components of MP 1	Jay's MP 1	Students' MP 1
A. Make sense of problems	4	1
B. Monitor the progress	9	9
C. Make sense of the solution	13	15
Total	26	24

To help students monitor their progress in solving problems, Jay asked the students to write their work on the notebooks, check each step of the work, and make an adjustment if anything does not make sense to themselves. In return, we observed students monitoring their progress. They often verified their strategies with their partners and checked each other's work. While we found the moments for all three components, some of them

Vignette 1

Jay: In my daughter's elementary school, they had eight 1st grade classes and ten 6th grade classes. That's a lot of classes. Unlike your school here, the number of students in different classes for the same grade is not the same.

Students: What?

Jay: So, the number of students in each class is different for all classes. But, here at Applewood Elementary School[1], as you guys had said before, there are 24 students in each class, right? So right now, how many students are in Class 1 of 3rd grade?

Students: 24 students.

Jay: There are 24 students, right? In 3rd grade, how many classes are there?

Student A: Three classes.

Jay: There are three classes, and each class has 24 students, right? Let's calculate the total number of students in 3rd grade in Applewood Elementary School.

[1] Every name of school and students is a pseudonym.

appeared more than others. For example, we observed more turns between Jay and the students discussing the solution strategies to make sense of the students' various solutions. We found that the number of occurrences for each component observed in Jay's teaching moves almost matched with the number of occurrences in students' MP components.

Jay tried to make sense of problems with the students. He continuously explained the situation for each of the three tasks by making it relevant to the students so that they could immerse themselves in the task. Shown in the Vignette 1, Jay created a classroom atmosphere that students could feel comfortable by talking about his own daughter's school and the number of students in her school as well as by comparing that with their own school's student numbers given in the task. After this, students explained the meaning of the problem to their partners and jumped into solving them.

Monitoring progress is another element we have noticed frequently throughout the lessons. The students progressively developed this proficiency as we observed them monitoring their work sometimes by explaining to their partners and other times by checking the answers with different methods. For example, the students deliberately looked for another way to find the solution. This sporadically made other students more confused. Such confusion, though, challenged other students to think logically to make sense of the different solutions.

Vignette 2

Jay: Student B told me, "I can calculate this differently from how Student C did." Let's listen to how Student B calculated it differently. Come forward, Student B.

Student B: There are ten of 3s so far. Since 3 times 10 is 30, up to this point, it is 30.

Jay: Student B, are you saying that you counted by 3s?

Student B: Yes

Jay: So far, how many of 3's is there? Ten. So, what Student B said was, because there are ten 3's. (Writing 3x10.). Everyone, what is ten of 3s?

Students: 30.

Jay: Yes, 30. Move on.

Student B: Then again one, two, three, four, five, six, seven, eight, nine, ten. Up to here, you do it two more times. (Student B writes '3x10=30' again underneath '3x10=30 – written by the teacher') Then you add the remaining four of 3s and it is 12.

Jay: Yes. Adding the remaining four of 3s is 12. Student B, you can use multiplication instead of addition, right? And when we all recited the times table, everyone counted up to 9 times, but you all know how to do 3x10, 4x10, 5x10, don't you? So, Student B didn't add every number, but he multiplied to find the overall sum. Student C, do you have anything to add on?

Jay facilitated whole class discussions using various talk moves (Chapin, et al., 2013). By using productive talk moves such as repeating, revoicing, and adding on, he asked the students if the discussed strategies and the solutions made sense. Shown in Vignette 2, Jay uses revoicing ("So, what Student B said was, because there are ten 3's.") and adding-on ("Student C, do you have anything to add on?") talk moves. These talk moves served for students' making sense of the solutions of others. We will discuss these talk moves again for the MP 3 (Construct viable arguments and critique the reasoning of others.).

MP 2. REASON ABSTRACTLY AND QUANTITATIVELY.

In Table 2, we present the frequency of the moments for two components–ability to decontextualize and ability to contextualize. This is one of the most prevalent proficiencies that we observed across the six multiplication lessons. When the students were asked to find the total number of students in all three tasks —task 1: total number of grade 3 in Applewood Elementary School, task 2: total number of grades 1-3 in Applewood Elementary School, and task 3: total number of students in their own made-up school, they decontextualized the situations and represented them using numbers and symbols immediately. They often manipulated the representations without attending to their referents in the context.

Jay often guided the students to contextualize the meaning of the numbers they were manipulating and the quantities that they found as the answer to the problem. He led discussions by asking questions within the problem situations for the students to make sense of the quantities in the context. For example, Jay questioned the whole class why Student C added 24s first by switching the order of the numbers (addends) in the given context. The information given to students (Figure 3) asked the students to find the sum of 25, 24, 24, 25, 24, 25, 24, 24, and 24. As shown in Vignette 3 and Figure 2 below, Student C grouped 24s and 25s and found the sums of each group. He referred to the number of students in each three grades when the students tried to explain their work by only using the numbers and symbols in their work without relating them to the situation.

Table 2. Frequencies of MP 2 occurrences

Components of MP 2	Jay's MP 2	Students' MP 2
A. Ability to *decontextualize*	6	12
B. Ability to *contextualize*	20	11
Total	31	23

Jay's actions helped students understand and flexibly use properties such as commutative property here and distributive property later. Shown in the above vignette, a student triggered "a rule" for changing the order of the operands that would not change the result (commutative property) without the teacher's introduction of the property. At that, Jay showed another example of commutative property of addition and more students used this property to find the sums of the student number in the tasks.

Vignette 3

Jay: Student B, are you saying that Student C just changed the number at the bottom?

Student B: You can change the addition at the top and bottom at the same time.

Jay: I will ask the question again. [pointing at Student C's work of 24+24+24+24 +24 +24: Figure 2]. Let's consider this together. Can you explain what Student B meant by changing the addition at the top and the bottom at the same time?

[Students are silent]

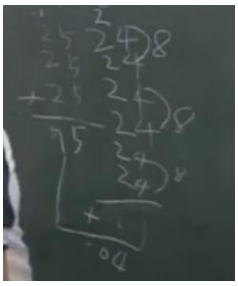


Figure 2. Student C's work

Jay: [Pointing at the number of students in grade 1: 25, 24, 24, grade 2: 25, 24, 25, and grade 3: 24, 24, 24 on the corner of the blackboard as a reference also shown in Figure 3] The number of students in grade 1, 2 and 3 is in this order, but Student C added student numbers of 3rd grade first, then he did for grade 1 [at this moment a student says "Oh, I get it."]. Then he did for grade 2, by switching the order, so looking at the order being switched, if you think about it a little deeper, [One student says, "there is a rule for that."] wouldn't there be a reason why Student C did "24+24+24..." first?

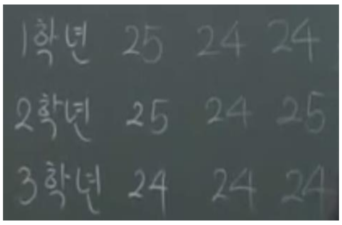


Figure 3. Applewood Elementary School grades 1-3 student numbers

MP 3. CONSTRUCT VIABLE ARGUMENTS AND CRITIQUE THE REASONING OF OTHERS.

Students became progressively familiarized with explaining and arguing various ways of problem solving over time. This seemed closely related to how Jay orchestrated the classroom discussions. Jay's facilitation of group discussions to help students construct their own knowledge was a potent teaching pedagogy driven by his constructivist beliefs (Richardson, 2003). Because Jay constantly provided students opportunities to discuss various strategies—most of which were invented by the students—it was not easy to mark a moment of MP 3. Thus, the number of frequencies here indicates longer blocks of time than other MPs' data. For this proficiency, we examined two major elements: 1) construct viable arguments and 2) critique the reasoning of others. For the first component, we examined how Jay helped students explain and discuss their thinking processes and how students explained their ideas. For the second, we examined Jay's guidance for students' listening and asking questions to others' explanations.

Table 3. Frequencies of MP 3 occurrences

Components of MP 3	Jay's MP 3	Students' MP 3
A. Construct viable arguments	4	5
B. Critique the reasoning of others	21	14
Total	25	19

As shown in Table 3, we observed a number of instances that Jay assisted the students to prepare for and engage in conversations. He promoted student engagement by creating a routine in which all students worked individually when a task was initially given, then calling up a few students to present their works on the front board. While these students wrote down their work on the board, the rest of the students continued to find more/different strategies for the problem or watched what was written on the board. Next, Jay asked either the students at the board to explain their own work or the audience to explain their peers' presented work. Creating this routine helped students be actively engaged in classroom discussions and feel comfortable presenting in front of others.

Vignette 4

Student B: Since you have to do 24 times 3, I first divided 24 by 2 because 12 can be split into two. I divided 24 into 12. Then 3 at the end, if you multiply 12 and 3, you get 36. Because 12 times 3 is 36 also, 36 added twice will give you 72 so the answer is 72.

Jay: Did you all hear what Student B said?

Students: Yes!

Jay: Do you agree with what he said? Can someone explain it to us?

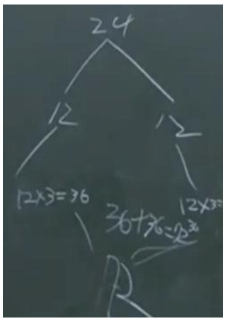


Figure 4. Decomposing 24 to multiply by 3

Throughout the lessons, we found that Jay showed mathematical talk moves that Chapin and her colleagues (2013) advocated to deepen students' mathematical understanding and learning during the classroom discussions. The talk moves include Say More, Revoicing, Repeating, Press for Reasoning, Agree or Disagree: … and Why, Add On, and Wait time. A couple instances were explained in MP 1 (in Vignette 2) earlier. For another example, Jay asked the students if they *agreed* and *why* it was true when Student B had multiplied 3 to each 12 after he decomposed 24 into 12 + 12 as shown in Vignette 4 and Figure 4.

A common complaint that we often hear from teachers is that students refuse to be called up to present their work. The students in this classroom, however, were far from the story. The students were eager to talk about their and other's work. When making decisions regarding the sequence of student presentations, some experts suggest connecting students' ideas. For example, a teacher may first call on a student with an incorrect answer to draw attention to a common mistake or misconception, then a student with more mathematically sophisticated strategies (Lampert, 2001, Smith & Stein, 2011). When Jay selected a student's work to be presented, he considered not only the sequence, but also inclusion of all students. He intended to call up more students to build their confidence in speaking out their thoughts as well as to show everyone's idea is respected.

MP 4. MODEL WITH MATHEMATICS.

In mathematics education, "representation" and "mathematical model" have been used interchangeably, meaning “a mathematical representation of the elements and relationships in an idealized version of a complex phenomenon. Mathematical models can be used to clarify and interpret the phenomenon and to solve problems” (NCTM, 2000, p. 70). However, this standard distinguishes the meaning of “Model with Mathematics" from representation by focusing on a great deal of application of mathematics (Colen, 2019; Malkevitch, 2012; NGA & CCSSO, 2010). For our analysis, we highlighted three important aspects of this proficiency: application, representation, and interpretation. In Table 4 below, we present how often we have observed these three elements throughout the lessons. This proficiency, though, could not be easily counted, because of the complexity of teaching as well as the characteristics of three tasks. First, teaching is a complex action and is not a clean-cut business. All actions during the classroom teaching are rather intertwined and the proficiencies are situated in the setting. Second, the tasks designed by Jay are real-world tasks that students can apply their mathematical knowledge to solve. Thus, regardless of the numbers in the table, MP 4 transpired in all 6 lessons. We only counted the significant instances that could be categorized for each component.

Table 4. Frequencies of MP 4 occurrences

Components of MP 4	Jay’s MP 4	Students’ MP 4
A. Apply mathematics	2	2
B. Simplify and represent	8	4
C. Analyze and interpret	3	2
Total	13	8

Besides providing the tasks relevant to the students and having them apply mathematical knowledge of repeated addition and multiplication, Jay encouraged the students to represent the task situation using mathematical expressions, diagrams, and tables. For example, Jay noticed that many students were confused when student B at the board explained his strategies using only numbers (See Figure 4). Jay first drew three long rectangles to represent each of the three 3rd grade classes of Applewood Elementary School. Then, he drew circles inside the rectangles to represent 24 students in the classroom as shown in Figure 5. Then he drew a segment to break 24 circles in each rectangle into 12 and 12 to explain Student B’s decomposition strategy shown in Figure 4. At this moment, several students nodded their heads or uttered “Oh, I get it now!” Such representation using tools such as diagrams or tables helped students understand other students' explanations and identify important quantities and relationships (NGA & CCSSO, 2010; Fennell & Rowan, 2001; NCTM, 2000).

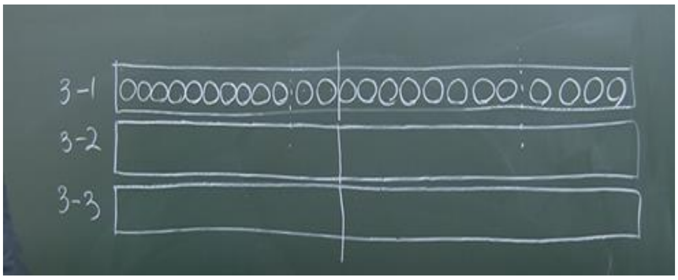


Figure 5. Representation of grade 3 students in Applewood Elementary School

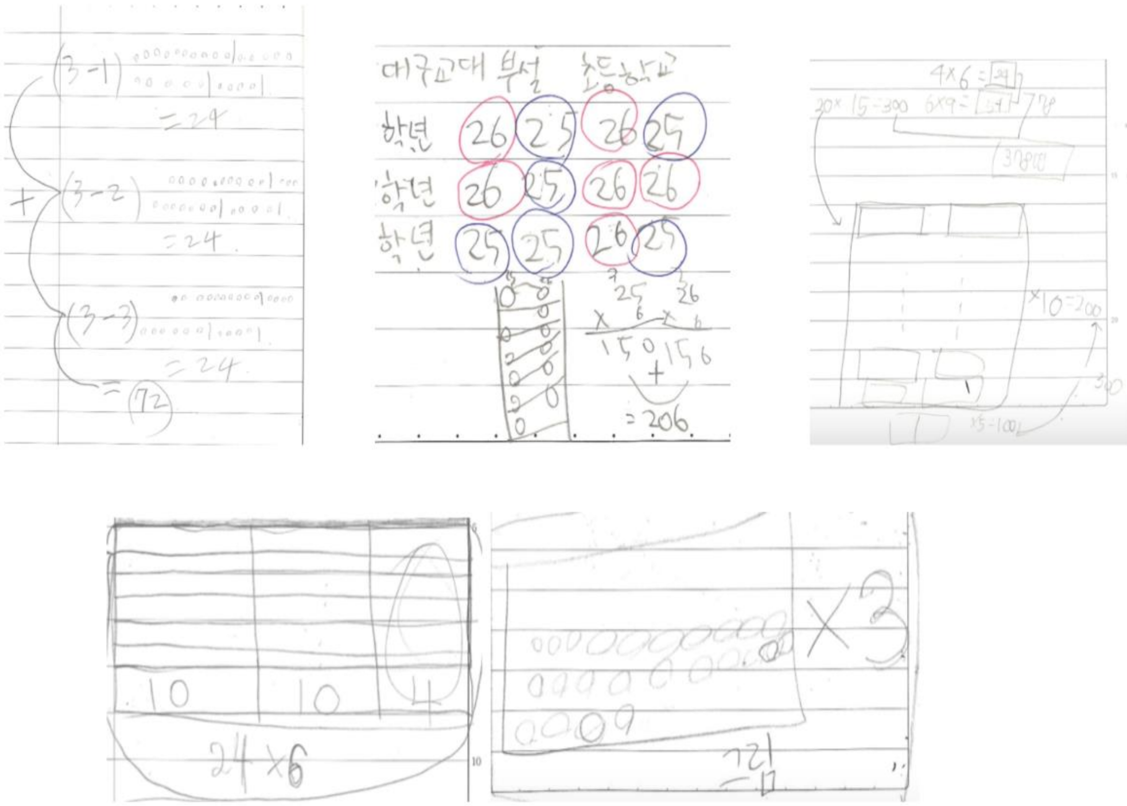


Figure 6. Examples of students' mathematical representation

When we examined the students' notebooks later, we found that several students used diagrams to represent the numbers that they dealt with as shown in Figure 6. Although we did not count any notebook examples for the frequency in the tables, it was significant that the teacher's one exemplary diagram usage influenced the students to draw diagrams for their better understanding of the problems mathematically and for describing the solutions to problems in easier ways. Not only the mathematics standards (NCTM, 2000; NGA & CCSSO, 2010), but also numerous studies have shown the effectiveness of representations such as diagrams for problem solving. A representation helps students make sense of the problem by organizing and representing the relevant information in the problem situation (Jitendra, 2002).

MP 5. LOOK FOR AND MAKE USE OF STRUCTURE

We watched four consecutive lesson videos several time and focused on two important components of MP 5 – discern a pattern or structure and see complicated things as single objects. As shown in Table 5 below, we noticed numerous moments that Jay helped students not only developing their capacity to discern a pattern or structure but also identifying and evaluating efficient strategies for solution, and the students identified different strategies for problem solving and used many different skills to determine the answer. But there is no "see complicated things as single objects or composed of several objects" moment from both Jay and the students. It may be due to the characteristics of the task in which the numbers or situations that the students worked with were not complicating.

Table 5. Frequencies of MP 5 occurrences

Components of MP 5	Jay's MP 5	Students' MP 5
A. Discern a pattern or structure	17	24
B. See complicated things as single objects	0	0
Total	17	24

From our analysis of MP 5, we found three important teaching moves that seemed to help students developing MP 5: 1) prompting the students to identify mathematical pattern or structure of the task in order to identify the most effective strategy, 2) encouraging students to justify/question the reasonableness of their/others' strategies, and 3) assisting students make connections between different strategies and student thinking.

These teaching moves fostered the students to consider various pathways to find the total number of students in Task 1. Some have constructed their own understanding of multiplication and others have deepened their understanding of the concept of multiplication based on what other students had presented. It is noticeable that important mathematical concepts emerged, while students were presenting their own thinking. Shown in Vignette 4 (for MP 3), Student B considered 24 as composition of two 12s. Instead of multiplying 24 by 3, he did 12x3+12x3 because he could calculate 12x3 in his head. This was the moment when the distributive property was unintentionally introduced to the students. Even if a third-grade student might have not learned this property yet, he developed his understanding of this concept that 24x3 = (12+12) x 3 = 12x3+12x3.

Jay not only asked Student B to explain what his idea was about, but he also asked other students to repeat what Student B had explained. When there still seemed to be confusion in some students and no other students volunteered to explain why this made sense, Jay represented this concept by drawing two groups of 12 circles which represent 12 students in each class (Figure 5).

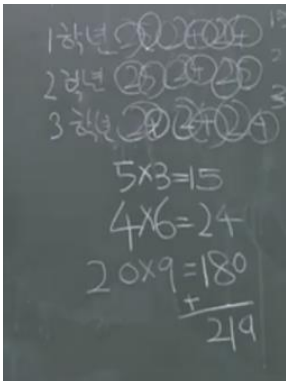


Figure 7. Examples of students' making use of structure

When solving the total number of students from grade 1 to grade 3, Student D decomposed 25 into 20+5 and 24 into 20+4 to make the problem easier for her. She counted 3 classes with 5 in ones place to find 3x5=15; 6 classes with 4 in ones' place to find 4x6=24; and every six class having 2 in tens' place to find 20x9=180. She added these three sums to find the total number of the students, 219 (Figure 7). We observed more students showed alternative ways, but often by picking up the previously shown strategies, as Jay encouraged them to consider other and effective strategies.

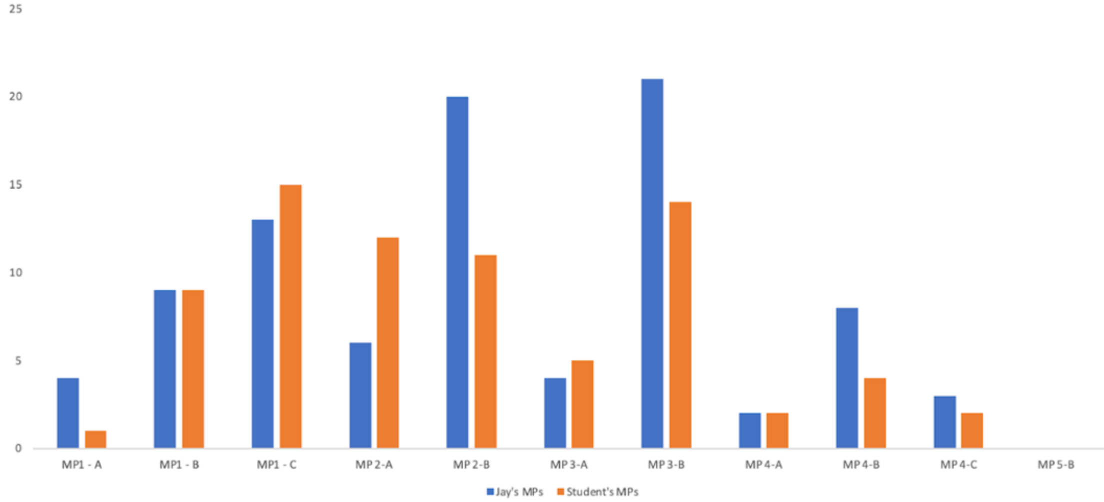
When Student E solved the total number of second grade students for 99, 99, and 99 students for three classes on Task 1.3, he used a multiplication of 99x3 and wrote 99x3 = 297 without further explanation. This drew Jay's attention. When Jay asked other students how they would solve 99x3, students showed different strategies, either using standard algorithm or invented strategies. Due to the lengthy transcript, we do not show it in a vignette, but in Appendix B. Briefly to say, an invented strategy, which most of the students agreed to use to find 99x3, was using the multiplication fact of 100x3 = 300. Several students explained 99x3 could be solved by 100x3-3 = 300-3 = 297 because 99 = 100-1.

V. DISCUSSION AND CONCLUSION

The purpose of this study is to provide some examples of practical teaching moves to pre- and in-service teachers. The authors of this article are mathematics teacher educators either in the U.S.A. or South Korea. Regardless of some differences in the standards, curriculum, and culture of the two countries, we commonly heard our preservice teachers' grumbling about the difficulty to implement the suggested mathematical practices. We aimed to unfold how a math teacher educator, who has experience of teaching elementary school students for about a decade, helped the young students develop their mathematical proficiencies. Researchers featured mathematical proficiencies differently by the demand or trend of the education field (Groves, 2012; Schoenfeld, 2007). Standards for Mathematical Practice (SMP) delineates expertise that mathematics teachers should seek to promote in their students. Among the eight suggested expertise by the SMP, we took account of five that emerged the most in Jay's classroom teaching. These five expertise or mathematical proficiencies (MPs) also are greatly emphasized in Korea's mathematics education curriculum (Ministry of Education, 2015).

We displayed Jay's teaching moves by connecting it to students' MPs using the MP Framework (Figure 1). In the vignettes, we provided Jay's specific teaching moves–either planned ahead of time or responsive to students' understanding–for each MP. We provided not only the qualitative data and interpretation, but also the frequencies of Jay's teaching moves for each component for every MP. When analyzing and interpreting the teaching moves in relation to students' MPs, we found several interesting things that we wanted to discuss here.

Table 6. Relationship between Jay's and students' MPs



First, Jay showed various teaching moves to advance students' MPs to some degree. Jay was a mathematics teacher educator in South Korea, where the distinct process standards were circulated. Despite no intention to implement any of the MPs, Jay carried out varieties of teaching moves linked with the MPs. One factor that reinforced such teaching moves was Jay's beliefs on mathematics teaching and learning. According to Jay, "I believe in students' ability to construct their own knowledge. In my teaching, I try to provide an environment where all students freely discuss their thinking" (Jay, personal communication, May 18, 2021). His beliefs on teaching and learning deriving from his deliberate research might have impacted his teaching moves that were aligned with the MPs. We identified the shared interest/focus of mathematics education across the nations. In recent decades, global projects, assessments, and research have increased (e.g., TIMSS, international conferences, etc.). Through such collaboration and communication, the borderline between the countries for mathematics education has been disappearing.

Second, we found that the teaching moves and students' MPs are closely related. When examining the numbers of occurrence for the components for each MP, we noticed that the numbers match. In other words, when we had a high number of occurrences for a Jay's MP component, we also achieved a high number of occurrences for the same MP component from the students. The relationship between Jay's and students' MPs per component is shown in Table 6. This finding implies the effects of a teacher's teaching on students' mathematical expertise. While previous studies investigated the relationship between teaching practices (or moves) on students' performance (e.g., Boaler, 1998; Carpenter et al., 1989, 2000; Cobb et al., 1991; Fennema et al., 1996; Fuson & Briars, 1990; Hiebert & Wearne, 1993; Spillane, 1999), we suggest further studies of the relationship between teaching and students' mathematical proficiencies or expertise.

Third, a certain teaching move does not infer one MP. From our findings, we noticed that a productive teaching move affects not only the closely related MP, but also other MPs. It might be due to the complexity of teaching and learning. All actions during the Jay's classroom teaching are rather intertwined and the proficiencies are situated in the setting. For example, Jay's talk moves during the whole class discussion had an effect not only on students' MP 3 but other MPs (MP 1, MP 2, and MP 5). Another example is from Vignette 4 when Student B decomposed 24 into 12 and 12 and multiplied it by 3. Jay pressed both Student B and all other students the reasoning of how it worked instead of delivering the knowledge of distributive property to them. Such talk moves supported students' MP 3 and MP 5.

The purpose of this study was to provide pre- and in-service teachers as well as teacher educators some exemplary teaching moves that were related with students' MPs. Although only the third-grade multiplication lessons were examined in this study, we encourage the readers to extend the teaching moves to different mathematical topics or to different grades.

We also suggest future studies to develop teaching moves, either by developing a framework or revising a previously developed framework. A more organized framework will allow the researchers to generalize effective teaching moves for various purposes such as students' performance, productive discussion, and mathematical proficiencies.

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Appendix A

Mathematical Proficiency Framework (adopted from the SMP; NGA & CCSSO, 2010)

MP 1 Make sense of problems and persevere in solving them.

A. **Make sense of problems:** Explain to themselves the meaning of a problem and looking for entry points; Analyze what is given to explain to themselves the meaning of the problem.
B. **Monitor the progress:** Plan a solution pathway instead of jumping to a solution; monitor the progress and change course if necessary and use various representations such as drawing diagrams of important features and relationships; graphing data and searching for regularity or trends; use concrete objects or pictures to help conceptualize and solve a problem.
C. **Make sense of the solution:** Students ask themselves, "Does this make sense?"; understand various approaches to solutions; identify correspondences between different approaches.

MP 2 Reason abstractly and quantitatively.

A. **Ability to decontextualize**: to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents; Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; Knowing and flexibly using different properties of operations and objects.
B. **Ability to contextualize**: to pause as needed during the manipulation process in order to probe into the referents for the symbols involved; Considering the units involved; Attending to the meaning of quantities, not just how to compute them

MP 3 Construct viable arguments and critique the reasoning of others.

A. **Construct viable arguments:** Understand and use stated assumptions, definitions, and previously established results in constructing arguments; Make conjectures and build a logical progression of statements to explore the truth of their conjectures; Analyze situations by breaking them into cases; Justify their conclusions, communicate them to others, and respond to the arguments of others; Recognize and use counterexamples; Reason inductively about data, making plausible arguments that take into account the context.
B. **Critique the reasoning of others:** Listen or read the arguments of others; Decide whether they make sense, and ask useful questions; Distinguish correct logic or reasoning from that which is flawed

MP 4 Model with mathematics.

A. **Apply the mathematics** they know to solve problems arising **in everyday life**, society, and the workplace.
B. **Simplify a complicated situation,** realizing that these may need revision later, by identifying important quantities in a practical situation; Map (represent) their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
C. **Interpret the model:** Analyze those relationships mathematically to draw conclusions; Interpret their mathematical results in the context of the situation; Reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

MP 5 Look for and make use of structure.

A. **Discern a pattern or structure**: look closely to discern a pattern or structure
B. **See complicated things**, such as some algebraic expressions, **as single objects or composed of several objects**.

Appendix B

Discussion after Student E's Strategy Was Presented

Jay: Okay, this time Student E is going to come up and share her work. The number of students seems to be a lot. Let's have a look at Student E's.

Student E: (Maybe he is judging that the students aren't ready for class) I think I have to wait. Three classes of Grade 1 have 90 students. So, 90 times 3 is 270. Then in Grade 2, there are 99 students in three classes, so 99 times 3 is 297. Then in Grade 3, 90 students in two classes and 99 students in one class so 90 times 2 is 180 and 99 times 1 is 99. 270 plus 297 is 567 and 180 plus 567 is 749. 749 plus 99 is eighty and forty-six so the total is 846.

Jay: Student E, I guess you said something wrong at the end?

Student B: Eighty forty-six.

Jay: You said eighty forty-six.

Student E: (Looking at his writing on the board) I wrote eight hundred.

Jay: He said eight hundred. Student E said this: How many students are there in one class in Grade 1?

Students: 90 students.

Jay: How many classes?

Students: Three classes.

Jay: So, if you look up there, there are 3 classes with 90 students each in Grade 1, therefore 270. In Grade 2, how many students are there in each class?

Students: 99 students.

Jay: Grade 2 has 3 classes with 99 students each. So, looking over there, it says 99 times 3. Student E wrote that 99 times 3 is 297, anyone can explain how 99 times 3 is 297? (Several students raise their hands.) Student F.

Student F: Three of 99s is 27. But …

Jay: Student F, say it again. 99 …

Student F: Three of 99s is 27.

Student B: Uhhh!

Jay: What do you mean?

Student F: 9 times 3 is 27.

Jay and student: I think you said 99 a while ago, Student F. (Writing 9x3=27 on the blackboard) If there are three of 9s then it is 27. This is what you said it.

Student F: Yes. But then it's 90 over there. If you change 9 to 90 then you have to put 0 next to 27. So, it becomes 270 and then the number in the ones place has to be multiplied, so 9 times 3 again and becomes 297.

Jay: I wrote it like this, but I can't write it like this, right? Can I do it like this? Or can't I?

Students: (as if in a chorus and loudly) No, you can't.

Jay: Student E said that 99 times 3 is 297 before.

Students: Yes.

Jay: But, in doing so, we get 54, right? I guess something went wrong. (Students start to raise their hand one by one.) Anyone can explain the reason why you can do it or can't do it? Student G. (After picking Student G, Student H puts his hand down and shakes her head sadly) Student H, don't be sad. Student G may be thinking something different than you. Student G, go ahead.

Student G: I don't think you can do it like that.

Jay: Why can't I?

Student G: 27 is in the tens place. 270 is in the hundreds place. You can't add the hundreds place and the tens place together. So that's why you can't do it like that.

Student H: Same.

Jay: Student H has the same idea too?

Student H : Yes.

Jay: When you add, you got to add the numbers in the same place. But just now we added numbers in the different places. We can't do that. So, in doing addition, we have to make sure that (writing the number in the correct position. [Figure]) The numbers are in the right place. You understand?

Students: Yes.

Jay: You should place the numbers in the correct position – add the numbers in the ones place together and add the numbers in the tens place together. As there are no hundreds, you can just write it here and you will get 297 as Student E did.

Student B: But teacher, there is an expression for this. When you times 20 and 15, 20 times 5 is 100 and 20 times 10 is 200, but the after-school tutor told me that I can write 20. I don't know the reason though. (Reconstructed with respect to the meaning)

Teacher: (After listening to Student B's explanation, writing on the board.) Student B said. Everyone, in calculation of 20 times 15, you can write 20 times 10 is 200. But, instead of writing 200, here you don't need to write the ones place and just write 20. I assume Student B's after-school tutor explained in this way

Student I: (With a loud voice) Oh, you don't have to write it.

Teacher: You don't have to write it, but he said he doesn't know why he doesn't have to. So, I'm going to explain. I'm explaining but I'm going to give you time to think about it. To help you guys to think, I'm going to give you guys a clue and the clue was in Student H 's explaining.

Students: Teacher, pick me up. (In an appealing voice, pleading to be called) Ah ha. I get it. Teacher, me. (There are some students who raise their hands quietly, and some writes something in their own notebooks. After about 20 seconds)

Teacher: Let's listen to Student I first and Student B can go ahead.

Student I: …… The number in the hundreds place is written together… I will do it on the blackboard. But from here I thought that only the place number(value) had to be in place. Because 0 means there is nothing, you can take this (the ones place in 200) out and you just have to put this and this in the correct place.

Student: Then, the 0 in the tens place can be taken out. (When he is saying, students are giving their feedback. Some students say that he can while others say he can't)

Student I: Usually when I multiply two-digit numbers with two-digit numbers, I just wrote the numbers in the correct place.

Student B: Teacher, the number is different, but I also put underneath the thousands place. I'll come up and do it. (Coming up to the board.) I have done it up to the thousands place. In doing this, you don’t have to take out the 0 at the bottom. That is how I learned it.

Teacher: (To Student I and Student B) Please sit down.

Student: I think I know.

Teacher: You think you know?

Student: Yes.